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A TREATISE

ON

INFINITESIMAL CALCULUS;

CONTAINING

DIFFERENTIAL AND INTEGRAL CALCULUS, CALCULUS OF VARIATIONS, APPLICATIONS TO ALGEBRA AND GEOMETRY, AND ANALYTICAL MECHANICS.

BY

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VOL. IV.

THE DYNAMICS OF MATERIAL SYSTEMS.

"Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires."—CH. DUPIN.

OXFORD:

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PREFACE.

ALTHOUGH I have entitled the present Volume, "Dynamics of Material Systems;" yet the investigations contained in it are far from comprising all which a complete treatise on that subject requires. They are indeed almost wholly confined to those particular systems in which the internal forces, brought into action, either effectively or potentially, by means of the external forces, enter in equal and opposite pairs; so that they disappear in the equations of motion formed on D'Alembert's principle. I say almost wholly, because, in the last Chapter but one of the Volume, the motion of the particles of an elastic body is to a certain extent discussed: and herein the elastic forces, which are internal forces, do not disappear, but enter as effective forces, the action of which is determined by Hooke's law, or by an equivalent assumption of a property of such matter. In all other cases, in a rigid body, in a rigid system which is maintained in a state of relative rest by rigid rods and similar modes of constraint, in systems wherein every mutual action of attraction or repulsion is accompanied by an equal and opposite reaction, the internal forces disappear from the equations of motion.

The expediency, nay, almost the necessity, of giving

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a geometrical image to complicated mechanical laws demanded the insertion of a preliminary Chapter, which should contain the required geometrical theorems. The importance of familiarity with the symbols of this Chapter and their symmetrical manipulation, with the linear and angular directions, with the geometrical forms, which being in tridimensional space are difficult of imagination, cannot be overestimated. A system of notation has also been hereby obtained, and this is preserved uniformly throughout the Treatise.

The motion of a system is of course more complicated than that of a single particle, and thus greater prominence has been given to the distinction between cinematics and dynamics than was necessary in the preceding Volume. A force surely cannot be perfectly apprehended as to its effects on the motion of a system, unless the effects have been previously examined, and, I may say, examined in all their generality. Hence arises the importance of the Chapter on cinematics, in which it is shewn that the most general motion of a rigid system is compounded of, and may be resolved into, a translation of any particle and a rotation about an axis which passes through that particle. The applications of this analysis of motion in the subsequent parts of the Treatise are many and various. The Analytical Table of Contents will sufficiently indicate the course of inquiry: it will manifest the logical sequence of the several parts; the first formation of the equations of motion by means of D'Alembert's principle: the theorems deduced from particular forms of these equations: the more general theorems and principles which they involve: the transformation of the equations of rotation into angular velocities: the consequent geometry of masses, and the theory of principal axes, and their

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distribution in space: the motion of a body subject to constraint either of a fixed axis or of a fixed point; that of a body perfectly free from all constraint; the theory of relative motion; and the theory of machines in motion and of *work* done thereby. I may, too, observe, that we are herein led to some curious properties of mechanical units, and to the mode of reducing all force-action to an uniform standard of mechanical work.

The concluding Chapter is the work of Mr. W. F. Donkin, M.A., F.R.S., of University College, and Savilian Professor of Astronomy, Oxford; "Theoretical Dynamics" is the subject of it; and the theory is discussed which assigns the number and the order of the differential equations of motion in the most general problem; the possibility of the solution of some or all of them; and the forms of the resulting integrals. The Lagrangian and Hamiltonian equations are investigated; and theorems, important in reference to these equations, discovered by Poisson, Jacobi, Professor Donkin himself, M. Bour, M. Liouville, are demonstrated. Perhaps no one is better able to expound the difficulties of the theory than the accomplished mathematician who has contributed the Chapter; he has studied the subject, and has made real advances in it.

Investigations of problems of a special character are introduced more generally than in the preceding Volume. Thus the subject of the precession and nutation of the earth's axis has been considered at some length; the apparent effects due to the earth's diurnal rotation on the action of the pendulum-experiment devised by Foucault, and of the gyroscope, on the deviation of heavy bodies, whether falling freely or projected with high velocities, are discussed with considerable minuteness. This course has been to a

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certain extent unavoidable; because cosmical phænomena, and machines devised to illustrate them, are the most simple and most appropriate examples of general mechanical processes; consequently it has been unnecessary to devise hard problems for the purpose of exhibiting the power of the equations, when they are best illustrated by the movements in which we ourselves daily take part. It is thought also that the utility of the work is hereby increased.

The general principle on which the equations of motion are formed is the same as that which is so frequently and so prominently stated in the preceding Volume; viz., the equality of the impressed and the expressed momentum on a single particle. This principle is indeed directly applicable to the determination of the motion of a material system, only when the internal forces which act on the several molecules are taken account of; and as the nature, the laws, and the action of these forces are generally unknown, some other mode of estimating the general results is required. If the system is so organized that the internal forces enter in equal and opposite pairs, they disappear in the equations of motion, and the circumstances are expressed without difficulty in a sufficient number of equations.

All the incidents of motion which arise out of continuous laws are expressed as infinitesimals. This is indeed the reason why the present and the preceding Volumes, in which mechanical subjects are treated, are included in a course of Infinitesimal Calculus. Infinitesimals are, as heretofore, stated and applied in their barest forms; and subject to the axiomatic properties of Art. 9, Vol. I. Infinitesimals and finite quantities are the *materies* of calculation according to the same laws. And it is submitted that continuous laws can only thus be adequately expressed symbolically.

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The object of the Author has been the construction of an uniform scientific Treatise, pervaded by one idea, and applying one principle. Thus, at the outset, a certain form is given to the equations of motion by D'Alembert's principle; and in that form they are applied to all subsequent purposes and processes; for they are directly applicable to all classes of dynamical problems. In some cases indeed they are conveniently applied in the transformed state, as they are known by the name of Euler's Equations; generally however all special artifices, however ingenious they may be, and whatever abridgement of work they may introduce, are avoided; the circumstances of a problem are resolved into their simplest elements, and these are expressed by the general equations and their integrals.

The object again is not to make new discoveries or to open new lines of research; but to use the present knowledge and the present materials; to digest, to arrange, to consolidate all into one harmonious Treatise; to make such additions as are necessary for the process; and to present all to a student on an uniform plan. The Treatise has arisen out of the want which the Author himself has frequently experienced in his professional employment; and the attempt to supply that want has given to the work its didactic character and its colloquial style.

The Author is of course under obligations to many writers on Mechanics and kindred subjects. These obligations he has attempted to acknowledge from time to time, as well as to specify the treatises wherein certain subjects have been originally or fully treated. It has however been impossible to satisfy the claims of such writers in all cases. In many cases, the Author has found, that theorems, to which he was led in the course of his investigation, had been previously

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discovered, and he is also bound to say, that many theorems which are attributed to certain authors have been known and proved long before the time of the writer to whom the credit is commonly given. This is a disappointment to which an inquirer in any branch of science must be liable. He will rejoice however to find that truth has advanced, although his share in the work may not be as large as he might expect. The benefit of the progress will be permanent, his disappointment will be temporary; and if he will take heed to use it aright, it will be an inducement to greater industry and further research.

PEMBROKE COLLEGE, OXFORD,
Nov. 14, 1861.

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ANALYTICAL MECHANICS.

PART III.

DYNAMICS; THE MOTION OF A MATERIAL SYSTEM.

CHAPTER I.

PRELIMINARY GEOMETRICAL INVESTIGATIONS.

ARTICLE 1.] In following the course suggested by the nature of the science of mechanics, the subject next for discussion is the motion of a material system; that is, of a system of material particles which are related to each other by means of certain forces of attraction, tension, and such like. These will be explained hereafter. This motion I shall consider in its greatest generality, and by the light of the best processes which modern science has discovered: we shall hereby be enabled to apply our principles to problems of great interest and of practical importance, and to their solution by most elegant methods. I shall also enuntiate and explain certain very general principles, which in their mathematical expression include all Dynamical problems. These will be introduced towards the close of our treatise; because I think that such and similar general propositions are more adequately apprehended, when they have been previously applied as it were piecemeal to particular problems. This is the course which I have taken heretofore, and which I shall still take, in the conviction that it is that which is best suited to a didactic treatise.

The general motion of a material system takes place in space; and is capable of determination only by means of properties of space; by means, that is, of systems of coordinates, or of some other equivalent mode of reference. It is necessary therefore for us to be prepared with a sufficient knowledge of these properties.

Moreover in the course of our treatise we shall often have occasion to translate mechanical results into analogous geometrical theorems, whereby we shall obtain a fertile interpreta-

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tion of mechanical laws from a geometrical point of view. Now although the theorems which we require may be found scattered here and there in treatises on geometry, yet it is almost necessary to collect them, that all may be seen in one conspectus. And there is also another reason for this introductory section; we shall preserve an uniform notation throughout: and this in complex formulæ is a matter of importance no less than of elegance; for when an algebraical equation or a geometrical form is difficult of comprehension, it is surely undesirable to increase the difficulty by a confused symbolism.

2.] I shall assume in my readers a knowledge of the ordinary equations of the straight line, and of the plane; and of their common properties: and I shall commence with the investigation of the transformation from one rectangular system of axes to another rectangular system, both of which originate at the same point. Let us suppose a point P to be (x, y, z), (ξ, η, ζ) in reference to these two systems respectively; and the systems to be connected by direction-cosines indicated in the following scheme:

	x	y	z
ξ	a_1	a_2	a_3
η	b_1	b_2	b_3
ζ	c_1	c_2	c_3

(1)

so that
$$\left.\begin{aligned} x &= a_1\xi + b_1\eta + c_1\zeta \\ y &= a_2\xi + b_2\eta + c_2\zeta \\ z &= a_3\xi + b_3\eta + c_3\zeta \end{aligned}\right\}; \tag{2}$$

and inversely,
$$\left.\begin{aligned} \xi &= a_1x + a_2y + a_3z \\ \eta &= b_1x + b_2y + b_3z \\ \zeta &= c_1x + c_2y + c_3z \end{aligned}\right\}. \tag{3}$$

As the systems are rectangular, we have

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = a_3^2 + b_3^2 + c_3^2 = 1, \tag{4}$$

$$a_1^2 + a_1^2 + a_1^2 = b_1^2 + b_2^2 + b_3^2 = c_1^2 + c_2^2 + c_3^2 = 1, \tag{5}$$

$$a_2a_3 + b_2b_3 + c_2c_3 = a_3a_1 + b_3b_1 + c_3c_1 = a_1a_2 + b_1b_2 + c_1c_2 = 0, \tag{6}$$

$$b_1c_1 + b_2c_2 + b_3c_3 = c_1a_1 + c_2a_2 + c_3a_3 = a_1b_1 + a_2b_2 + a_3b_3 = 0. \tag{7}$$

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From the last two equations of (6) we have,

$$\left.\begin{aligned} a_2a_1+b_2b_1+c_2c_1 &= 0 \\ a_3a_1+b_3b_1+c_3c_1 &= 0 \end{aligned}\right\}; \tag{8}$$

$$\therefore \quad \frac{a_1}{b_2c_3-c_2b_3} = \frac{b_1}{c_2a_3-a_2c_3} = \frac{c_1}{a_2b_3-b_2a_3}$$

$$= \pm \frac{\{a_1^2+b_1^2+c_1^2\}^{\frac{1}{2}}}{\{(b_2c_3-c_2b_3)^2+(c_2a_3-a_2c_3)^2+(a_2b_3-b_2a_3)^2\}^{\frac{1}{2}}}$$

$$= \pm 1. \tag{9}$$

As to the double sign in the right-hand member of this last equation I would observe, that it indicates the two orders in which the positive axes of the two systems of coordinates may be taken: that is, if both are taken in the same order, when ξ and η coincide with x and y respectively, ζ will also coincide with z: but if they are taken in a contrary order, ζ will coincide with $-z$ when ξ and η coincide with x and y. Let us assume the order of the two systems to be the same: so that when the two systems are coincident the following simultaneous conditions must be satisfied:

$$\left.\begin{aligned} a_1 &= 1, & a_2 &= 0, & a_3 &= 0, \\ b_1 &= 0, & b_2 &= 1, & b_3 &= 0, \\ c_1 &= 0, & c_2 &= 0, & c_3 &= 1; \end{aligned}\right\} \tag{10}$$

and for the fulfilment of these conditions the upper sign must be taken in (9): so that from (9), and similarly from the other pairs of equations of (6), we have

$$\frac{b_2c_3-c_2b_3}{a_1} = \frac{c_2a_3-a_2c_3}{b_1} = \frac{a_2b_3-b_2a_3}{c_1} = 1, \tag{11}$$

$$\frac{b_3c_1-c_3b_1}{a_2} = \frac{c_3a_1-c_3a_1}{b_2} = \frac{a_3b_1-b_3a_1}{c_2} = 1, \tag{12}$$

$$\frac{b_1c_2-c_1b_2}{a_3} = \frac{c_1a_2-a_1c_2}{b_3} = \frac{a_1b_2-a_2b_1}{c_3} = 1. \tag{13}$$

These results may also be deduced from (5) and (7), which are equations inverse to (4) and (6).

3.] In the preceding Article the new system of coordinate axes of (ξ, η, ζ) is connected with the original system of (x, y, z) by means of nine direction cosines: those, that is, of (1): these however are related by six equations of condition, either (4) and (6), or (5) and (7): so that only three of the nine direction-

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cosines are independent. It is however to be observed that the transformation is thus effected by means of symmetrical linear equations; in many cases the advantage of employing such formulæ is greater than the inconvenience of introducing many variables which are not independent; but in other cases it is more convenient to introduce as few variables as possible; and I proceed therefore to explain Euler's process of transformation, in which only three new quantities are required.

Let, as heretofore, x, y, z refer to the original system, and ξ, η, ζ to the transformed system.

(1) Let the system of axes be turned about the axis of z in a positive direction through an angle ψ: see Fig. 1; and let x', y', z' be the values of x, y, z, when this transformation has taken place; so that

$$\left.\begin{aligned} x &= x' \cos\psi - y' \sin\psi, \\ y &= x' \sin\psi + y' \cos\psi, \\ z &= z'. \end{aligned}\right\} \tag{14}$$

(2) Let the system of x', y', z' be turned through an angle θ about the line ON, which is the axis of x'; and let x'', y'', z'' be the coordinates when this transformation has taken place; so that

$$\left.\begin{aligned} x' &= x'', \\ y' &= y'' \cos\theta - z'' \sin\theta, \\ z' &= y'' \sin\theta + z'' \cos\theta. \end{aligned}\right\} \tag{15}$$

(3) Let the system of x'', y'', z'' be turned about the axis of z'' in a positive direction through an angle ϕ; and let ξ, η, ζ be the coordinates when this transformation has taken place; so that

$$\left.\begin{aligned} x'' &= \xi \cos\phi - \eta \sin\phi, \\ y'' &= \xi \sin\phi + \eta \cos\phi, \\ z'' &= \zeta. \end{aligned}\right\} \tag{16}$$

Then by these successive transformations the system of axes will be transformed in the most general manner possible; and substituting in (14) from (15) and (16), we have

$$x = \xi(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta) + \eta(-\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta) + \zeta\sin\psi\sin\theta, \tag{17}$$

$$y = \xi(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) + \eta(-\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta) - \zeta\cos\psi\sin\theta, \tag{18}$$

$$z - \xi\sin\phi\sin\theta + \eta\cos\phi\sin\theta + \zeta\cos\theta, \tag{19}$$

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whereby the relations between the old and new coordinates are expressed in terms of three undetermined quantities θ, ϕ, and ψ.

4.] The comparison of (17), (18) and (19) with (2) indicates the following equivalences:

$$\left.\begin{aligned} a_1 &= \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta, \\ b_1 &= -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta, \\ c_1 &= \sin\psi\sin\theta, \end{aligned}\right\} \qquad (20)$$

$$\left.\begin{aligned} a_2 &= \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta, \\ b_2 &= -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta, \\ c_2 &= -\cos\psi\sin\theta, \end{aligned}\right\} \qquad (21)$$

$$\left.\begin{aligned} a_3 &= \sin\phi\sin\theta, \\ b_3 &= \cos\phi\sin\theta, \\ c_3 &= \cos\theta; \end{aligned}\right\} \qquad (22)$$

these equations also satisfy the conditions (4) ... (7); and from them we have

$$\cos\theta = c_3, \qquad \tan\phi = \frac{a_3}{b_3}, \qquad \tan\psi = -\frac{c_1}{c_2}; \qquad (23)$$

so that the nine direction-cosines are expressed by means of three quantities θ, ϕ, ψ.

5.] In the course of our work we shall frequently require for illustration surfaces of the second degree. The most general form of the equation of which is

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{D}yz + 2\mathrm{E}zx + 2\mathrm{F}xy + 2\mathrm{G}x + 2\mathrm{H}y + 2\mathrm{J}z + \mathrm{K} = 0: \quad (24)$$

but as we shall need only central surfaces, and these referred to the centre as the origin, we had better reduce (24) to the most simple form which the equation of such surfaces admits of.

Let (x', y', z') be the new origin, and be the centre; then substituting $x+x'$, $y+y'$, $z+z'$ severally for x, y, z, (24) becomes

$$\begin{aligned} &\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{D}yz + 2\mathrm{E}zx + 2\mathrm{F}xy \\ &+ 2(\mathrm{A}x' + \mathrm{F}y' + \mathrm{E}z' + \mathrm{G})\,x + 2(\mathrm{F}x' + \mathrm{B}y' + \mathrm{D}z' + \mathrm{H})\,y \\ &\qquad\qquad + 2(\mathrm{E}x' + \mathrm{D}y' + \mathrm{C}z' + \mathrm{J})\,z \\ &+ \mathrm{A}x'^2 + \mathrm{B}y'^2 + \mathrm{C}z'^2 + 2\mathrm{D}y'z' + 2\mathrm{E}z'x' + 2\mathrm{F}x'y' \\ &\qquad\qquad + 2\mathrm{G}x' + 2\mathrm{H}y' + 2\mathrm{J}z' + \mathrm{K} = 0; \quad (25) \end{aligned}$$

as (x', y', z') is the centre, this equation is to be unaltered when

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for x, y, z we substitute $-x, -y, -z$; therefore the coefficients of x, y, z must vanish; so that

$$\left.\begin{aligned} \mathrm{A}x' + \mathrm{F}y' + \mathrm{E}z' + \mathrm{G} &= 0, \\ \mathrm{F}x' + \mathrm{B}y' + \mathrm{D}z' + \mathrm{H} &= 0, \\ \mathrm{E}x' + \mathrm{D}y' + \mathrm{C}z' + \mathrm{J} &= 0; \end{aligned}\right\} \tag{26}$$

whence we have finite values for x', y', z'; unless

$$\mathrm{ABC} - \mathrm{AD}^2 - \mathrm{BE}^2 - \mathrm{CF}^2 + 2\mathrm{DEF} = \nabla \text{ (say)} = 0, \tag{27}$$

in which case the values of x', y', z' are infinite. Let us suppose ∇ to be finite: then the equation to the surface becomes

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{D}yz + 2\mathrm{E}zx + 2\mathrm{F}xy + \mathrm{K}' = 0; \tag{28}$$

wherein K' is the constant term, and represents the last two lines of (25): and as is evident from (26),

$$\mathrm{K}' = \mathrm{G}x' + \mathrm{H}y' + \mathrm{J}z' + \mathrm{K}; \tag{29}$$

and if we substitute the values of x', y', z', which are determined by (26), we have

$$\begin{aligned} \mathrm{K}'\nabla = {} & \mathrm{G}^2(\mathrm{D}^2 - \mathrm{BC}) + \mathrm{H}^2(\mathrm{E}^2 - \mathrm{CA}) + \mathrm{J}^2(\mathrm{F}^2 - \mathrm{AB}) \\ & + 2\mathrm{HJ}(\mathrm{AD} - \mathrm{EF}) + 2\mathrm{JG}(\mathrm{BE} - \mathrm{FD}) + 2\mathrm{GH}(\mathrm{CF} - \mathrm{DE}) - \mathrm{K}\nabla, \quad (30) \\ = {} & \nabla' \text{ (say)}. \quad (31) \end{aligned}$$

In passing I would observe, that ∇ is the determinant of the three equations (26), when the last terms are omitted; and that ∇' is, omitting a factor, the determinant of the four equations (26) and (29). This condition has been already determined in Ex. 3, Art. 355, Vol. I. Ed. 2.

If $\nabla = 0$, the coordinates of the centre are infinite: the surface in this case is non-central, and is a paraboloid, or one of its degenerate varieties.

If $\nabla' = 0$, the equation to the surface is

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 + 2\mathrm{D}yz + 2\mathrm{E}zx + 2\mathrm{F}xy = 0, \tag{32}$$

and the centre is on the surface. The surface is therefore a cone, or one of its degenerate varieties.

6.] We can further reduce the general equation to central surfaces by means of another transformation of coordinate axes. Let the centre still be the origin, and let another system of rectangular axes originate at it. Let us omit the accent on K' in (28), and for x, y, z let us substitute the values given in (2). Then (28) becomes

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$$\begin{aligned}
&\xi^2\{\mathrm{A}a_1^2+\mathrm{B}a_2^2+\mathrm{C}a_3^2+2\mathrm{D}a_2a_3+2\mathrm{E}a_3a_1+2\mathrm{F}a_1a_2\}\\
&+\eta^2\{\mathrm{A}b_1^2+\mathrm{B}b_2^2+\mathrm{C}b_3^2+2\mathrm{D}b_2b_3+2\mathrm{E}b_3b_1+2\mathrm{F}b_1b_2\}\\
&+\zeta^2\{\mathrm{A}c_1^2+\mathrm{B}c_2^2+\mathrm{C}c_3^2+2\mathrm{D}c_2c_3+2\mathrm{E}c_3c_1+2\mathrm{F}c_1c_2\}\\
&+2\eta\zeta\{\mathrm{A}b_1c_1+\mathrm{B}b_2c_2+\mathrm{C}b_3c_3\\
&\qquad+\mathrm{D}(b_2c_3+b_3c_2)+\mathrm{E}(b_3c_1+b_1c_3)+\mathrm{F}(b_1c_2+b_2c_1)\}\\
&+2\zeta\xi\{\mathrm{A}c_1a_1+\mathrm{B}c_2a_2+\mathrm{C}c_3a_3\\
&\qquad+\mathrm{D}(c_2a_3+c_3a_2)+\mathrm{E}(c_3a_1+c_1a_3)+\mathrm{F}(c_1a_2+c_2a_1)\}\\
&+2\xi\eta\{\mathrm{A}a_1b_1+\mathrm{B}a_2b_2+\mathrm{C}a_3b_3\\
&\qquad+\mathrm{D}(a_2b_3+a_3b_2)+\mathrm{E}(a_3b_1+a_1b_3)+\mathrm{F}(a_1b_2+a_2b_1)\}+\mathrm{K}=0. \quad (33)
\end{aligned}$$

In this equation nine direction-cosines are involved, and these thus far are subject to only six conditions; viz. (4) and (6), or (5) and (7) of Art. 2. Three other equations therefore are necessary for their complete determination; assuming the following conditions to be possible and sufficient, let us suppose the coefficients of $\eta\zeta$, $\zeta\xi$, and $\xi\eta$ in (33) to vanish: so that we have

$$\left.\begin{aligned}
&\mathrm{A}b_1c_1+\mathrm{B}b_2c_2+\mathrm{C}b_3c_3+\mathrm{D}(b_2c_3+b_3c_2)+\mathrm{E}(b_3c_1+b_1c_3)+\mathrm{F}(b_1c_2+b_2c_1)=0,\\
&\mathrm{A}c_1a_1+\mathrm{B}c_2a_2+\mathrm{C}c_3a_3+\mathrm{D}(c_2a_3+c_3a_2)+\mathrm{E}(c_3a_1+c_1a_3)+\mathrm{F}(c_1a_2+c_2a_1)=0,\\
&\mathrm{A}a_1b_1+\mathrm{B}a_2b_2+\mathrm{C}a_3b_3+\mathrm{D}(a_2b_3+a_3b_2)+\mathrm{E}(a_3b_1+a_1b_3)+\mathrm{F}(a_1b_2+a_2b_1)=0.
\end{aligned}\right\}\quad(34)$$

Also let the new coefficients of ξ^2, η^2, ζ^2 in the transformed equation severally be A', B', C'; so that

$$\left.\begin{aligned}
\mathrm{A}a_1^2+\mathrm{B}a_2^2+\mathrm{C}a_3^2+2\mathrm{D}a_2a_3+2\mathrm{E}a_3a_1+2\mathrm{F}a_1a_2&=\mathrm{A}',\\
\mathrm{A}b_1^2+\mathrm{B}b_2^2+\mathrm{C}b_3^2+2\mathrm{D}b_2b_3+2\mathrm{E}b_3b_1+2\mathrm{F}b_1b_2&=\mathrm{B}',\\
\mathrm{A}c_1^2+\mathrm{B}c_2^2+\mathrm{C}c_3^2+2\mathrm{D}c_2c_3+2\mathrm{E}c_3c_1+2\mathrm{F}c_1c_2&=\mathrm{C}';
\end{aligned}\right\}\quad(35)$$

whereby (if these equations are possible) the transformed equation is

$$\mathrm{A}'\xi^2+\mathrm{B}'\eta^2+\mathrm{C}'\zeta^2+\mathrm{K}=0. \quad (36)$$

Now the last two equations of (34) may be put into the following forms:

$$\left.\begin{aligned}
(\mathrm{A}a_1+\mathrm{F}a_2+\mathrm{E}a_3)c_1+(\mathrm{F}a_1+\mathrm{B}a_2+\mathrm{D}a_3)c_2+(\mathrm{E}a_1+\mathrm{D}a_2+\mathrm{C}a_3)c_3=0,\\
(\mathrm{A}a_1+\mathrm{F}a_2+\mathrm{E}a_3)b_1+(\mathrm{F}a_1+\mathrm{B}a_2+\mathrm{D}a_3)b_2+(\mathrm{E}a_1+\mathrm{D}a_2+\mathrm{C}a_3)b_3=0;
\end{aligned}\right\}\quad(37)$$

and our hypothesis requires these to coexist with the second and third of (7), Article 2; viz. with

$$\left.\begin{aligned}
a_1c_1+a_2c_2+a_3c_3&=0,\\
a_1b_1+a_2b_2+a_3b_3&=0.
\end{aligned}\right\}\quad(38)$$

But if we have two pairs of equations of the forms

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$$\left.\begin{aligned} lx + my + nz &= 0 \\ l\xi + m\eta + n\zeta &= 0 \end{aligned}\right\}, \qquad \left.\begin{aligned} \mathrm{L}x + \mathrm{M}y + \mathrm{N}z &= 0 \\ \mathrm{L}\xi + \mathrm{M}\eta + \mathrm{N}\zeta &= 0 \end{aligned}\right\},$$

from the first pair we have

$$\frac{l}{y\zeta - \eta z} = \frac{m}{z\xi - \zeta x} = \frac{n}{x\eta - \xi y};$$

and from the second,

$$\frac{\mathrm{L}}{y\zeta - \eta z} = \frac{\mathrm{M}}{z\xi - \zeta x} = \frac{\mathrm{N}}{x\eta - \xi y};$$

$$\therefore \quad \frac{l}{\mathrm{L}} = \frac{m}{\mathrm{M}} = \frac{n}{\mathrm{N}}. \tag{39}$$

As (37) and (38) are pairs of equations of the same form as these, we have

$$\frac{\mathrm{A}a_1 + \mathrm{F}a_2 + \mathrm{E}a_3}{a_1} = \frac{\mathrm{F}a_1 + \mathrm{B}a_2 + \mathrm{D}a_3}{a_2} = \frac{\mathrm{E}a_1 + \mathrm{D}a_2 + \mathrm{C}a_3}{a_3} \tag{40}$$

$$= \frac{\mathrm{A}a_1^2 + \mathrm{B}a_2^2 + \mathrm{C}a_3^2 + 2\mathrm{D}a_2a_3 + 2\mathrm{E}a_3a_1 + 2\mathrm{F}a_1a_2}{a_1^2 + a_2^2 + a_3^2} \tag{41}$$

$$= \mathrm{A}', \text{ from (35)}; \tag{42}$$

(41) being inferred from (40) by operating on the numerators and denominators of (40) severally with the factors a_1, a_2, a_3, and by adding numerators and denominators.

Similarly from the third and first, and from the first and second of (34), we have

$$\frac{\mathrm{A}b_1 + \mathrm{F}b_2 + \mathrm{E}b_3}{b_1} = \frac{\mathrm{F}b_1 + \mathrm{B}b_2 + \mathrm{D}b_3}{b_2} = \frac{\mathrm{E}b_1 + \mathrm{D}b_2 + \mathrm{C}b_3}{b_3} = \mathrm{B}',$$

$$\frac{\mathrm{A}c_1 + \mathrm{F}c_2 + \mathrm{E}c_3}{c_1} = \frac{\mathrm{F}c_1 + \mathrm{B}c_2 + \mathrm{D}c_3}{c_2} = \frac{\mathrm{E}c_1 + \mathrm{D}c_2 + \mathrm{C}c_3}{c_3} = \mathrm{C}'.$$

As these last equations are of precisely the same form as (42), let us take a type-expression of all; and assume X to be the type of A', B', C'; and t_n to be the type of a_n, b_n, c_n: so that we have the following typical form:

$$\left.\begin{aligned} (\mathrm{A}-\mathrm{X})t_1 + \mathrm{F}t_2 + \mathrm{E}t_3 &= 0, \\ \mathrm{F}t_1 + (\mathrm{B}-\mathrm{X})t_2 + \mathrm{D}t_3 &= 0, \\ \mathrm{E}t_1 + \mathrm{D}t_2 + (\mathrm{C}-\mathrm{X})t_3 &= 0; \end{aligned}\right\} \tag{43}$$

whence by cross multiplication,

$$(\mathrm{A}-\mathrm{X})(\mathrm{B}-\mathrm{X})(\mathrm{C}-\mathrm{X}) - \mathrm{D}^2(\mathrm{A}-\mathrm{X}) - \mathrm{E}^2(\mathrm{B}-\mathrm{X}) - \mathrm{F}^2(\mathrm{C}-\mathrm{X}) + 2\mathrm{DEF} = 0; \tag{44}$$

which equation is the condition of the coexistence of the three equations of (43).

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As (44) is a cubic in x it has three roots; and these are the values of A', B', C' which are the coefficients of ξ^2, η^2, ζ^2 in equation (36): so that we have

$$(A-x)(B-x)(C-x)-D^2(A-x)-E^2(B-x)-F^2(C-x)+2DEF$$
$$= (A'-x)(B'-x)(C'-x) = 0, \quad (45)$$

of which the x-differential is

$$(B-x)(C-x)+(C-x)(A-x)+(A-x)(B-x)-D^2-E^2-F^2$$
$$= (B'-x)(C'-x)+(C'-x)(A'-x)+(A'-x)(B'-x). \quad (46)$$

7.] As the roots of the cubic (44) are the coefficients of ξ^2, η^2, ζ^2 in the reduced equation of the surface, and as these coefficients must be real quantities, the possibility of the preceding reduction depends on the reality of these roots; and this is demonstrated by the following process, due to Cauchy.

Let r_1 and r_2 be the two roots of

$$(A-x)(B-x)-F^2 = 0,$$

so that
$$r_1 = \frac{A+B}{2}+\left\{\left(\frac{A-B}{2}\right)^2+F^2\right\}^{\frac{1}{2}},$$
$$r_2 = \frac{A+B}{2}-\left\{\left(\frac{A-B}{2}\right)^2+F^2\right\}^{\frac{1}{2}};$$

r_1 and r_2 are evidently real quantities. In (44) let us substitute for x,

(1) $+\infty$; the result is negative;
(2) r_1 ; the result is positive;
(3) r_2 ; the result is negative;
(4) $-\infty$; the result is positive;

therefore the roots of (44) lie respectively between $+\infty$ and r_1; between r_1 and r_2; between r_2 and $-\infty$; and are all real. Thus the assumptions made in (34) are demonstrated to be legitimate; and A', B', C' are real quantities, which are determined by the equation (44). Henceforth we shall suppose them to be known.

Also from (43) another form of the cubic equation may be found, which is for many purposes more useful than (44). The several equations of (43) may be put into the following form:

$$\frac{t_1}{D}+\frac{t_2}{E}+\frac{t_3}{F} = \frac{t_1}{EF}\left\{x-A+\frac{EF}{D}\right\}$$
$$= \frac{t_2}{FD}\left\{x-B+\frac{FD}{E}\right\} = \frac{t_3}{DE}\left\{x-C+\frac{DE}{F}\right\}. \quad (47)$$

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Whence we have

$$\frac{EF}{D(X-A)+EF}+\frac{FD}{E(X-B)+FD}+\frac{DE}{F(X-C)+DE}-1=0, \quad (48)$$

which is a cubic equation in terms of X, and is indeed the same as (44) *.

And the values of the corresponding direction-cosines may thus be found: from (42) we have

$$\left.\begin{aligned}(A-A')a_1 + Fa_2 + Ea_3 &= 0,\\ Fa_1 + (B-A')a_2 + Da_3 &= 0,\\ Ea_1 + Da_2 + (C-A')a_3 &= 0;\end{aligned}\right\} \quad (49)$$

and taking these equations two and two together, we have

$$\left.\begin{aligned}\frac{a_1}{(B-A')(C-A')-D^2}&=\frac{a_2}{DE-F(C-A')}=\frac{a_3}{FD-E(B-A')},\\ \frac{a_1}{DE-F(C-A')}&=\frac{a_2}{(C-A')(A-A')-E^2}=\frac{a_3}{EF-D(A-A')},\\ \frac{a_1}{FD-E(B-A')}&=\frac{a_2}{EF-D(A-A')}=\frac{a_3}{(A-A')(B-A')-F^2};\end{aligned}\right\} \quad (50)$$

from any one of which the direction-cosines corresponding to A' may be found. Also we have

$$\frac{a_1^2}{(B-A')(C-A')-D^2}=\frac{a_1a_2}{DE-F(C-A')}=\frac{a_1a_3}{FD-E(B-A')}$$

$$=\frac{a_2^2}{(C-A')(A-A')-E^2}=\frac{a_3^2}{(A-A')(B-A')-F^2} \quad (51)$$

$$=\frac{1}{(B-A')(C-A')+(C-A')(A-A')+(A-A')(B-A')-D^2-E^2-F^2} \quad (52)$$

$$=\frac{1}{(B'-A')(C'-A')}; \quad (53)$$

the denominators of (52) and (53) being equal, by reason of (46).

Also from (50) we have

$$a_1 : a_2 : a_3 = \frac{1}{EF-D(A-A')} : \frac{1}{FD-E(B-A')} : \frac{1}{DE-F(C-A')}; \quad (54)$$

this last system is also evident by reason of (47).

* On the proof that all the roots of (44) and (48) are real, see also a paper by Kummer in Crelle's Journal, Vol. xxvi. p. 268.

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We have thus two systems of symmetrical equations for determining the values of a_1, a_2, a_3 which correspond to A'. In a similar way two symmetrical systems may be determined in terms of B' and C', whereby the corresponding values of b_1, b_2, b_3, c_1, c_2, c_3 will be found; and therefore generally as these values will be determinate, so will the position of the three lines perpendicular to each other to which A', B', C' correspond be also determinate; and the equation to the surface will be of the form

$$\text{A}'\xi^2 + \text{B}'\eta^2 + \text{C}'\zeta^2 + \text{K} = 0. \tag{55}$$

This is the most simple form to which the equation of a central surface of the second degree can be reduced. The three rectangular axes to which it is referred are called Principal Axes. These names are specially given to those parts of the coordinate axes which are intercepted between the origin and the surface. The three planes passing through the centre, which are perpendicular to the principal axes, are called Principal Planes: they are the three coordinate planes of the equation (55).

8.] As the equation of a central surface of the second degree will be applied hereafter for the purpose of illustrating certain mechanical laws, it is necessary also to demonstrate other properties of principal axes and principal planes. In the first place I shall shew that the central radii vectores of these surfaces which coincide with the principal axes have singular values; that is, are maxima or minima, either totally or partially.

Let us take the equation (28) to be the equation to central surfaces; and let (x, y, z) on its surface be the extremity of a central radius vector r; then

$$r^2 = x^2 + y^2 + z^2;$$

and as r is to have a singular value,

$$r\,\text{D}r = x\,dx + y\,dy + z\,dz = 0: \tag{56}$$

but the differentials of these variables are connected also by the differential of (28), whereby we have

$$(\text{A}x + \text{F}y + \text{E}z)\,dx + (\text{F}x + \text{B}y + \text{D}z)\,dy + (\text{E}x + \text{D}y + \text{C}z)\,dz = 0; \tag{57}$$

and therefore from (56) and (57),

$$\frac{\text{A}x + \text{F}y + \text{E}z}{x} = \frac{\text{F}x + \text{B}y + \text{D}z}{y} = \frac{\text{E}x + \text{D}y + \text{C}z}{z}. \tag{58}$$

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Let l, m, n be the direction-cosines of the singular radius vector r: so that

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r;$$

and (58) becomes

$$\frac{\mathrm{A}l + \mathrm{F}m + \mathrm{E}n}{l} = \frac{\mathrm{F}l + \mathrm{B}m + \mathrm{D}n}{m} = \frac{\mathrm{E}l + \mathrm{D}m + \mathrm{C}n}{n}. \qquad (59)$$

Now these equations are in form identical with (40); and therefore the singular radii vectores are coincident with the principal axes; that is, with those lines for which, when taken as coordinate axes, the terms in the equation to the surface involving $\eta\zeta$, $\zeta\xi$, $\xi\eta$ disappear.

Let each term of (59) be equal to s; so that we have

$$\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2 + 2\mathrm{D}mn + 2\mathrm{E}nl + 2\mathrm{F}lm = s,$$

$$\text{and} \quad \left.\begin{aligned} (\mathrm{A}-s)l + \mathrm{F}m + \mathrm{E}n &= 0, \\ \mathrm{F}l + (\mathrm{B}-s)m + \mathrm{D}n &= 0, \\ \mathrm{E}l + \mathrm{D}m + (\mathrm{C}-s)n &= 0; \end{aligned}\right\} \qquad (60)$$

whence we have the cubic equation

$$(\mathrm{A}-s)(\mathrm{B}-s)(\mathrm{C}-s) - \mathrm{D}^2(\mathrm{A}-s) - \mathrm{E}^2(\mathrm{B}-s) - \mathrm{F}^2(\mathrm{C}-s) + 2\mathrm{DEF} = 0, \quad (61)$$

which is identical with (44): and of which therefore the three real roots are A', B', C', which are given by the equations (35); and the corresponding values of l, m, n are a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3, because the equations for the determination of the three different values of l, m, n which correspond to the three roots of (61) are the same as those by which, in the preceding Article, the direction-cosines of the principal axes have been determined.

It is also evident from the form of the equation that the three singular radii vectores are at right angles to each other. Let us take the cubic which arises from (60) in the form given in (48): and let us take the equations which correspond to B' and C'; whereby we have

$$\frac{\mathrm{EF}}{\mathrm{D}(\mathrm{B}'-\mathrm{A}) + \mathrm{EF}} + \frac{\mathrm{FD}}{\mathrm{E}(\mathrm{B}'-\mathrm{B}) + \mathrm{FD}} + \frac{\mathrm{DE}}{\mathrm{F}(\mathrm{B}'-\mathrm{C}) + \mathrm{DE}} - 1 = 0,$$

$$\frac{\mathrm{EF}}{\mathrm{D}(\mathrm{C}'-\mathrm{A}) + \mathrm{EF}} + \frac{\mathrm{FD}}{\mathrm{E}(\mathrm{C}'-\mathrm{B}) + \mathrm{FD}} + \frac{\mathrm{DE}}{\mathrm{F}(\mathrm{C}'-\mathrm{C}) + \mathrm{DE}} - 1 = 0;$$

and subtracting the latter from the former, we have

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$$(\mathrm{C}'-\mathrm{B}')\left\{\frac{1}{\{\mathrm{D}(\mathrm{B}'-\mathrm{A})+\mathrm{EF}\}\{\mathrm{D}(\mathrm{C}'-\mathrm{A})+\mathrm{EF}\}}\right.$$
$$+\frac{1}{\{\mathrm{E}(\mathrm{B}'-\mathrm{B})+\mathrm{FD}\}\{\mathrm{E}(\mathrm{C}'-\mathrm{B})+\mathrm{FD}\}}$$
$$\left.+\frac{1}{\{\mathrm{F}(\mathrm{B}'-\mathrm{C})+\mathrm{DE}\}\{\mathrm{F}(\mathrm{C}'-\mathrm{C})+\mathrm{DE}\}}\right\}=0; \quad (62)$$

and by reason of the equations which are analogous to (54) this becomes

$$(\mathrm{C}'-\mathrm{B}')\{b_1c_1+b_2c_2+b_3c_3\}=0;$$

and as B' is not generally equal to C', we must have

$$b_1c_1+b_2c_2+b_3c_3=0;$$

and therefore the two corresponding singular radii vectores are perpendicular to each other: in the same manner it may be shewn that the other singular radius vector is at right angles to each of these: so that the three form a system of rectangular axes; and if the equation to the surface is referred to them as its coordinate axes, its equation is (55).

It is also evident that the normals at the points where these principal axes meet the surface are coincident with the axes.

9.] The theory of principal axes and planes may also be derived from another property of surfaces of the second degree. I shall in the first place demonstrate that the locus of the middle points of a system of parallel chords is a plane.

Let us take (28) to be the equation to the surface; and let the equations to one of a system of parallel chords be

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r,$$

where (x, y, z) is a point on the surface, (x', y', z') is a point through which the chord passes, and r is the distance between these two points. For x, y, z in (28) let us substitute $x'+lr$, $y'+mr$, $z'+nr$ respectively, and let us arrange the result in terms of r; then (28) becomes

$$r^2\{\mathrm{A}l^2+\mathrm{B}m^2+\mathrm{C}n^2+2\mathrm{D}mn+2\mathrm{E}nl+2\mathrm{F}lm\}$$
$$+2r\{(\mathrm{A}l+\mathrm{F}m+\mathrm{E}n)x'+(\mathrm{F}l+\mathrm{B}m+\mathrm{D}n)y'+(\mathrm{E}l+\mathrm{D}m+\mathrm{C}n)z'\}$$
$$+\mathrm{A}x'^2+\mathrm{B}y'^2+\mathrm{C}z'^2+2\mathrm{D}y'z'+2\mathrm{E}z'x'+2\mathrm{F}x'y'+\mathrm{K}=0. \quad (63)$$

This is a quadratic equation in terms of r, and has two roots; and therefore from a point (x', y', z') two radii vectores can be drawn to a surface of the second degree along the same straight line.

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Let us suppose (x', y', z') to be the middle point of the chord: then the two values of r are equal and of opposite directions and signs, and consequently the term involving the first power of r in (63) must vanish: hence we have the condition

$$(\mathrm{A}l+\mathrm{F}m+\mathrm{E}n)x'+(\mathrm{F}l+\mathrm{B}m+\mathrm{D}n)y'+(\mathrm{E}l+\mathrm{D}m+\mathrm{C}n)z'=0; \quad (64)$$

As the chords are all parallel to each other, and to a central radius vector whose equations are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}, \quad (65)$$

l, m, n are constant: therefore (64) is the equation to a plane passing through the centre, of which the current coordinates are x', y', z': and therefore the middle points of a system of parallel chords is a plane passing through the centre of the surface.

The plane and the line whose equations are (64) and (65) are called relatively to each other a conjugate plane and a conjugate diameter.

Now it is evident that generally a diameter will not be perpendicular to its conjugate plane. Let us examine whether this relation between them is ever possible; and, if so, the circumstances under which it may exist.

If (65) is perpendicular to the plane (64),

$$\frac{\mathrm{A}l+\mathrm{F}m+\mathrm{E}n}{l}=\frac{\mathrm{F}l+\mathrm{B}m+\mathrm{D}n}{m}=\frac{\mathrm{E}l+\mathrm{D}m+\mathrm{C}n}{n} \quad (66)$$

$$=\mathrm{A}l^2+\mathrm{B}m^2+\mathrm{C}n^2+2\mathrm{D}mn+2\mathrm{E}nl+2\mathrm{F}lm$$

$$=s, \quad (67)$$

where s is the coefficient of r^2 in (63).

As (66) are identical with (59) they involve similar conclusions.

There are therefore three diameters which are respectively perpendicular to their conjugate planes; and these are the principal diameters, and their conjugate planes are the principal planes of the surface.

We have thus considered the properties of principal axes under three different aspects: (1) if the surface is referred to them as coordinate axes, its equation takes the form (36), and has no term containing the products of the variables: (2) they are singular radii vectores: (3) they are diameters which are perpendicular to their conjugate planes.

Principal axes may be defined by either one of these pro-

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perties; and all three mutually involve each other; and are in fact identical in the geometrical conception of infinitesimals.

10.] Let us next consider those cases in which the roots of the cubic equation (44) have particular values.

(1) Let two roots be equal: say, let $A' = B'$; then (46) vanishes when $X = A' = B'$; and we have

$$(B-A')(C-A') + (C-A')(A-A') + (A-A')(B-A') - D^2 - E^2 - F^2 = 0. \quad (68)$$

Also as $a_1, a_2, a_3, b_1, b_2, b_3$ cannot be infinite, by reason of (52) we must have

$$(B-A')(C-A') - D^2 = (C-A')(A-A') - E^2 = (A-A')(B-A') - F^2 = 0; \quad (69)$$

that is, $$A - A' = \frac{EF}{D}, \qquad B - A' = \frac{FD}{E}, \qquad C - A' = \frac{DE}{F}; \quad (70)$$

and consequently,

$$A' = A - \frac{EF}{D} = B - \frac{FD}{E} = C - \frac{DE}{F}; \quad (71)$$

in which case the direction-cosines a_1, a_2, a_3, and similarly the direction-cosines b_1, b_2, b_3 are indeterminate; but c_1, c_2, c_3, which correspond to the unequal root C', are determinate as heretofore. In this case the equation to the surface is

$$A'(\xi^2 + \eta^2) + C'\zeta^2 + K = 0. \quad (72)$$

The principal axis of ζ being determinate, any two axes in the plane of (ξ, η) perpendicular to each other are the other principal axes. Equation (72), in this case, represents a surface of revolution, whose axis of revolution is the ζ- axis. If A' were the unequal root of (44), and $B' = C'$, then the reduced equation is

$$A'\xi^2 + C'(\eta^2 + \zeta^2) + K = 0, \quad (73)$$

which also represents a surface of revolution: the axis of ξ is the determined axis, and is the axis of revolution of the surface, the position of the other two axes being indeterminate.

(2) Let all the roots of (44) be equal: that is, let $A' = B' = C'$: then differentiating (69),

$$2A' = B + C = C + A = A + B;$$

$$\therefore \quad A = B = C; \quad (74)$$

and from (69), $$D = E = F = 0; \quad (75)$$

$$\therefore \quad A' = B' = C' = A;$$

and then all the direction-cosines are indeterminate, and any

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system of rectangular axes originating at the centre is a system of principal axes. And the equation to the surface is

$$\xi^2 + \eta^2 + \zeta^2 + \frac{K}{A'} = 0; \tag{76}$$

and if $K = -a^2 A'$, this is the equation to a sphere whose radius is a.

(3) Let one root of (44) be zero; say, $c' = 0$; then we have

$$ABC - AD^2 - BE^2 - CF^2 + 2DEF = 0; \tag{77}$$

and the equation to the surface becomes

$$A'\xi^2 + B'\eta^2 + K = 0; \tag{78}$$

and therefore $\zeta = \frac{0}{0}$. As (77) is the same expression as (27), the centre of the surface is at an infinite distance. Also (78) is the equation of a central conic in the plane of (ξ, η); therefore the surface is a cylinder whose trace on the plane of (ξ, η) is the conic (78).

(4) Let two roots of (44) be zero; say, $B' = C' = 0$; then, besides the condition (77), we have

$$BC + CA + AB - D^2 - E^2 - F^2 = 0; \tag{79}$$

and the equation to the surface becomes

$$A'\xi^2 + K = 0; \tag{80}$$

and as $\eta = \zeta = \frac{0}{0}$, it represents two planes parallel to the plane of (η, ζ).

(5) Let all the roots of (44) be zero; so that $A' = B' = C' = 0$; then the equation to the surface becomes

$$K = 0; \tag{81}$$

which represents a plane at an infinite distance.

We may express these several equations in a more convenient form.

If ∇ does not vanish, and if $\nabla' = 0$, in which case the surface is central, and the constant K in the reduced equation (36) vanishes, then we have

$$A'\xi^2 + B'\eta^2 + C'\zeta^2 = 0; \tag{82}$$

and if A', B', C' are all positive, the only values of ξ, η, ζ which satisfy the equation are $\xi = \eta = \zeta = 0$; that is, the equation represents a point at the origin.

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If one coefficient, say C', is negative, and A' and B' are positive, then if

$$\text{A}' = \frac{1}{a^2}, \qquad \text{B}' = \frac{1}{b^2}, \qquad \text{C}' = -\frac{1}{c^2},$$

(82) becomes

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 0; \tag{83}$$

which is the equation to an elliptical cone, and all plane sections of it perpendicular to the axis of ζ are ellipses; and if $a = b$, the surface is a right circular cone, whose axis of rotation is the axis of ζ.

If however V' does not vanish, the equation to the surface is

$$\text{A}'\xi^2 + \text{B}'\eta^2 + \text{C}'\zeta^2 + \text{K} = 0. \tag{84}$$

If A', B', C' and K are all positive, the equation does not admit of geometrical interpretation. Let us therefore assume K to be negative: so that with obvious substitutions, and with all the varieties of sign which the quantities admit of, the equation may take either of the forms,

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1, \tag{85}$$

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1, \tag{86}$$

$$\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1; \tag{87}$$

which severally represent an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. Let us assume $a > b > c$; then of (85) degenerate species are, (1) an oblate spheroid, when $a = b$; (2) a prolate spheroid, when $b = c$; (3) a sphere, when $a = b = c$. And if $a = b$, the hyperboloids of revolution are particular species of (86) and (87). We have not however space or occasion to enter into all these particulars, or into the nature and forms of the surfaces. This information must be obtained from treatises wherein the properties of these surfaces are specially treated of. The omission of this and similar matter is necessarily incidental to preliminary chapters which must be incomplete. It is our intention to demonstrate for the most part only those geometrical theorems which will be required in the sequel; not because other theorems are in themselves unimportant, but because the interpretation of our mechanical results will not require them.

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11.] I will first take the cone whose equation is (83); but will, for convenience of symbols, use x, y, z for the coordinates of any point on its surface; so that its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0; \tag{88}$$

and I shall observe with respect to it that the vertex is its centre, and that, like other surfaces of the second order which are represented by (84), it has three principal axes; which are respectively the axis of the cone, commonly so called, or, as we may call it, the internal axis; and two lines through the vertex which are respectively parallel to the major and minor axes of those elliptic sections whose planes are perpendicular to the axis of the cone; these are called the external axes.

The plane which contains the internal axis and the major axis of the principal elliptic sections is the plane of greatest section of the cone; and that which contains the internal axis and the minor axis of the elliptic sections is the plane of least section. The principal axes of a cone of the second degree, when its equation is given in the most general form, are determined by the process of Art. 6; for this is applicable to equation (28), whether $\kappa' = 0$ or not.

At the vertex of the cone (88) let straight lines be drawn perpendicular to the tangent planes, these will all lie in a second cone which is called the supplementary or reciprocal cone; its equation may thus be found. The equation to the tangent plane of (88) is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} - \frac{z\zeta}{c^2} = 0; \tag{89}$$

and the equations to the line through the vertex perpendicular to it are

$$\frac{a^2\xi}{x} = \frac{b^2\eta}{y} = -\frac{c^2\zeta}{z}; \tag{90}$$

therefore squaring these, and multiplying the terms of (88) respectively by them, we have

$$a^2\xi^2 + b^2\eta^2 - c^2\zeta^2 = 0; \tag{91}$$

which is the equation required, and represents a cone of the second degree, which has the same internal axis as (88), but whose major and minor external axes are respectively the minor and major external axes of (88).

For the construction of the second cone (91) it is evident that as every line through its vertex is perpendicular to a tangent

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plane of (88), so is every tangent plane of (91) perpendicular to a line through the vertex of (88). Thus the cones have corresponding or reciprocal properties, and to a tangent plane of the first cone and to its line of contact correspond a line through the vertex of the second and the tangent plane along that line.

Again, to the first cone let two tangent planes be drawn, and let the corresponding lines on the second cone be taken; the plane containing these lines is perpendicular to the line of intersection of the two tangent planes of the former cone; and the tangent planes of the second cone along these lines are perpendicular to the lines on the first cone; and their line of intersection is perpendicular to the plane through the lines of contact on the first cone, so that to a line and its polar plane on the first cone correspond a plane and its polar relatively to the second cone.

It is evident therefore that properties relative to the angles contained by certain planes and right lines on the first cone will give rise to properties of corresponding right lines and planes in the second cone; in other words the properties of cones of the second degree are double; and the principle of duality is established.

12.] Assuming that a cone admits of being cut by a plane such that the section is a circle, I propose to determine the position of the circular planes of (88).

Let us suppose a^2 to be greater than b^2; and let (88) be put into the form

$$x^2+y^2+z^2 = m^2(z^2-n^2y^2), \tag{92}$$

as we may do by the following substitutions;

$$m^2 = \frac{a^2+c^2}{c^2}; \qquad n^2 = \frac{c^2}{b^2}\,\frac{a^2-b^2}{a^2+c^2}. \tag{93}$$

Then (92) may be resolved into the two factors

$$(x^2+y^2+z^2)^{\frac{1}{2}} = km(z-ny),$$

$$(x^2+y^2+z^2)^{\frac{1}{2}} = \frac{m}{k}(z+ny);$$

and as k is an undetermined quantity, these are satisfied by

$$(1)\quad x^2+y^2+z^2 = 0, \qquad z = ny, \tag{94}$$

$$(2)\quad x^2+y^2+z^2 = 0, \qquad z = -ny; \tag{95}$$

each of which pairs of equations represents a plane section of a sphere; that is, represents a circle; so that the sections of the cone made by the two planes, $z = ny$ and $z = -ny$, are

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circles; and it is evident that the sections made by all planes parallel to either of these will also be circles; for this reason these planes are called the cyclic planes of the cone, and their equations are

$$z = \pm \frac{c}{b}\left(\frac{a^2-b^2}{a^2+c^2}\right)^{\frac{1}{2}} y; \tag{96}$$

they pass through the axis of x which is parallel to the major axis of the elliptic sections perpendicular to the axis of the cone; and they make with the plane of (x,y)

$$\pm \tan^{-1}\frac{c}{b}\left(\frac{a^2-b^2}{a^2+c^2}\right)^{\frac{1}{2}}. \tag{97}$$

Now if through the vertex of the cone (88) two lines are drawn perpendicular to these cyclic planes, as the line of intersection of the cyclic planes is perpendicular to the plane of least section of (88), so will these lines lie in the plane of the greatest section of (91); and because every plane perpendicular to one of these right lines cuts the reciprocal one in a conic, one of whose foci is on this right line, these lines are called focal lines. The analytical proof of this property of focal lines is contained in the preceding equations; a geometrical proof will be found in the Memoir of M. Chasles, entitled, "Sur les propriétés des cones du second degré," and contained in the VIth volume of the Memoirs of the Royal Academy of Brussels.

13.] We must also investigate certain properties of the ellipsoid, as we shall frequently require this surface for the purposes of illustration and interpretation, but our description of the properties will be incomplete, because we shall demonstrate those only which are wanted hereafter.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{98}$$

where $a^2 > b^2 > c^2$; so that a is the greatest, and c is the least of all central radii vectores; b, however, has also a critical value, for it is the semi-axis minor of the elliptic trace in the plane of (x, y), and for that plane is a minimum, but it is the semi-axis major of the elliptic trace in the plane of (y,z), and for that plane is a maximum; we shall immediately determine lines on the surface which will indicate the regions for which b is a maximum, and for which b is a minimum, and the dividing line of these regions will indicate certain singular positions of central radii vectores.

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Let us in the first place inquire into the position of the cyclic planes of the ellipsoid; viz. those planes which cut the ellipsoid in circles, and we will take the central equation (98), and suppose the plane to pass through the origin; let (98) be transformed to the system of axes of ξ, η, ζ of Art. 2; and let us suppose the plane of (ξ, η) to be the cyclic plane; so that from (2) we have the following substitutions,

$$x = a_1\xi + b_1\eta, \qquad y = a_2\xi + b_2\eta, \qquad z = a_3\xi + b_3\eta; \tag{99}$$

and therefore from (98) we have

$$\xi^2\left(\frac{a_1^2}{a^2} + \frac{a_2^2}{b^2} + \frac{a_3^2}{c^2}\right) + 2\xi\eta\left\{\frac{a_1b_1}{a^2} + \frac{a_2b_2}{b^2} + \frac{a_3b_3}{c^2}\right\}$$
$$+\eta^2\left\{\frac{b_1^2}{a^2} + \frac{b_2^2}{b^2} + \frac{b_3^2}{c^2}\right\} = 1. \tag{100}$$

Now this is to represent a circle; therefore

$$\frac{a_1^2}{a^2} + \frac{a_2^2}{b^2} + \frac{a_3^2}{c^2} = \frac{b_1^2}{a^2} + \frac{b_2^2}{b^2} + \frac{b_3^2}{c^2}; \tag{101}$$

$$\frac{a_1b_1}{a^2} + \frac{a_2b_2}{b^2} + \frac{a_3b_3}{c^2} = 0. \tag{102}$$

As the positions of the axes of ξ and η in the plane of (ξ, η) are indeterminate, let us suppose the axis of ξ to coincide with the line of intersection of the two planes of (ξ, η) and of (x, y), so that $a_3 = 0$; therefore from (7),

$$a_1b_1 + a_2b_2 = 0;$$

and therefore from (102)

$$a_1b_1\left\{\frac{1}{a^2} - \frac{1}{b^2}\right\} = 0; \tag{103}$$

which is satisfied by either $a_1 = 0$, or $b_1 = 0$.

(1) Let $a_1 = 0$; therefore $a_2 = 1$ and $b_2 = 0$; so that the plane of section passes through the axis of y; and if θ is the angle between the plane of section and the plane of (x, y), so that $b_1 = \cos\theta$, $b_3 = \sin\theta$, $b_2 = 0$, from (101) we have

$$\tan\theta = \pm\left\{\frac{\dfrac{1}{b^2} - \dfrac{1}{a^2}}{\dfrac{1}{c^2} - \dfrac{1}{b^2}}\right\}^{\frac{1}{2}}, \tag{104}$$

$$= \pm\left\{\frac{B' - A'}{C' - B'}\right\}^{\frac{1}{2}}, \tag{105}$$

according to the notation of equation (84); so that there are

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two planes of section equally inclined to the plane of (x, y), and passing through the axis of y. The mean principal axis of the ellipsoid is evidently the radius of the circle, and the cyclic planes are related to the ellipsoid in the manner indicated by the lines in Fig. 2, where $\text{UOA} = \text{U'OA'} = \theta$; $\text{OU} = \text{OU'} = b$.

Thus all the central radii vectores in these two cyclic planes are equal; and are equal to the mean semi-axis of the ellipsoid. For all parts of the surface of the ellipsoid contained between these two planes towards the maximum axis, the radii vectores are greater than b; and for all parts towards the minimum axis, the radii vectores are less than b; these cyclic planes therefore divide the ellipsoid into four parts, corresponding to two of which at B b is a minimum, and corresponding to the other two it is a maximum.

Also all planes parallel to the central cyclic planes determined by (104) are cyclic planes; for the conditions involved in (101) and (102) depend only on the coefficients of x^2, y^2, z^2 in (98), and these quantities are not changed by a change of origin, if the new axes are parallel to the old. And the four points at which planes touch the ellipsoid are the umbilics. This is also manifest from Ex. I. Art. 408. Vol. I. Ed. 2.

(2) Let $b_1 = 0$; therefore $a_2 b_2 = 0$; and $b_2^2 + b_3^3 = 1$; either $a_2 = 0$, or $b_2 = 0$.

Let $a_2 = 0$; therefore $a_1 = 1$, so that the cutting plane passes through the axis of x; and (101) becomes

$$\frac{1}{a^2} = \frac{b_2^2}{b^2} + \frac{1-b_2^2}{c^2}; \qquad \therefore \quad b_2^2 = \frac{a^2 b^2 - b^2 c^2}{a^2 b^2 - a^2 c^2};$$

which is greater than unity; and therefore this result is impossible.

Let $b_2 = 0$; therefore $b_3 = 1$; $a_1^2 + a_2^2 = 1$; and the cutting plane passes through the axis of z; then (101) becomes

$$\frac{a_1^2}{a^2} + \frac{1-a_1^2}{b^2} = \frac{1}{c^2}; \qquad \therefore \quad a_1^2 = \frac{a^2(b^2-c^2)}{c^2(b^2-a^2)};$$

which is a negative quantity; and therefore a_1 is impossible. The only cyclic planes then are those determined in the first case, and these two planes pass through the mean principal axis of the ellipsoid.

The cyclic planes of a cone, and indeed of all the surfaces of the second order, may be determined by the method of this Article.

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I may observe also that the process which has been applied to the cone is also applicable to the ellipsoid; for the equation (98) may be thrown into the form

$$k^2(x^2+y^2+z^2-b^2) = x^2-n^2z^2,$$

where

$$k^2 = \frac{a^2}{a^2-b^2}, \qquad n^2 = \frac{a^2(b^2-c^2)}{b^2(a^2-b^2)};$$

and then the equation to the ellipsoid is satisfied by the pairs of simultaneous equations

$$(1) \quad x^2+y^2+z^2 = b^2, \qquad x = nz;$$

$$(2) \quad x^2+y^2+z^2 = b^2, \qquad x = -nz.$$

Each of which pair represents a plane section of the sphere whose radius is b, and therefore represents a circular section of the surface.

14.] Let us also inquire somewhat briefly into the relations which exist between an axis and its conjugate plane relatively to the ellipsoid (98). We have already investigated the condition which generally exists between a radius vector and the plane which bisects all chords of the ellipsoid which are parallel to that axis; but some further properties of axes in conjugate relations to each other will be required in the sequel.

Let the equations to a radius vector be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}; \tag{106}$$

then, by reason of equation (64), the equation to its conjugate plane is

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0; \tag{107}$$

and therefore if the equation to a plane is

$$\mathrm{L}x + \mathrm{M}y + \mathrm{N}z = 0, \tag{108}$$

the equations to the axis conjugate to it are

$$\frac{x}{a^2\mathrm{L}} = \frac{y}{b^2\mathrm{M}} = \frac{z}{c^2\mathrm{N}}; \tag{109}$$

if the line (106) meets the surface of the ellipsoid at the point (x', y', z'), then we have

$$\frac{l}{x'} = \frac{m}{y'} = \frac{n}{z'};$$

and (107) becomes

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0; \tag{110}$$

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which is evidently the plane parallel to that which touches the ellipsoid at (x', y', z'); so that if a tangent plane be drawn to an ellipsoid at a given point, the central plane parallel to that plane is conjugate to the axis drawn to the point of contact.

Now if three axes of an ellipsoid are such that each is the axis conjugate to the plane which contains the other two, these lines form a system of conjugate axes. And if three planes are such that the line of intersection of any two is the conjugate axis of the third, these planes form a system of conjugate planes. Of such systems we have already had an instance in the principal axes and the principal planes. Let us determine the relations which exist generally between these lines and planes.

Let (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3) be the three points on the ellipsoid to which the system of conjugate axes corresponds; so that the equations to the three axes are

$$\left.\begin{aligned} \frac{x}{x_1} &= \frac{y}{y_1} = \frac{z}{z_1}, \\ \frac{x}{x_2} &= \frac{y}{y_2} = \frac{z}{z_2}, \\ \frac{x}{x_3} &= \frac{y}{y_3} = \frac{z}{z_3}; \end{aligned}\right\} \tag{111}$$

and the equations to the conjugate planes are

$$\left.\begin{aligned} \frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} &= 0, \\ \frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} &= 0, \\ \frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2} &= 0. \end{aligned}\right\} \tag{112}$$

But since the first of (111) coincides with the line of intersection of the second and third of (112), we have

$$\frac{x_1}{a^2(y_2z_3 - z_2y_3)} = \frac{y_1}{b^2(z_2x_3 - x_2z_3)} = \frac{z_1}{c^2(x_2y_3 + y_2x_3)}; \tag{113}$$

which equations are equivalent to the two equations,

$$\left.\begin{aligned} \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} &= 0, \\ \frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} &= 0; \end{aligned}\right\} \tag{114}$$

and as the other two lines of (111) must coincide with the lines

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of intersection of the other planes of (112), we shall, in addition to (114), have also the equation

$$\frac{x_2 x_3}{a^2} + \frac{y_2 y_3}{b^2} + \frac{z_2 z_3}{c^2} = 0; \tag{115}$$

these are three relations between the coordinates of the extremities of three conjugate axes.

By a similar process it may be shewn, that if

$$\left.\begin{aligned} \mathrm{L}_1 x + \mathrm{M}_1 y + \mathrm{N}_1 z &= 0, \\ \mathrm{L}_2 x + \mathrm{M}_2 y + \mathrm{N}_2 z &= 0, \\ \mathrm{L}_3 x + \mathrm{M}_3 y + \mathrm{N}_3 z &= 0, \end{aligned}\right\} \tag{116}$$

are the equations to three planes of a conjugate system, then

$$\left.\begin{aligned} a^2 \mathrm{L}_2 \mathrm{L}_3 + b^2 \mathrm{M}_2 \mathrm{M}_3 + c^2 \mathrm{N}_2 \mathrm{N}_3 &= 0, \\ a^2 \mathrm{L}_3 \mathrm{L}_1 + b^2 \mathrm{M}_3 \mathrm{M}_1 + c^2 \mathrm{N}_3 \mathrm{N}_1 &= 0, \\ a^2 \mathrm{L}_1 \mathrm{L}_2 + b^2 \mathrm{M}_1 \mathrm{M}_2 + c^2 \mathrm{N}_1 \mathrm{N}_2 &= 0. \end{aligned}\right\} \tag{117}$$

The equations (114) and (115) contain apparently nine unknown quantities, but as the equations are homogeneous these are equivalent to only four; and as these are subject to only three conditions, the system is indeterminate; the number of systems of conjugate axes is therefore unlimited; if, however, one axis is given the other two are determinate.

15.] Again, a system of conjugate axes may be defined by the following equations;

$$\left.\begin{aligned} x_1 &= a l_1, \\ y_1 &= b m_1, \\ z_1 &= c n_1; \end{aligned}\right\} \quad \left.\begin{aligned} x_2 &= a l_2, \\ y_2 &= b m_2, \\ z_2 &= c n_2; \end{aligned}\right\} \quad \left.\begin{aligned} x_3 &= a l_3, \\ y_3 &= b m_3, \\ z_3 &= c n_3; \end{aligned}\right\} \tag{118}$$

in which cases the equation of the ellipse gives

$$l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1. \tag{119}$$

and from (114) and (115) we have

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = l_3 l_1 + m_3 m_1 + n_3 n_1 = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0; \tag{120}$$

and from these six equations we have the inverse systems

$$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1; \tag{121}$$

$$m_1 n_1 + m_2 n_2 + m_3 n_3 = n_1 l_1 + n_2 l_2 + n_3 l_3 = l_1 m_1 + l_2 m_2 + l_3 m_3 = 0. \tag{122}$$

Also we have theorems analogous to (11), (12), and (13) of Art. 2. Now these relations are useful for the proof of many properties of conjugate axes; thus, let r_1, r_2, r_3 be three conjugate axes; then

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 &= a^2 (l_1^2 + l_2^2 + l_3^2) + b^2 (m_1^2 + m_2^2 + m_3^2) \\ &\qquad + c^2 (n_1^2 + n_2^2 + n_3^2) \\ &= a^2 + b^2 + c^2; \end{aligned} \tag{123}$$

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that is, the sum of the squares of three conjugate axes is constant.

16.] Let there be three central radii vectores of an ellipsoid mutually at right angles to each other; then the sum of the squares of their reciprocals is constant.

Let r_1, r_2, r_3 be the three central radii vectores, of which let the direction cosines be (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3); then we have

$$\left.\begin{aligned} \frac{1}{r_1^2} &= \frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}, \\ \frac{1}{r_2^2} &= \frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}, \\ \frac{1}{r_3^2} &= \frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2}; \end{aligned}\right\} \tag{124}$$

and therefore by addition

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{125}$$

17.] The normal and the tangent plane drawn at any point of a central surface of the second degree meet each of the principal planes at a point and along a straight line respectively, which are such that in each principal plane the point is the pole, and the straight line is the corresponding polar, relatively to a certain determinate conic in that principal plane.

Let us take the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{126}$$

to be the typical case; and consider the normal and the tangent plane at the point (x, y, z); and let us also take the principal plane of (x, y), which is that of the greatest and mean principal axes. Then the normal pierces this plane at $\left(\frac{a^2-c^2}{a^2}x, \frac{b^2-c^2}{b^2}y\right)$; and the equation to the line of intersection of the tangent plane and the plane of (x, y) is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1; \tag{127}$$

this is evidently the polar of the pole $\left(\frac{a^2-c^2}{a^2}x, \frac{b^2-c^2}{b^2}y\right)$ relatively to the conic

$$\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1. \tag{128}$$

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By a similar process we may shew that the conics in the other principal planes are expressed by the equations

$$\frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=1, \tag{129}$$

$$\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2}=1; \tag{130}$$

of these equations (128) is that of an ellipse in the plane of (x, y); (129) of an hyperbola in the plane of (x, z); and (130) of a curve which is wholly imaginary in the plane of (y, z). These curves are called the excentrical or the focal conics of the ellipsoid (126); and for this reason; the vertices of (128) are the foci of the elliptic sections of the ellipsoid by the principal planes of (y, z) and (x, z); and the foci of it are the foci of the elliptic section of the plane of (x, y): also the vertices of (129) are the foci of the elliptic sections of the ellipsoid in the planes of (x, y) and of (z, y); and the foci are the foci of the elliptic section made by the principal plane of (x, z). The third curve is imaginary, although its foci are, as in the other two cases, real. Also the hyperbola (129) passes through the umbilics of the ellipsoid.

18.] Now we call those surfaces of the second degree confocal, the principal sections of which are confocal; hence it appears that all surfaces of the second degree, which have the same focal conics, are confocal.

Thus the general equation of all surfaces of the second degree confocal with (126) is

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1; \tag{131}$$

the equations to whose focal conics are (128), (129), and (130). And if (131) passes through a given point (x', y', z'), we have from it the cubic equation in θ

$$(\theta+a^2)(\theta+b^2)(\theta+c^2)-x'^2(\theta+b^2)(\theta+c^2)-y'^2(\theta+c^2)(\theta+a^2)$$
$$-z'^2(\theta+a^2)(\theta+b^2)=0;$$

in which if we substitute for θ successively $+\infty$, $-c^2$, $-b^2$, $-a^2$, the results are severally $+$, $-$, $+$, $-$; so that the roots lie respectively between $+\infty$ and $-c^2$, $-c^2$ and $-b^2$, $-b^2$ and $-a^2$; in which cases (131) represents respectively an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. Thus, at the point (x', y', z') these three confocal surfaces intersect. We have also proved (see Vol. I. Art. 411. Ed. 2.) that they intersect at right angles and along their lines of curvature.

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Thus, at the common points of intersection of these three surfaces, their normals are at right angles at each other. It is also evident that these surfaces intersect in eight points, one in each octant of space about their centre.

Now if $\theta = -c^2$, (131) requires that $z = 0$, and we have

$$\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1,$$

which is the equation to the focal conic in the plane of (x, y); similarly if $\theta = -b^2$, and if $\theta = -a^2$, we have the focal conics in the planes of (x, z) and (y, z) respectively; whence it appears that the focal conics are only particular cases of the surface of the second order confocal with (126.)

And therefore surfaces which are confocal may also be described as those which have the same focal conics.

19.] If from any point (ξ, η, ζ) an enveloping cone is drawn to the ellipsoid (126), the principal axes of that cone coincide with the normals of the three confocal surfaces of the second degree which intersect at the vertex of the cone.

By Ex. 2. Art. 355. Vol. I. Ed. II. the equation to the cone whose vertex is (ξ, η, ζ), and which envelopes the ellipsoid (126), is

$$\left\{\frac{x\xi}{a^2} + \frac{y\eta}{b^2} + \frac{z\zeta}{c^2} - 1\right\}^2 - \left(\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) = 0. \quad (132)$$

For the sake of abbreviation let

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} - 1 = \text{K}; \quad (133)$$

so that (132) on expansion becomes

$$\left(\frac{\xi^2}{a^2} - \text{K}\right)\frac{x^2}{a^2} + \left(\frac{\eta^2}{b^2} - \text{K}\right)\frac{y^2}{b^2} + \left(\frac{\zeta^2}{c^2} - \text{K}\right)\frac{z^2}{c^2}$$
$$+ \frac{2\eta\zeta}{b^2c^2}yz + \frac{2\zeta\xi}{c^2a^2}zx + \frac{2\xi\eta}{a^2b^2}xy + \ldots\ldots = 0; \quad (134)$$

the other terms being omitted because the position of the principal axes of the cone depends on the first six terms only of the expanded equation.

In this case equation (48) becomes

$$\frac{\xi^2}{a^2(a^2\text{X} + \text{K})} + \frac{\eta^2}{b^2(b^2\text{X} + \text{K})} + \frac{\zeta^2}{c^2(c^2\text{X} + \text{K})} = 1; \quad (135)$$

also from (133) we have

$$\frac{\xi^2}{a^2\text{K}} + \frac{\eta^2}{b^2\text{K}} + \frac{\zeta^2}{c^2\text{K}} = 1 + \frac{1}{\text{K}};$$

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therefore by subtraction

$$\frac{\xi^2}{a^2+\frac{\kappa}{x}}+\frac{\eta^2}{b^2+\frac{\kappa}{x}}+\frac{\zeta^2}{c^2+\frac{\kappa}{x}}=1. \qquad (136)$$

Now κ and x are functions of the coordinates of the given vertex (ξ, η, ζ) and are therefore known: hence if we describe the surface of the second degree whose equation is

$$\frac{x^2}{a^2+\frac{\kappa}{x}}+\frac{y^2}{b^2+\frac{\kappa}{x}}+\frac{z^2}{c^2+\frac{\kappa}{x}}=1, \qquad (137)$$

(136) shews that that surface passes through the vertex of the enveloping cone; and this surface is evidently confocal with the original ellipsoid (126); and as x has three values which are the roots of (135), so, as we have shewn in Art. 18, the equation (137) represents three surfaces which are severally an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, all of which are confocal with (126); and which intersect orthogonally at (ξ, η, ζ).

For the determination of the principal axes of the cone, let us take the system of direction-cosines of Art. 6, and the forms of them given in (54); in the case of (134) these take the following values

$$\frac{a_1}{\xi}\left(a^2+\frac{\kappa}{x}\right)=\frac{a_2}{\eta}\left(b^2+\frac{\kappa}{x}\right)=\frac{a_3}{\zeta}\left(c^2+\frac{\kappa}{x}\right); \qquad (138)$$

and to fix our thoughts let us suppose x in this equation to be that root of (135) which corresponds to the ellipsoid. Now the direction-cosines of the normal to the ellipsoid (137) at the point (ξ, η, ζ) are proportional to

$$\frac{\xi}{a^2+\frac{\kappa}{x}}, \qquad \frac{\eta}{b^2+\frac{\kappa}{x}}, \qquad \frac{\zeta}{c^2+\frac{\kappa}{x}};$$

and a comparison of these values with (138) shews that the principal axis (a_1, a_2, a_3), the position of which is determined by (138), coincides with the normal of the confocal ellipsoid which passes through the point. This axis is generally the internal axis of the cone. By a similar process we may shew that the two external axes of the cone, which correspond to the two other roots of the cubic, are normal to the two confocal hyperboloids which intersect at the given vertex.

The normals to these two hyperboloids are, as we have shewn,

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tangents to the lines of curvature on the ellipsoid at the point (ξ, η, ζ); and therefore if a cone envelopes an ellipsoid, and if through the vertex of the cone an ellipsoid be described confocal with the former ellipsoid, the normal to the ellipsoid, and the tangents to the two lines of curvature on it, are the principal axes of the enveloping cone.

M. Chasles has also proved that the generating lines of the confocal hyperboloid of one sheet which passes through the vertex are the focal lines of the cone.

Now these same properties are true if instead of the ellipsoid (126) we had taken any other surface confocal with it; and therefore are true if the focal conics are the directors of the cone, because the focal conics are the limiting forms of the confocal surfaces.

Hence also it follows that if two surfaces of the second degree have the same focal conic, and if from any point in space as vertex two cones are described enveloping these surfaces, these cones have the same principal axes, and the same focal lines.

20.] Another principle of duality also arises from the theory of reciprocation which has been explained within the limits of plane geometry in Section 3, Chapter XIII. Vol. I. 2nd Ed.; and which I must here extend to the geometry of surfaces, so far as the sequel requires.

Let the equation to an ellipsoid be

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1. \tag{139}$$

and from every point of it as a pole let the polar plane be taken relatively to the sphere

$$x^2 + y^2 + z^2 = r^2; \tag{140}$$

the general equation of the polar plane is

$$x\xi + y\eta + z\zeta = r^2: \tag{141}$$

we propose to find the envelope of these planes; differentiating (139) and (141), we have

$$x\,d\xi + y\,d\eta + z\,d\zeta = 0,$$

$$\frac{\xi}{a^2}\,d\xi + \frac{\eta}{b^2}\,d\eta + \frac{\zeta}{c^2}\,d\zeta = 0;$$

whence we have

$$\frac{a^2x}{\xi} = \frac{b^2y}{\eta} = \frac{c^2z}{\zeta};$$

and eliminating ξ, η, ζ, we have

$$a^2x^2 + b^2y^2 + c^2z^2 = r^4; \tag{142}$$

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which is the equation to another ellipsoid, and is called the sphero-polar reciprocal of (139).

Now it is evident that a tangent plane of (142) corresponds to a point of (139); and to the intersection of two tangent planes of (142) corresponds a line passing through the two corresponding points of (139); and to a point on (142) corresponds a tangent plane of (139). Also to a tangent line of (142) corresponds a tangent line of (139). These surfaces therefore have reciprocal properties, and to a plane a line and a point on either, a point a line and a plane on the other severally correspond, so that all properties admit of being doubled. It is manifest that the theory of the reciprocal cone which has been explained in Art. 11. is a particular case of this principle.

I may also observe that the sphero-polar reciprocal of any surface of the second degree is also another surface of the second degree; but as we shall require only the simple form which has just been discussed, the more general case may be omitted.

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CHAPTER II.

THE CINEMATICS OF A RIGID BODY. ANGULAR VELOCITIES; THEIR COMPOSITION AND RESOLUTION, AND RELATION TO LINEAR VELOCITIES.

21.] SEVERAL times in the course of the treatise on Mechanics allusion has been made to a division of the subject into two parts, Cinematics and Dynamics proper. In the former of these motion and its incidents are discussed apart from all consideration of the action of forces which produce that motion; the affections of pure motion are investigated; as, for instance, it is shewn that motion takes place in time and space; that a particle moves with a certain velocity, and that velocity depends on time and space. In the latter motion is considered as the effect of certain producing causes, and the relations between it as the effect and force as the cause are investigated; thus the laws of motion and the equations of motion belong to Dynamics proper. In the exposition of the first principles of Dynamics, Chapter VII. Vol. III, this division of the subject is made, and the parts are treated separately; but it was unnecessary to bring the division into more special prominence, because the Cinematics of a moving particle do not present to the mind images difficult of formation. Every one can form a conception, more or less perfect, of the motion of a single particle; it describes a certain line, which is its path, and we can easily imagine that path; it moves with a certain velocity, and if its velocity varies, it is not difficult to conceive the rate of variation. But the motion of a system of particles is more complex; we can indeed follow the path of any one particle of it; it describes a line, just as if it were not connected with the other particles; but what is the motion of all the particles of the system relatively to it? Let us however confine our attention to the motion of a rigid body, which is a system of particles of invariable form. The body can, as it were, pirouette about any one particle in all ways, and it is difficult to imagine and to trace the motion of any other particle. So if a body rotates about an axis absolutely fixed, we can easily picture to ourselves

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the path described by every particle; it is a circle, the plane of which is perpendicular to the fixed axis, and the centre of which is in that axis. But when the body has the most general motion of which it is capable, our conception is for the most part very obscure. Hence arises the necessity of resolving that motion into its simplest elements; so that when we have an adequate conception of each separate element, we may combine them, and thus obtain an adequate conception of the motion of the whole body. We must therefore first discuss the several motions of which a rigid body is capable, independently of the forces which produce these motions; and subsequently consider the relations which subsist between these effects and their causes. In the present Chapter we shall confine our attention to the former part, viz. the Cinematics of a rigid body; and in the following Chapters we shall consider the Dynamics proper, the fundamental axioms, and the theorems deducible from them.

22.] Let us imagine a rigid body or a system of material particles of invariable form to be in motion. The form of this system will be definite if (1) the distances of three particles from each other which are not on the same straight line are given; (2) the distances of every other particle from each of these three particles is given; so that, as the system is rigid, if the positions of the first three particles are determined, the position of every other particle is also determined, and that of the whole body is also known. Thus it is sufficient for us to consider the motion of the first three particles.

Let the three particles of the system which form a triangle, and relatively to which every other particle is known, be P, Q, R, and let these be joined by straight lines. Now if the motion is such that the sides of this triangle are always parallel to their original positions, it is plain that the line joining any other point to each of these three points is also parallel to its original position; such a motion is said to be a motion of translation of the body or system. In this case the paths of all particles are equal and parallel lines, and are described with equal and parallel velocities; and the motion of the whole body will be easily inferred from that of any one particle. As the incidents of such a motion have been fully discussed in the previous volume of our work it is unnecessary to say more on this part of the subject.

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If however the paths described by the several particles are not equal and parallel the body has another motion besides that of translation. Let us consider its nature.

Now let P, Q, R be the positions of the three particles of the body, to which the position of every other particle is referred at a given time; and let P′, Q′, R′ be their positions after a certain motion; suppose moreover that the paths described by these three particles in their motion are not equal and parallel; we may analyse the motion by the following process: first let all the three particles be moved so as to describe paths equal and parallel to that described by P; that is, let us suppose a motion of translation of the whole body such that every particle of it moves over a space equal and parallel to PP′; let the positions taken by Q and R after this motion be Q″, R″; through P′, Q″, R″ let a plane be drawn, which is manifestly parallel to the original plane PQR; and also let a plane be drawn through the three final positions P′, Q′, R′; let these two planes intersect along the line P′N; let the body rotate about the line P′N, until the plane Q″R″P′ is brought into the plane Q′R′P′; and, if it is necessary, let the body again rotate about a line passing through P′ and perpendicular to the plane P′Q′R′, until Q″ and R″ coincide respectively with Q′ and R′; by these several motions the body will have passed from its first to its final position. The motions are three; the first is a motion of translation; the other two are motions of turning or of rotation about certain axes; and as the motion has been of the most general kind, so may all motion be resolved into separate motions of the kinds which we have mentioned.

This motion of rotation requires careful consideration. It always takes place about a certain straight line or axis. If a body rotates all points along the axis are at rest so far as the motion of rotation is concerned; they may move by reason of other circumstances, but they do not move by reason of the rotation of the body about that axis; and the straight line along which the quiescent points are is called the axis of rotation.

Also this axis may or may not meet the body. If it meets the body, the particles of the body along the axis are at rest; if it does not meet the body, all the particles of the body move by reason of the rotation.

Many rotations about different axes may co-exist; we must

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consider how this is, and investigate laws by which these may be combined into one or more resultants.

Now the most simple rotation is that of a body rotating about an axis fixed absolutely; that is, relatively to it and to space. In this case every particle of the body describes a circle in a plane perpendicular to the axis; and the body being rigid, the times in which the circles are described are the same for all the particles; and their relative position is not changed by or during the motion.

Imagine a particle m at a distance r from the axis of a rotating body; and through the fixed axis and containing it let a plane be drawn fixed in space; then the position of the particle may be determined at any instant by means of r and the angle at which r is inclined to this fixed plane. Thus in Fig. 3, let P be the place of m at the time t; let Oz be the rotation-axis, fixed relatively to the body and to space; through it let the plane zox be drawn, and let it be fixed in space, so that when the body rotates, this plane as well as the axis remains fixed; let OP be drawn at right angles to Oz; OP $= r$, PO$x = \theta$; then r and θ are sufficient to determine the place of m.

Firstly, let us suppose the body to rotate uniformly about the axis; that is, let us suppose θ to receive equal increments in equal times; let ω be the angle by which θ is increased, that is through which r revolves, in an unit of time; then if θ is the angle through which r has revolved in t units of time,

$$\theta = \omega t; \tag{1}$$

so that if OP coincides with Ox when $t = 0$, PO$x = \theta = \omega t$. And from (1) we have

$$\omega = \frac{\theta}{t}. \tag{2}$$

We must enlarge our language; let us take our nomenclature from that of motion of translation. Since the linear velocity of a particle moving uniformly is the linear space described by it in an unit of time; so let the angle through which an uniformly rotating body rotates in an unit of time be called the angular velocity of the body. Thus ω is the angular velocity of the body, and is defined mathematically by (2). I must observe that the angular velocity is independent of r and is the same for all parts of the body.

Secondly, suppose the body not to rotate uniformly about the axis, so that the radius vector of any particle does not

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describe equal angles in equal times; then the angular velocity is a function of the time. Let the time be resolved into infinitesimal elements; and let us suppose the angular velocity at the time t to be ω, and to be $\omega + d\omega$ at the time $t + dt$; and let $d\theta$ be the angle through which the body has rotated in the time dt. Then since ω is the angular velocity at the beginning of dt, and $\omega + d\omega$ is the angular velocity at the end of dt, the *mean* angular velocity with which $d\theta$ has been described in $\omega + \phi\, d\omega$ if ϕ is a proper fraction; and ϕ is positive or negative according as the angular velocity is increasing or decreasing; so that by reason of (1)

$$d\theta = (\omega + \phi\, d\omega)\, dt;$$

and omitting the infinitesimal of the second order,

$$d\theta = \omega\, dt; \tag{3}$$

thus $d\theta$ is the angle described in dt units of time by the body rotating with the angular velocity ω at the beginning of dt; and therefore dividing both sides by dt, we have

$$\omega = \frac{d\theta}{dt}; \tag{4}$$

and therefore ω or $\frac{d\theta}{dt}$ is the angle described in an unit of time, and is the angular velocity of the body.

Thus in both cases, of uniform and of continuously varying angular velocity, angular velocity is the angle described by the radius vector of any particle in an unit of time; and is the ratio of the angle described in a given time to the time in which it is described; in the case of varying velocity this ratio is the ratio of two infinitesimals.

The unit angular velocity is that of a body which rotates through an unit angle in an unit of time; and if the angular velocity of a body is ω, ω is a number designating the number of unit angles through which the body rotates in an unit of time.

23.] Again let a rigid body rotate about a fixed axis; at a given instant the angular velocity is the same for all particles of the body; but the linear velocity is evidently not the same for all; the linear velocity of those at a greater distance from the axis is greater than of those at a less distance; the relation between the angular and the linear velocities of a particle is thus found.

Let us take a particle m at a distance r from the axis. Let ω be the angular velocity, $d\theta$ be the angle described by r in the

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time dt, and let ds be the space described by m; then $ds = r\,d\theta$; and

$$\frac{ds}{dt} = r\frac{d\theta}{dt}, \tag{5}$$

$$= r\omega; \tag{6}$$

so that the linear velocity of m is the product of the angular velocity and the radius of m, and therefore varies directly as the distance of m from the rotation-axis. If therefore $r = 1$, the angular velocity is identical with the linear velocity.

24.] Hence is derived the principle on which angular velocities are measured; if two bodies rotate with angular velocities such that the particles in each at an unit distance from the axis describe equal spaces in equal times, the angular velocities of the bodies being uniform during that time, these angular velocities are said to be equal. And this mode of determining equal angular velocities being adopted, it is evident that one angular velocity may be double, or treble, or n times another. If the equal spaces are described by each particle in the same direction, the angular velocities are equal and in the same direction; but if the equal spaces are described in opposite directions, the angular velocities are equal and opposite. Angular velocities may therefore be affected with signs. Thus if ω represents the angular velocity with which a body rotates in a given direction, $-\omega$ will represent the equal angular velocity of a body rotating in the opposite direction. As angular velocities have rotation-axes, intensities, and directions, it is evidently desirable to have some geometrical representative of them, as of linear velocities. This is supplied by a straight line on a principle similar to that by which the line-representatives of couples are determined in Statics. Along the rotation-axis let a length be taken containing the same number of linear units as ω contains angle-units; then this line by its position and its length represents the axis of rotation and the intensity of the angular velocity. Let a point on this rotation-axis be taken as a fixed pole; as the body may rotate about this axis in either of two directions, so may the line-representative of the angular velocity be measured in either of two opposite directions, and therefore we must choose a principle by which direction of rotation may be determined. Let it be this; if, as we look along the axis from the pole, the body rotates from left to right, like the hands of a watch when we face it, let that ro-

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tation be called positive, and let its line-representative be measured from the pole in the direction in which we look; but if the body rotates from right to left, that is in the direction opposite to that of the motion of the hands of a watch, let that rotation be negative, and let the line-representative be measured from the pole in a direction opposite to that along which we look. Thus in Fig. 4; let O be the pole and Ox the rotation-axis; as we look from O towards x let the body rotate as the hands of a watch which we face; that is, in the direction of the letters PQRS, then that rotation is positive, and its line-representative is to be measured from O towards the right; let OA be that line, then OA is as to direction and length a representative of the angular velocity. If, on the contrary, when we look from O towards x, the rotation of the body is in the opposite direction, then the line OA is to be measured along Ox produced backwards; that is, OA′ is the line-representative of the angular velocity; this principle of interpretation is in accordance with that of Art. 251. Vol. I. Ed. 2. I may observe that if we look from O towards A′, that is towards the left instead of the right, OA′ is the representative of the rotation in the second case according to the principle we have adopted, for as we look from O towards A′ the body rotates in the same direction as the hands of a watch; thus the line-representative is independent of the direction in which we look from O. We shall hereafter use straight lines as adequate representatives of angular velocities.

25.] Thus much as to single angular velocities, and their line-representatives. We will now investigate the circumstances of a body which rotates with many simultaneous angular velocities; that is, we suppose a body to rotate about a determinate axis with a given angular velocity, and another angular velocity to be communicated to it; what change of motion is due to the addition of this new angular velocity, and what is the combined resultant of the new and the original angular velocities? And again, we shall suppose other angular velocities to be communicated, and we shall have to determine the resultant of all of them. At present we say nothing about the source of these velocities or the mode of communication; we shall consider only the combined effect of them as expressed angular velocities. The problem therefore is the composition and resolution of angular velocities; and we shall in order consider (1) those which have the same rotation-axes; (2) those whose rotation-

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axes meet in a point; (3) those whose rotation-axes do not meet; and, as a special case of (2), those whose rotation-axes are parallel, that is, meet at an infinite distance.

Let a body rotate about the axis ox, Fig. 5, with two positive angular velocities ω_a, ω_b, whose representatives are OA and OB; let P be the place of any element of the body, which we will take to be in the plane of the paper. Let PM be drawn from P at right angles to ox, and let PM $= r$; as both the angular velocities are positive, all elements of the body lying in the plane of the paper and above ox move from below to above the plane of the paper; and all elements lying below ox move from above to below; therefore the spaces through which P passes from below to above the plane of the paper in the time dt are $\omega_a r\,dt$ due to ω_a and $\omega_b r\,dt$ due to ω_b; so that the whole space over which P passes in $dt = (\omega_a + \omega_b)\,r\,dt$.

Now suppose ω to be the resultant of ω_a and ω_b; then if the body rotates with this velocity, P will pass in the time dt over a space due to ω which is equal to the spaces due to ω_a and to ω_b; but the space passed over by P due to the angular velocity ω in the time dt is $\omega\,r\,dt$; therefore

$$\omega\,r\,dt = (\omega_a + \omega_b)\,r\,dt;$$

$$\therefore \quad \omega = \omega_a + \omega_b; \tag{7}$$

that is, the resultant angular velocity is the sum of the two component angular velocities.

If one of the components, say ω_b, is negative, the positive space passed over by P in the time dt will manifestly be $(\omega_a - \omega_b)\,r\,dt$; and we shall have

$$\omega = \omega_a - \omega_b; \tag{8}$$

and if $\omega_b = -\omega_a$, then from (7) $\omega = 0$, and two equal and opposite angular velocities neutralize each other.

Similarly if a body rotates about a given axis with coaxal angular velocities $\omega_1, \omega_2, \ldots\ldots \omega_n$, and if Ω is the resultant angular velocity,

$$\Omega = \omega_1 + \omega_2 + \ldots\ldots + \omega_n,$$

$$= \Sigma.\omega; \tag{9}$$

where $\Sigma.\omega$ expresses the algebraical sum of the several component angular velocities.

In all these cases the line-representative of the resultant is the algebraical sum of the line-representatives of the components.

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26.] Next let us suppose the body to move with two simultaneous angular velocities ω_a and ω_b, whose axes OA and OB intersect each other in the point O.

Let us take O to be the pole, and OA, OB, Fig. 6, to be the line-representatives of the angular velocities ω_a ω_b respectively; so that due to ω_a all particles of the body in the plane of the paper which are above the line OA pass from below to above the plane of the paper, and all those below OA pass from above to below; similarly by means of ω_b all particles to the right of OB pass from above to below and all those to the left from below to above. Let us take a point P within the angle BOA, and from P let PL and PK be drawn perpendicular to OA and OB respectively; let $OM = x$, $MP = y$, $BOA = \gamma$; then $PK = x \sin\gamma$, $PL = y \sin\gamma$. Let us investigate the paths described by P in the time dt, which are due to these two angular velocities; the upward path of P due to $\omega_a = \omega_a y \sin\gamma\, dt$; and the downward path of P due to $\omega_b = \omega_b x \sin\gamma\, dt$: so that the resultant upward path of $P = (y\,\omega_a - x\,\omega_b) \sin\gamma\, dt$. Now let us suppose P to be at rest under the effects of the two angular velocities, then

$$\frac{x}{\omega_a} = \frac{y}{\omega_b}; \tag{10}$$

and replacing ω_a and ω_b by their representatives OA and OB,

$$\frac{x}{OA} = \frac{y}{OB}; \tag{11}$$

but either of these is the equation to a straight line passing through O; and as all particles along it are at rest, it is the axis of the resultant angular velocity. From (11) it appears that it is the diagonal of the parallelogram of which OA and OB are the containing sides; so that the axis of the resultant angular velocity lies along the diagonal of the parallelogram of which the line-representatives of the component angular velocities are the containing sides.

The intensity of the resultant may be found as follows; let us suppose it to be ω_c; then as the path which A, or indeed any particle on the line OA, describes in dt in the case of the component angular velocities is that due to ω_b only, and is $\omega_b\, OA \sin\gamma\, dt$, and as the path described by A in dt in the case of the resultant angular velocity is $\omega_c\, OA \sin COA\, dt$, these paths are to be equal; and therefore

$$\omega_b \sin\gamma = \omega_c \sin COA; \tag{12}$$

similarly if we equate to each other the two paths described by

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B in the cases of the component and of the resultant angular velocities, we shall have

$$\omega_a \sin \gamma = \omega_c \sin \text{COB}; \tag{13}$$

from either of which equations it appears that ω_c is represented in length by the diagonal OC. Hence it follows that if a body rotates with two simultaneous velocities, whose line-representatives meet in a point and are the adjacent sides of a parallelogram, the resultant angular velocity is equivalently replaced in all respects by that diagonal of the parallelogram which abuts at the point of insersection of the line-representatives of the component angular velocities.

Hence if, as in Fig. 7, CO is produced to C′, so that C′O = CO, then if OA, OB, OC′ are the line-representatives of three simultaneous angular velocities, the body is at rest; because OC, which represents the resultant of ω_a and ω_b, represents an angular velocity which is neutralized by that of which OC′ is the line-representative; and therefore if AOC′ = β, BOC′ = α, from (12) and (13) we have

$$\frac{\omega_a}{\sin \alpha} = \frac{\omega_b}{\sin \beta} = \frac{\omega_c}{\sin \gamma}; \tag{14}$$

and therefore if a body rotates with three simultaneous angular velocities whose axes meet in a point, and whose line representatives are parallel and equal to the three sides of a triangle, the angular velocities neutralize each other, and the body remains at rest.

Hence also it follows that if a body has two simultaneous angular velocities ω_a and ω_b about two axes which intersect at an angle γ, these are equivalent to a single angular velocity ω_c, which is given by the equation

$$\omega_c^2 = \omega_a^2 + 2\omega_a\omega_b \cos \gamma + \omega_b^2; \tag{15}$$

the rotation-axis of which makes angles α and β with the rotation-axes of ω_b and ω_a are such that

$$\frac{\omega_a}{\sin \alpha} = \frac{\omega_b}{\sin \beta} = \frac{\omega_c}{\sin \gamma}. \tag{16}$$

27.] Now suppose a body to have two simultaneous angular velocities ω_x and ω_y about two axes intersecting each other at right angles; then, if ω is the resultant angular velocity,

$$\omega^2 = \omega_x^2 + \omega_y^2; \tag{17}$$

and, if α is the angle between the axes of ω and ω_x,

$$\left.\begin{aligned} \omega_x &= \omega \cos \alpha, \\ \omega_y &= \omega \sin \alpha; \end{aligned}\right\} \tag{18}$$

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so that angular velocities may by means of their line-representatives be resolved and compounded according to the projective laws of pure geometry, the laws of resolution and composition of statical pressures, and of dynamical linear velocities.

Similarly if a body has three simultaneous angular velocities $\omega_x, \omega_y, \omega_z$ about three axes which intersect at right angles; then, if ω is the resultant angular velocity,

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2; \tag{19}$$

and if α, β, γ are the angles which the axis of the resultant angular velocity makes with those of the component angular velocities,

$$\frac{\omega_x}{\cos\alpha} = \frac{\omega_y}{\cos\beta} = \frac{\omega_z}{\cos\gamma} = \omega. \tag{20}$$

28.] The machine represented in Fig. 8 may probably facilitate the conception of simultaneous angular velocities, and their combined effects; the machine is delineated in its primary state of rest. O is the centre of a sphere, through which a horizontal axis AA′ passes, the ends of which are in a horizontal circular ring ABA′B′, so that the sphere can rotate about this axis. To this horizontal ring two pivots are attached at B and B′, the line joining which is perpendicular to AA′; these pivots work in a vertical circular ring CBC′B′, so that the ring containiug the sphere can rotate about the axis BB′; the ring CC′ has also two pivots at C and C′, which are fixed points, the line joining which is vertical and is at right angles to both AA′ and BB′; and the vertical ring can rotate about CC′ as an axis. The three lines AA′, BB′, CC′ intersect in O the centre of the sphere, and thus form a system of rectangular axes in space, the origin of which is at the centre of the sphere. Now this is the state of the machine at rest, and the problem is this; let an angular velocity ω_x be given to the sphere about the line AA′; to the ring ABA′B′ carrying the sphere let an angular velocity ω_y be given about the axis BB′; and to the ring CBC′B′ carrying the former ring and the sphere let an angular velocity ω_z be given about the axis CC′; then the sphere has from its connexion with the rings all these angular velocities simultaneously, and the question is, what line of particles is at rest? What is the axis of rotation, and what is the angular velocity of the sphere about it? Try to follow in your mind the path described by any particle of the sphere, when it moves with all these simultaneous angular velocities; and try thence to determine the line of

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quiescent particles. Probably in the difficulty of doing so, you will perceive the necessity of such composition and resolution of angular velocities as we have just explained. Let us assume the three angular velocities to be positive; then the resultant angular velocity will be given by equation (19) and the direction-cosines of its axis by (20). This may be thus exhibited; let us suppose $\omega_x = \omega_y = \omega_z$; then $\alpha = \beta = \gamma$; let a diameter of the sphere be drawn making equal angles with AA′, BB′, CC′ in its original position; and at the poles where this diameter meets the surface, let the surface be divided into three equal lunes, and let them be coloured respectively red, yellow, and blue; then it will be found that the sphere will rotate about this axis which is equally inclined to the three lines AA′, BB′, CC′, and the rotating sphere will, if its angular velocity is great enough, appear white; whereas if the resultant rotation-axis does not pass through the point where the differently coloured lunes intersect, the colour of the rotating sphere will be that of the lune in which the rotation-axis pierces the sphere.

29.] Hence we can deduce the single resultant angular velocity of many angular velocities, whose rotation-axes pass through a given point: let that point be the origin, and at it let a system of coordinate axes originate; let the several angular velocities be $\omega_1, \omega_2, \ldots\ldots \omega_n$; and let their rotation-axes be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, ... $(\alpha_n, \beta_n, \gamma_n)$; let each angular velocity be resolved into three components along the three coordinate axes; so that those, whose rotation-axis is the axis of x, are $\omega_1 \cos \alpha_1$, $\omega_2 \cos \alpha_2$, ... $\omega_n \cos \alpha_n$; and if $\Sigma.\omega \cos \alpha$ is the sum of all these,

$$\left.\begin{aligned} &\Sigma.\omega \cos \alpha = \omega_1 \cos \alpha_1 + \omega_2 \cos \alpha_2 + \ldots\ldots + \omega_n \cos \alpha_n; \\ &\text{similarly} \\ &\Sigma.\omega \cos \beta = \omega_1 \cos \beta_1 + \omega_2 \cos \beta_2 + \ldots\ldots + \omega_n \cos \beta_n, \\ &\Sigma.\omega \cos \gamma = \omega_1 \cos \gamma_1 + \omega_2 \cos \gamma_2 + \ldots\ldots + \omega_n \cos \gamma_n. \end{aligned}\right\} \quad (21)$$

Let Ω be the resultant angular velocity; and let a, b, c be the direction-angles of its rotation-axis; then

$$\Omega \cos a = \Sigma.\omega \cos \alpha, \quad \Omega \cos b = \Sigma.\omega \cos \beta, \quad \Omega \cos c = \Sigma.\omega \cos \gamma; \quad (22)$$

$$\therefore \quad \Omega^2 = (\Sigma.\omega \cos \alpha)^2 + (\Sigma.\omega \cos \beta)^2 + (\Sigma.\omega \cos \gamma)^2; \quad (23)$$

and
$$\frac{\Sigma.\omega \cos \alpha}{\cos a} = \frac{\Sigma.\omega \cos \beta}{\cos b} = \frac{\Sigma.\omega \cos \gamma}{\cos c} = \Omega; \quad (24)$$

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which equations give the intensity of the resultant angular velocity, and the direction-cosines of its rotation-axis.

30.] The experiment with the pendulum, devised by Foucault to exhibit to the eye the rotation of the earth about its axis, is a simple application of the laws of resolution and composition of angular velocities which have been investigated. Let us suppose the earth to be a perfect sphere, of which a plane section through the poles is drawn in Fig. 9; P and P′ being the north and south poles, C being the centre, and WCE being the intersection of the plane of the paper and the plane of the equator. Let a pendulum be suspended at the north pole, and so that it may vibrate and turn freely in all directions. Now if the pendulum is at rest, and were suspended from a point not rotating with the earth, but fixed absolutely in space, the earth would rotate under the pendulum from west to east in 24 hours; and the apparent effect to a person on the earth would be a complete rotation of the pendulum through 360° from east to west in the same time. A point of suspension however not fixed to the earth cannot be obtained, and of course if the point of suspension is joined to the earth, all moves together, and the pendulum has no apparent rotatory motion. Let however the pendulum be suspended from a point fixed to the earth; and let it vibrate in a plane; then as the earth turns, the point of suspension turns, and the pendulum turns, but the *plane of vibration* is not affected by these rotations; it is as stationary as if the point of suspension were absolutely fixed; so that in the course of 24 hours, if the pendulum vibrates so long, the plane of vibration will apparently pass in succession over all the meridians from east to west, because the earth in that time performs a complete revolution from west to east under the pendulum; and the angular velocity of the earth will be the apparent angular velocity of the plane of vibration of the pendulum. A similar phænomenon will be presented by a pendulum at the south pole, but the direction of the apparent rotation of the plane of vibration will be from west to east. At the equator no such effect takes place. For suppose a pendulum to be suspended at the equator, and its plane of vibration to be, say, north and south; as the earth rotates about its axis, it is evident that neither the point of suspension of the pendulum nor the plane of vibration has any

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rotation; the point of suspension of the pendulum is carried round in a circle, and the plane of vibration continues north and south. At the equator therefore no effects of the earth's rotation such as we have described will be exhibited by a pendulum. The full effect is exhibited at either pole; and no effect at the equator. Now let us take any place A, whose latitude is λ; and let AC be drawn to the centre of the earth. Let ω be the angular velocity of the earth about its axis; then $\omega \sin \lambda$ is the angular velocity about the line AC, which is the normal to the earth's surface at the point A; so that the plane of vibration of the pendulum at A will undergo a displacement from east to west similar to that which takes place at P, but more slowly; for whereas the time of a complete revolution at P $= \dfrac{2\pi}{\omega} = 24$ hours, the time of a complete revolution at A $= \dfrac{2\pi}{\omega \sin \lambda}$; so that

$$\text{The time of revolution at A} = \frac{24 \text{ hours}}{\sin \lambda}. \qquad (25)$$

This law has been verified by numerous observations made at various places on the earth; for although the vibration of the pendulum has not been continued through 24 hours, yet the arcs described by the plane of vibration in a given time have been found to vary in different latitudes as the sines of these latitudes.

31.] We proceed now to consider the resolution and composition of angular velocities, the rotation-axes of which do not meet; and we will first consider the particular case of angular velocities whose axes are parallel, and about which separately the body rotates in the same direction. Let the angular velocities be ω_a and ω_b; and let their poles be O and O′, and their axes OA, O′B; OO′ being perpendicular to each of these lines; see Fig. 10; let P be the place of any particle in the line OO′, and to fix our thoughts let us take it between O and O′; let OO′ $= c$, OP $= x$, PO′ $= y$; then the downward path of P in the time dt which is due to ω_a is $x\,\omega_a\,dt$, and the upward path in the same time due to ω_b is $y\,\omega_b\,dt$; so that the downward path of P in the time dt

$$= (x\,\omega_a - y\,\omega_b)\,dt.$$

Now suppose Q to be a point in OO′ which under the effects of the two angular velocities ω_a and ω_b remains at rest; then

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if x and y are the respective distances of Q from OA and from O′B, $x\omega_a - y\omega_b = 0$; whence

$$\frac{x}{\omega_b} = \frac{y}{\omega_a}; \tag{26}$$

whereby Q is determined; and as every point in the line through Q perpendicular to OO′ is at rest, so QC is the axis of the resultant angular velocity; and (26) shews that it divides the distance between the axes of the two component angular velocities into two parts which are to each other inversely as the angular velocities.

Let ω be the resultant angular velocity; then in the case of the component angular velocities, the downward path of O′ in the time $dt = \omega_a c\, dt$; and in the case of the resultant angular velocity, the downward path of O′ $= \omega y\, dt$; these are of course equal; whence we have

$$\omega_a c = \omega y; \tag{27}$$

similarly, if we equate the two paths of O in the two cases, we have

$$\omega_b c = \omega x; \tag{28}$$

whence

$$\frac{\omega}{c} = \frac{\omega_a}{y} = \frac{\omega_b}{x}; \tag{29}$$

$$= \frac{\omega_a + \omega_b}{y + x},$$

$$= \frac{\omega_a + \omega_b}{c}; \tag{30}$$

$$\therefore \quad \omega = \omega_a + \omega_b; \tag{31}$$

that is, the resultant angular velocity is the sum of the component angular velocities.

A similar theorem is true, whatever is the number of the component angular velocities which have parallel axes.

If one of the component angular velocities, say ω_b, is negative, the paths of all particles between O and O′ due to both ω_a and ω_b will be downward; let us then, see Fig. 11, consider the path taken by a point P in the line OO′ produced; now the line-representative of ω_b is O′B, which is drawn from O′ in a direction opposite to that in which OA is drawn from O. Let OP $= x$, O′P $= y$; then the downward path described by P in dt due to ω_a is $\omega_a x\, dt$; and the upward path due to ω_b is $\omega_b y\, dt$; therefore the whole downward path in the time dt

$$= (\omega_a x - \omega_b y)\, dt.$$

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Let Q be a point in oo′ which remains at rest; then, if oQ $= x$, o′Q $= y$,

$$\omega_a x = \omega_b y; \tag{32}$$

so that the line oo′ is divided externally into two parts which are inversely proportional to the component angular velocities.

Also let ω be the resultant angular velocity whose rotation-axis is QC; then equating the downward paths of o which are due to ω_b and to ω respectively, we have

$$\omega_b c = \omega x; \tag{33}$$

and equating the downward paths of o′ which are due to ω_a and to ω respectively,

$$\omega_a c = \omega y; \tag{34}$$

$$\therefore \quad \frac{\omega}{c} = \frac{\omega_b}{x} = \frac{\omega_a}{y},$$

$$= \frac{\omega_b - \omega_a}{x - y},$$

$$= \frac{\omega_b - \omega_a}{c};$$

$$\therefore \quad \omega = \omega_b - \omega_a; \tag{35}$$

that is, the resultant angular velocity is the excess of the larger component over the less; and has therefore the same sign as the larger.

Hence if a body moves with many angular velocities $\omega_1, \omega_2, \ldots \omega_n$, all of which have parallel rotation-axes, and if Ω is the resultant angular velocity,

$$\Omega = \omega_1 + \omega_2 + \ldots\ldots + \omega_n$$

$$= \Sigma.\omega; \tag{36}$$

where $\Sigma.\omega$ is the algebraical sum of the several components; but more will be said hereafter on this subject.

32.] If however the difference between ω_a and ω_b is infinitesimal, then ω is also infinitesimal; and if $\omega_a = \omega_b$, $\omega = 0$, and the resultant angular velocity vanishes. In this case however $x = y$; which can be only if $x = y = \infty$. Here then a paradox presents itself; when two component angular velocities with parallel axes are equal and have opposite signs, the resultant angular velocity is zero, and its axis is at an infinite distance. We must return to first principles.

Consider Fig. 12, wherein oA and o′B are the line-representatives of two equal angular velocities which have opposite directions: let oo′ $=$ c, and take any particle P in the line oo′:

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let OP $= x$, O'P $= y$; then the downward path of P in the time dt due to ω_a and to $-\omega_a$

$$\begin{aligned} &= \omega_a x\,dt + \omega_a y\,dt, \\ &= \omega_a (x+y)\,dt, \\ &= \omega_a c\,dt; \end{aligned} \tag{37}$$

and therefore is the same, whatever is the place of P. Thus all particles of the body are advanced in the time dt along a distance equal to $\omega_a c\,dt$ and perpendicular to the plane containing the two parallel axes of the component angular velocities. The effect therefore of a body moving with such a pair of equal and opposite angular velocities is a displacement of translation of the body over a distance proportional to the product of either angular velocity and the perpendicular distance between the two axes. M. Poinsot, to whom we are indebted for the laws of composition and resolution of angular velocities, calls such a pair of equal and opposite angular velocities a couple of angular velocities, and the product $\omega_a c$ he calls the moment of the couple. The analogy is evident between these theorems and those of statical couples.

Hence a couple of angular velocities gives a body a displacement of translation equal to $\omega_a c\,dt$ in the time dt, and along a line perpendicular to the plane of the axes of the couple.

Hence also it is evident that a couple may be equivalently replaced by any other equimomental couple provided that the planes of the axes of the couples are either parallel or identical. And the geometrical representation of a couple is a straight line whose length is proportional to the moment of the couple, and which is perpendicular to the plane of the axes of the couple.

33.] Lastly, let us consider the most general case; that in which a body moves with many simultaneous angular velocities, the rotation-axes of which do not pass through one and the same point, and are not parallel.

As the signs of angular velocity are arbitrary, it is convenient to affect them with those which are best suited to a system of coordinate axes in space. Let those angular velocities be considered positive with which, having for their rotation-axes severally the axes of x, y, z, the body turns from the y-axis to the z-axis, from the z-axis to the x-axis, from the x-axis to the y-axis respectively; and let those be negative with which the

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body rotates in contrary directions. This system is evidently cyclical, and is easily remembered.

Let the angular velocities be $\omega_1, \omega_2, \ldots \omega_n$; and let a point o rigidly connected with the body be the origin; at it let a system of rectangular coordinates fixed in space originate; and let the direction-angles of the rotation-axes be $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \ldots \alpha_n, \beta_n, \gamma_n$; let $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots (x_n, y_n, z_n)$ be points severally on the rotation-axis of each, and let $p_1, p_2, \ldots p_n$ be the perpendicular distances from o on the several rotation-axes. And of all these quantities let ω, (α, β, γ), (x, y, z), p be the types. Let us consider the type velocity ω. At the origin let a pair of equal and opposite angular velocities be introduced, each of which is equal to ω, and the rotation-axis of which is parallel to that of ω; and from o let the perpendicular distance p be drawn to the rotation-axis of ω; so that instead of the original ω, we have ω at o equal to the original ω and with a rotation-axis parallel to that of the original ω, and a couple of angular velocities, each of which is ω, and the distance between whose axes is p; so that $p\omega$ is the moment of the couple; and the effect of which is a displacement of the body in the time dt over a distance equal to $\omega p\, dt$ in a line through the origin perpendicular to the plane which contains the rotation-axis of ω. Let a similar process be performed on all the angular velocities; then we have a system of angular velocities the rotation-axes of which pass through o, and also a system of couples of angular velocities, the effects of which are severally a displacement of translation of the body.

Let Ω be the resultant angular velocity of all those which act at o; let $\Omega_x, \Omega_y, \Omega_z$ be its axial components; and let a, b, c be the direction-angles of its rotation-axis; then

$$\begin{aligned} \Omega_x &= \omega_1 \cos \alpha_1 + \omega_2 \cos \alpha_2 + \ldots\ldots + \omega_n \cos \alpha_n, \\ &= \Sigma.\omega \cos \alpha; \end{aligned} \tag{38}$$

$$\Omega_y = \Sigma.\omega \cos \beta; \tag{39}$$

$$\Omega_z = \Sigma.\omega \cos \gamma; \tag{40}$$

$$\begin{aligned} \therefore\quad \Omega^2 &= \Omega_x^2 + \Omega_y^2 + \Omega_z^2, \\ &= (\Sigma.\omega \cos \alpha)^2 + (\Sigma.\omega \cos \beta)^2 + (\Sigma.\omega \cos \gamma)^2; \end{aligned} \tag{41}$$

and
$$\frac{\Sigma.\omega \cos a}{\cos a} = \frac{\Sigma.\omega \cos \beta}{\cos b} = \frac{\Sigma.\omega \cos \gamma}{\cos c} = \Omega; \tag{42}$$

whereby the intensity and the direction-cosines of the rotation-axis of the resultant angular velocity through o are known.

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As to the couple of angular velocities which arises from ω, the moment of the couple is $p\omega$; and as p is the perpendicular from the origin on a line passing through (x, y, z), whose direction-angles are (α, β, γ), we have

$$p^2 = (y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2. \quad (43)$$

Now the displacement of translation which the body undergoes by virtue of this couple of angular velocities is along a line perpendicular to the rotation-axis of ω and to p; so that its direction-cosines are

$$\frac{y\cos\gamma - z\cos\beta}{p}, \quad \frac{z\cos\alpha - x\cos\gamma}{p}, \quad \frac{x\cos\beta - y\cos\alpha}{p}. \quad (44)$$

Let $\Delta\sigma$ be the space through which the body is displaced in the time dt by reason of this couple of angular velocities; then

$$\Delta\sigma = \omega p\, dt; \quad (45)$$

and the direction-cosines of $\Delta\sigma$ are given by (44); so that if $\Delta\xi, \Delta\eta, \Delta\zeta$ are the axial projections of $\Delta\sigma$,

$$\left.\begin{aligned} \Delta\xi &= \omega\,(y\cos\gamma - z\cos\beta)\,dt, \\ \Delta\eta &= \omega\,(z\cos\alpha - x\cos\gamma)\,dt, \\ \Delta\zeta &= \omega\,(x\cos\beta - y\cos\alpha)\,dt. \end{aligned}\right\} \quad (46)$$

A result similar to this is true for each component angular velocity; and therefore if σ is the whole space through which the origin is transferred, and if ξ, η, ζ are the axial projections of σ,

$$\left.\begin{aligned} \xi &= \Sigma.\omega\,(y\cos\gamma - z\cos\beta)\,dt, \\ \eta &= \Sigma.\omega\,(z\cos\alpha - x\cos\gamma)\,dt, \\ \zeta &= \Sigma.\omega\,(x\cos\beta - y\cos\alpha)\,dt; \end{aligned}\right\} \quad (47)$$

and

$$\sigma^2 = \xi^2 + \eta^2 + \zeta^2; \quad (48)$$

and the direction-cosines of σ are

$$\frac{\Sigma.\omega(y\cos\gamma - z\cos\beta)\,dt}{\sigma}, \quad \frac{\Sigma.\omega(z\cos\alpha - x\cos\gamma)\,dt}{\sigma}, \quad \frac{\Sigma.\omega(x\cos\beta - y\cos\alpha)\,dt}{\sigma}; \quad (49)$$

whereby we have the resultant motions, both of translation of the origin, and of rotation about an axis through it, of a body moving with many simultaneous angular velocities.

34.] If the simultaneous angular velocities $\omega_1, \omega_2, \ldots \omega_n$ are such that the body is at rest, then $\Omega = 0$, and $\sigma = 0$; so that we have the six conditions

$$\left.\begin{aligned} \Sigma.\omega\cos\alpha &= 0, \\ \Sigma.\omega\cos\beta &= 0, \\ \Sigma.\omega\cos\gamma &= 0; \end{aligned}\right\} (50) \qquad \left.\begin{aligned} \Sigma.\omega\,(y\cos\gamma - z\cos\beta) &= 0, \\ \Sigma.\omega\,(z\cos\alpha - x\cos\gamma) &= 0, \\ \Sigma.\omega\,(x\cos\beta - y\cos\alpha) &= 0; \end{aligned}\right\} (51)$$

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from these equations theorems can be deduced similar to those of Art. 57, Vol. III. Also if the angular velocities are capable of composition into a single angular velocity, we must have

$$\xi \Omega_x + \eta \Omega_y + \zeta \Omega_z = 0; \tag{52}$$

which equation shews that the resultant line of displacement is perpendicular to the rotation-axes of the resultant angular velocity; and as the resultant displacement may be replaced by two equal and opposite angular velocities whose rotation-axes are perpendicular to the line of displacement, we may take these rotation-axes to be parallel to that of Ω, whereby we shall have three angular velocities with parallel rotation-axes which may be compounded into a single angular velocity.

If the axes of all the simultaneous couples are parallel, then

$$\left.\begin{aligned} \Omega \cos a &= \cos \alpha\, \Sigma.\omega, \\ \Omega \cos b &= \cos \beta\, \Sigma.\omega, \\ \Omega \cos c &= \cos \gamma\, \Sigma.\omega; \end{aligned}\right\} \tag{53}$$

$$\therefore \quad a = \alpha, \qquad b = \beta, \qquad c = \gamma; \tag{54}$$

$$\Omega = \Sigma.\omega; \tag{55}$$

that is, the resultant angular velocity is equal to the sum of all the component angular velocities, and its rotation-axis is parallel to the rotation-axes of the components. Also

$$\left.\begin{aligned} \xi &= (\cos \gamma\, \Sigma.\omega\, y - \cos \beta\, \Sigma.\omega\, z)\, dt, \\ \eta &= (\cos \alpha\, \Sigma.\omega\, z - \cos \gamma\, \Sigma.\omega\, x)\, dt, \\ \zeta &= (\cos \beta\, \Sigma.\omega\, x - \cos \alpha\, \Sigma.\omega\, y)\, dt. \end{aligned}\right\} \tag{56}$$

In this case (52) is satisfied; and the angular velocities are capable of reduction to a single angular velocity; let Ω be the resultant, and let $(\bar{x}, \bar{y}, \bar{z})$ be a point on its rotation-axis; then as it produces the same effect as all the components taken in combination, and as the direction-angles of its axis are α, β, γ,

$$\left.\begin{aligned} \xi &= (\cos \gamma\, \Omega\, \bar{y} - \cos \beta\, \Omega\, \bar{z})\, dt, \\ \eta &= (\cos \alpha\, \Omega\, \bar{z} - \cos \gamma\, \Omega\, \bar{x})\, dt, \\ \zeta &= (\cos \beta\, \Omega\, \bar{x} - \cos \alpha\, \Omega\, \bar{y})\, dt; \end{aligned}\right\} \tag{57}$$

which are severally equal to the values given in (56). But α, β, γ are manifestly indeterminate; so that we have

$$\bar{x} = \frac{\Sigma.\omega x}{\Sigma.\omega}, \qquad \bar{y} = \frac{\Sigma.\omega y}{\Sigma.\omega}, \qquad \bar{z} = \frac{\Sigma.\omega z}{\Sigma.\omega}. \tag{58}$$

$(\bar{x}, \bar{y}, \bar{z})$ is called the *centre* of the parallel angular velocities.

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35.] From the preceding Articles it appears that if a body is moving with many simultaneous angular velocities, the resultant motion consists (1) of a determinate angular velocity Ω, the rotation-axis of which is determinate in direction but not in position; and (2) of a displacement of translation of an arbitrarily chosen point on the axis of the resultant angular velocity Ω, the line of motion and amount of which are given by equations (47) and (48). Now whatever is the position of this arbitrarily chosen point, the resultant angular velocity is the same in intensity and direction, but the amount and the line of the displacement of translation of the point varies with the point. In the infinitesimal time dt the displacement is infinitesimal, but equations (47) shew that it varies with the origin. Let us therefore inquire what is the position of the origin, when that displacement is the least; the form of (47) indicates that it does not admit of a maximum, but it may be a minimum.

Let the directions of the coordinate axes be unchanged; and let (x_0, y_0, z_0) be the origin for which the displacement of translation is the least; so that for x, y, z in (47) we must substitute $x-x_0$, $y-y_0$, $z-z_0$; and let ξ_0, η_0, ζ_0 be the projections of the displacement σ_0 of the new origin; then

$$\begin{aligned}
\xi_0 &= \Sigma.\omega\,\{(y-y_0)\cos\gamma-(z-z_0)\cos\beta\}\,dt\,;\\
&= \Sigma.\omega(y\cos\gamma-z\cos\beta)\,dt-y_0\,\Sigma.\omega\cos\gamma\,dt+z_0\,\Sigma.\omega\cos\beta\,dt\,;\\
&= \xi-(y_0\,\Omega_z-z_0\,\Omega_y)\,dt,\\
\eta_0 &= \eta-(z_0\,\Omega_x-x_0\,\Omega_z)\,dt,\\
\zeta_0 &= \zeta-(x_0\,\Omega_y-y_0\,\Omega_x)\,dt\,;
\end{aligned}\tag{59}$$

$$\therefore\quad \sigma_0^2 = \{\xi-(y_0\Omega_z-z_0\Omega_y)dt\}^2+\{\eta-(z_0\Omega_x-x_0\Omega_z)dt\}^2 + \{\zeta-(x_0\Omega_y-y_0\Omega_x)dt\}^2. \tag{60}$$

Thus σ_0^2 is a function of x_0, y_0, z_0 which are three independent variables; whence equating to zero the three partial differentials of σ_0, as in Art. 72, Vol. III, we shall have

$$\frac{x_0-\dfrac{\zeta\,\Omega_y-\eta\,\Omega_z}{\Omega^2\,dt}}{\Omega_x} = \frac{y_0-\dfrac{\xi\,\Omega_z-\zeta\,\Omega_x}{\Omega^2\,dt}}{\Omega_y} = \frac{z_0-\dfrac{\eta\,\Omega_x-\xi\,\Omega_y}{\Omega^2\,dt}}{\Omega_z}\,; \tag{61}$$

which are the equations to a straight line; this line therefore is the locus of points for which the displacement of translation due to the simultaneous angular velocities is a minimum. The line is evidently parallel to the rotation-axis of Ω; and passes through a point whose coordinates are

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$$\frac{\zeta\Omega_y - \eta\Omega_z}{\Omega^2 dt}, \qquad \frac{\xi\Omega_z - \zeta\Omega_x}{\Omega^2 dt}, \qquad \frac{\eta\Omega_x - \xi\Omega_y}{\Omega^2 dt}; \tag{62}$$

which give the following geometrical construction. At any point O taken arbitrarily, let the displacement of translation σ and its axial components ξ, η, ζ be drawn; also let the rotation-axis of Ω be drawn. Let ϕ be the angle between these two lines; through O draw a line perpendicular to both of them; and along that perpendicular from O take a length OD $= p$, such that

$$p = \frac{\sigma \sin\phi}{\Omega\, dt}; \tag{63}$$

then a line drawn through D parallel to the rotation-axis of Ω is that whose equations are (61).

36.] If the origin is taken on the line, $p = 0$; and therefore $\sin\phi = 0$; so that the line of displacement lies along the rotation-axis of Ω. In this case therefore the system of simultaneous angular velocities is reduced to (1) an angular velocity Ω about a determinate axis, and (2) a displacement of translation σ along this axis; this axis is called the central axis of the system. In the infinitesimal time dt the body rotates through an angle $\Omega\, dt$ about the axis, and moves along the axis over a distance σ which is given by equation (48). Thus it has a helical motion, like a screw: while it rotates with a given angular velocity, it also advances along the rotation-axis with a determinate linear velocity. This is one of the most simple images which the motion of a rigid body admits of.

I may observe that the equations (61) of the central axis may be found by investigating the locus of points at which the displacement of translation lies along the rotation-axis of Ω.

Now when a body has a continuous motion the system of the central axes forms a ruled surface in space, and another ruled surface in the body; all the generators of the second successively coincide with those of the first. In imagining such a motion let us suppose that of translation or the sliding of the axis to take place before the rotation about the axis; let two generators be placed on each other in their corresponding positions, and let the sliding along them take place over the distance σ; and then let the body turn about the common generator of the two surfaces through the angle $\Omega\, dt$; by this means the next two generators will be brought into coincidence; and the corresponding sliding and rotation will again take place; and

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so on; whereby the two surfaces will be successively brought into contact with each other along their generators. If one of the surfaces is developable, the other is; and if $\sigma = 0$ throughout the motion, in which case there is no sliding, the two surfaces are evidently cones. This also is the case when one point in the body is fixed, so that all the axes pass through that point. The line of contact of the two surfaces is called the instantaneous sliding axis.

The case however in which the body has a fixed point, and in which the ruled surfaces become cones, deserves further illustration. Let o, see Fig. 13, be the fixed point of the body; and from it let a sphere be described of any radius; and let the cones which are respectively fixed in space and fixed in the body be cut by the surface of the sphere in the curves 1s and 1s′; let 1 be the point where the instantaneous axis meets the spherical surface; let s and s' be the arcs of the two curves 1s and 1s′ respectively; of which let the length-elements be ds and ds'; let o1 be the instantaneous line of contact of the two cones, and let the time be divided into equal infinitesimal elements dt; also let the curves s and s' be divided into elements corresponding to successive dt's, and so that the corresponding elements are equal. Let os be the cone fixed in space, and os′ that fixed in the body; then the motion of the body will be represented by the rolling, without sliding, of the latter cone on the former; the line of contact of the two cones being always the instantaneous axis of the body. Hereafter we shall find the equations to these cones, and thus have their relative magnitudes and position. In the mean time I may observe that the rolling may take place in many ways. One cone may roll outside the other, as in Fig. 13; or the moveable cone may roll inside, as in Fig. 14; or again as in Fig. 15, where the moveable cone is larger than the fixed cone and rolls on the outside of it. Or again either of the cones may degenerate into a plane, as in Fig. 16; and either of the cones may become a straight line, in which case the axis of rotation is fixed. Or the cones may become identical, in which case the position of the rotation-axis is indeterminate.

Also the vertex of either cone may be at an infinite distance, in which case the cone becomes a cylinder; and as the instantaneous axes are all parallel, the path of every particle is in a plane perpendicular to the axis of the cylinder.

37.] Let us consider other properties of the central axis.

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From (59) we have

$$\frac{\xi_0 \Omega_x + \eta_0 \Omega_y + \zeta_0 \Omega_z}{\Omega} = \frac{\xi \Omega_x + \eta \Omega_y + \zeta \Omega_z}{\Omega}; \tag{64}$$

and by a process similar to that of Art. 71, Vol. III, it may be shewn that this quantity is constant, through whatever angle the system of coordinate axes is turned. But if ϕ is the angle between σ corresponding to any origin and the rotation-axis of ω,

$$\cos\phi = \frac{\xi \Omega_x + \eta \Omega_y + \zeta \Omega_z}{\sigma \Omega};$$

$$\therefore \quad \sigma \cos\phi = \text{a constant} = \sigma_0; \tag{65}$$

where σ_0 is the displacement of translation along the central axis; that is, the projections of the displacements of translation of all points of the body on the central axis are equal. If therefore a plane is drawn perpendicular to the central axis, the distances of all points of the body from this fixed plane are increased or diminished by the same quantity in the time dt. And the line of this quantity is, as shewn in Art. 35, that along which the displacement of translation is the least. Hereby we have the following construction for the direction of the central axis. Through any point in space draw straight lines equal and parallel to the actual displacements of all the points of a body due to the time dt; then the ends of these lines will be all in one plane which is perpendicular to the central axis.

Again, all the properties of reciprocal lines which are demonstrated in Vol. III, Art. 70—87, are true, mutatis mutandis, of angular velocities; indeed the principle of duality is completely applicable to these theorems. X, Y, Z, R, L, M, N, G are to be changed into Ω_x, Ω_y, Ω_z, Ω, ξ, η, ζ, σ respectively, and the enunciations of the theorems are to be changed accordingly; that is, pressures are to be changed into angular velocities, and moments are to be changed into displacements of translation. Thus, it is there proved that any system of pressures may be reduced into two pressures acting along lines at right angles to each other: hence it follows that every system of angular velocities may be reduced into two angular velocities whose axes are perpendicular to each other.

Again, as all moment-centres of equal moment lie on a cylindrical surface whose axis lies along the central axis; so all points for which the displacement of translation is the same lie on a cylindrical surface; also all the lines of equal displacement corresponding to points on a circle whose plane is perpendicular

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to the central axis lie on the surface of a hyperboloid of revolution.

So for all points on a line perpendicular to the central axis, the lines of displacement are in the surface of a hyperbolic paraboloid.

From the theorems of Art. 85, Vol. III, we have the following deductions. If a body moves with many simultaneous angular velocities, they may be reduced to two others whose rotation-axes are such that if one is given, the other is the reciprocal line; and the perpendicular line to these two axes passes through and is perpendicular to the central axis of the system.

And from Art. 86, it follows that every system of angular velocities may be replaced by two equal angular velocities, whose rotation-axes are perpendicular to each other, and each of which is inclined at 45° to the central axis, and the axes are perpendicular to a line which is bisected at right angles by the central axis; each of the angular velocities $= \frac{\Omega}{2^{\frac{1}{2}}}$; and the length of the perpendicular distance between the axes $= \frac{2\sigma_0}{\Omega}$.

38.] We have in the next place to determine the motion of translation of any particle of a body which rotates about a given determinate axis; and I will consider the body to have a fixed point, through which of course the rotation-axis passes, so that the displacement of a particle may be due to the rotation only; and I shall consider the displacements which take place during the infinitesimal time dt.

Let the fixed point of the body be taken as the origin, and a system of coordinate axes fixed in space originate at it; let ω be the angular velocity with which the body rotates about the axis, of which let the direction-angles be (α, β, γ); let (x, y, z) be the place of the particle P, the displacement of which in the time dt is to be calculated; let ds be the displacement, of which let dx, dy, dz be the axial projections; let $\omega_x, \omega_y, \omega_z$ be the axial components of ω; so that

$$\frac{\omega_x}{\cos\alpha} = \frac{\omega_y}{\cos\beta} = \frac{\omega_z}{\cos\gamma} = \omega. \tag{66}$$

Let p be the perpendicular distance from (x, y, z) on the rotation-axis (α, β, γ); then as $\omega\, dt$ is the infinitesimal angle through which the body rotates in the time dt,

$$ds = p\,\omega\, dt; \tag{67}$$

and

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$$p^2 = (z\cos\beta - y\cos\gamma)^2 + (x\cos\gamma - z\cos\alpha)^2 + (y\cos\alpha - x\cos\beta)^2. \quad (68)$$

Now ds is perpendicular to the rotation-axis, and to the line drawn from the origin to (x, y, z); also dx, dy, dz are proportional to its direction-cosines; therefore

$$\left.\begin{aligned} \cos\alpha\, dx + \cos\beta\, dy + \cos\gamma\, dz &= 0, \\ x\,dx + \quad y\,dy + \quad z\,dz &= 0; \end{aligned}\right\} \quad (69)$$

$$\therefore \quad \frac{dx}{z\cos\beta - y\cos\gamma} = \frac{dy}{x\cos\gamma - z\cos\alpha} = \frac{dz}{y\cos\alpha - x\cos\beta} \quad (70)$$

$$= \frac{ds}{p}; \quad (71)$$

$$\therefore \quad \left.\begin{aligned} dx &= (z\cos\beta - y\cos\gamma)\,\omega\, dt = (z\,\omega_y - y\,\omega_z)\,dt, \\ dy &= (x\cos\gamma - z\cos\alpha)\,\omega\, dt = (x\,\omega_z - z\,\omega_x)\,dt, \\ dz &= (y\cos\alpha - x\cos\beta)\,\omega\, dt = (y\,\omega_x - x\,\omega_y)\,dt; \end{aligned}\right\} \quad (72)$$

$$\therefore \quad \frac{ds^2}{dt^2} = (z\,\omega_y - y\,\omega_z)^2 + (x\,\omega_z - z\,\omega_x)^2 + (y\,\omega_x - x\,\omega_y)^2. \quad (73)$$

In all these expressions there has been an ambiguity of sign, which I have omitted; that sign has been taken which is in accordance with the principle of signs of angular velocities determined in Article 33; for suppose the body to rotate about the axis of x only; so that $\omega_y = \omega_z = 0$; then for a particle in the first octant of space, dz, the increment of z, will be positive, and dy, the increment of y, will be negative; similarly for single rotations about the other coordinate axes; and equations (72) are in accordance with these conditions.

Hence the equations to the tangent line of the path which the particle m at (x, y, z) is taking at the time t are

$$\frac{\xi - x}{z\,\omega_y - y\,\omega_z} = \frac{\eta - y}{x\,\omega_z - z\,\omega_x} = \frac{\zeta - z}{y\,\omega_x - x\,\omega_y}. \quad (74)$$

Hence also if a body has a fixed point at the origin, and rotates with angular velocities ω_x, ω_y, ω_z about its three coordinate axes respectively, the rotation-axis of the body is the locus of the points which are at rest, and its equations consequently are

$$\frac{\xi}{\omega_x} = \frac{\eta}{\omega_y} = \frac{\zeta}{\omega_z}. \quad (75)$$

As an exact comprehension of (72) is of great importance, let us investigate them by another and a more elementary process.

From (x, y, z), the place of m, let perpendiculars r_x, r_y, r_z be drawn to the three rectangular axes of x, y, z respectively; and let θ be the angle between r_x and the plane of (xy), ϕ the angle

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between r_y and the plane of (yz), ψ the angle between r_z and the plane of (zx); so that $d\theta$, $d\phi$, $d\psi$ are positive according to our hypothesis of signs for small rotations about the axes of x, y, z respectively. Let rotations through infinitesimal angles take place successively about these axes, and let the changes of the variables due to these rotations be calculated. For a rotation about the axis of x through $d\theta$ we have

$$\begin{aligned} y &= r_x \cos\theta, & z &= r_x \sin\theta, \\ dy &= -r_x \sin\theta\, d\theta, & dz &= r_x \cos\theta\, d\theta, \\ &= -z\, d\theta; & &= y\, d\theta; \end{aligned}$$

so that the infinitesimal variations of y and z are respectively $-z\,d\theta$, and $y\,d\theta$; similarly for a rotation through an angle $d\phi$ about the axis of y, the variations of z and x are respectively $-x\,d\phi$ and $z\,d\phi$; and for a rotation through $d\psi$ about the axis of z, the variations of x and y are respectively $-y\,d\psi$ and $x\,d\psi$; so that if dx, dy, dz are the total variations of x, y, and z due to these combined rotations,

$$\left.\begin{aligned} dx &= z\,d\phi - y\,d\psi, \\ dy &= x\,d\psi - z\,d\theta, \\ dz &= y\,d\theta - x\,d\phi. \end{aligned}\right\} \tag{76}$$

39.] We shall hereafter find it convenient to refer a body and its motion to two sets of coordinate axes at the same time; one of which is assumed to be fixed in space, and the other to be fixed in the body and to be moving with it. At first I will assume these two systems to originate at the same fixed point of the body so that the body is capable of only a motion of rotation about an axis passing through this fixed point; and I will suppose a point P to be (x, y, z) and (ξ, η, ζ) in reference respectively to the systems fixed in space and fixed in the body; and I shall suppose the two systems to be related by the scheme of direction-cosines given in Art. 2. As the body moves about an axis, which passes through the origin, x, y, z and the nine direction-cosines vary, but ξ, η, ζ do not vary; so that from (2), Art. 2, we have for the variations in the time dt

$$\left.\begin{aligned} \frac{dx}{dt} &= \xi\frac{da_1}{dt} + \eta\frac{db_1}{dt} + \zeta\frac{dc_1}{dt}, \\ \frac{dy}{dt} &= \xi\frac{da_2}{dt} + \eta\frac{db_2}{dt} + \zeta\frac{dc_2}{dt}, \\ \frac{dz}{dt} &= \xi\frac{da_3}{dt} + \eta\frac{db_3}{dt} + \zeta\frac{dc_3}{dt}; \end{aligned}\right\} \tag{77}$$

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which values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ express at the time t the axial components of the velocity of the element which is at (x, y, z).

Let $\omega_x, \omega_y, \omega_z$ be the axial components of the angular velocity ω of the body along the fixed axes of x, y, z respectively; then from (72), Art. 38,

$$\frac{dx}{dt} = z\omega_y - y\omega_z;$$
$$= \omega_y(a_3\xi + b_3\eta + c_3\zeta) - \omega_z(a_2\xi + b_2\eta + c_2\zeta);$$

$$\left.\begin{aligned}\frac{dx}{dt} &= \xi(a_3\omega_y - a_2\omega_z) + \eta(b_3\omega_y - b_2\omega_z) + \zeta(c_3\omega_y - c_2\omega_z),\\ \frac{dy}{dt} &= \xi(a_1\omega_z - a_3\omega_x) + \eta(b_1\omega_z - b_3\omega_x) + \zeta(c_1\omega_z - c_3\omega_x),\\ \frac{dz}{dt} &= \xi(a_2\omega_x - a_1\omega_y) + \eta(b_2\omega_x - b_1\omega_y) + \zeta(c_2\omega_x - c_1\omega_y).\end{aligned}\right\} \quad (78)$$

In (77) and (78) (ξ, η, ζ) is the place of any particle, and therefore ξ, η, ζ are indeterminate; so that the systems of equations are identical; hence we may equate coefficients, and we have

$$\left.\begin{aligned}\frac{da_1}{dt} &= a_3\omega_y - a_2\omega_z, & \frac{db_1}{dt} &= b_3\omega_y - b_2\omega_z, & \frac{dc_1}{dt} &= c_3\omega_y - c_2\omega_z;\\ \frac{da_2}{dt} &= a_1\omega_z - a_3\omega_x, & \frac{db_2}{dt} &= b_1\omega_z - b_3\omega_x, & \frac{dc_2}{dt} &= c_1\omega_z - c_3\omega_x;\\ \frac{da_3}{dt} &= a_2\omega_x - a_1\omega_y, & \frac{db_3}{dt} &= b_2\omega_x - b_1\omega_y, & \frac{dc_3}{dt} &= c_2\omega_x - c_1\omega_y.\end{aligned}\right\} \quad (79)$$

These formulæ are important, and it is necessary to understand their meaning; we have arrived at them indirectly, and therefore let us prove them by another process.

The circumstances of the body are these. A point in it is fixed and is the origin; at it originate (1) a system of coordinate axes (x, y, z) fixed absolutely in space; (2) a system of coordinate axes (ξ, η, ζ) fixed in the moving body: the body rotates with an angular velocity ω about an axis such that the axial components of ω along the axes of x, y, z are $\omega_x, \omega_y, \omega_z$; at the time t the direction-cosines of the axis of ξ are a_1, a_2, a_3; What are the changes of these quantities due to the angular velocity ω? Take a point (x, y, z) on the axis of ξ at a distance ρ from the origin; then

$$\frac{x}{a_1} = \frac{y}{a_2} = \frac{z}{a_3} = \rho;$$

$$\therefore \quad \frac{dx}{dt} = \rho\frac{da_1}{dt}, \qquad \frac{dy}{dt} = \rho\frac{da_2}{dt}, \qquad \frac{dz}{dt} = \rho\frac{da_3}{dt};$$

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and replacing $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ by their values given in (72), we have

$$\left.\begin{aligned} \frac{da_1}{dt} &= \frac{1}{\rho}(z\,\omega_y - y\,\omega_z) \\ &= a_3\omega_y - a_2\,\omega_z, \\ \frac{da_2}{dt} &= a_1\omega_z - a_3\,\omega_x, \\ \frac{da_3}{dt} &= a_2\omega_x - a_1\,\omega_y; \end{aligned}\right\}$$

and the other six equivalents of (79) may be found by similar processes.

Thus, if $\rho = 1$, $\frac{da_1}{dt}$, $\frac{da_2}{dt}$, $\frac{da_3}{dt}$ are severally the axial components of the velocity of the particle on the axis of ξ at an unit distance from the origin at the time t.

Also from (79) we have

$$\left.\begin{aligned} \left(\frac{da_1}{dt}\right)^2 + \left(\frac{db_1}{dt}\right)^2 + \left(\frac{dc_1}{dt}\right)^2 &= \omega_y^2 + \omega_z^2, \\ \left(\frac{da_2}{dt}\right)^2 + \left(\frac{db_2}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2 &= \omega_z^2 + \omega_x^2, \\ \left(\frac{da_3}{dt}\right)^2 + \left(\frac{db_3}{dt}\right)^2 + \left(\frac{dc_3}{dt}\right)^2 &= \omega_x^2 + \omega_y^2; \end{aligned}\right\} \tag{80}$$

40.] Hereby also the axial components $\omega_x, \omega_y, \omega_z$ may be determined in terms of the t-differentials of the direction-cosines. Let us multiply the three equations in the last horizontal row of (79) by a_2, b_2, c_2 respectively and add, then

$$\omega_x = a_2\frac{da_3}{dt} + b_2\frac{db_3}{dt} + c_2\frac{dc_3}{dt}; \tag{81}$$

also let us multiply the three equations in the middle row of (79) by a_3, b_3, c_3 respectively and add; then

$$-\omega_x = a_3\frac{da_2}{dt} + b_3\frac{db_2}{dt} + c_3\frac{dc_2}{dt}; \tag{82}$$

these two values of ω_x are in accordance with the first equation of (6) Art. 2.

Hence, and from similar processes, we have

$$\omega_x = a_2\frac{da_3}{dt} + b_2\frac{db_3}{dt} + c_2\frac{dc_3}{dt} = -\left\{a_3\frac{da_2}{dt} + b_3\frac{db_2}{dt} + c_3\frac{dc_2}{dt}\right\}; \tag{83}$$

$$\omega_y = a_3\frac{da_1}{dt} + b_3\frac{db_1}{dt} + c_3\frac{dc_1}{dt} = -\left\{a_1\frac{da_3}{dt} + b_1\frac{db_3}{dt} + c_1\frac{dc_3}{dt}\right\}; \tag{84}$$

$$\omega = a_1\frac{da_2}{dt} + b_1\frac{db_2}{dt} + c_1\frac{dc_2}{dt} = -\left\{a_2\frac{da_1}{dt} + b_2\frac{db_1}{dt} + c_2\frac{dc_1}{dt}\right\}; \tag{85}$$

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Hence also the equations (75) to the rotation-axis may be expressed in terms of the t-differentials of the direction-cosines of the axes fixed in the body.

Also, if ω_ξ, ω_η, ω_ζ are the axial components along the axes of ξ, η, ζ of the angular velocity of the body at the time t,

$$\left.\begin{aligned} \omega_\xi &= a_1\,\omega_x + a_2\,\omega_y + a_3\,\omega_z, \\ \omega_\eta &= b_1\,\omega_x + b_2\,\omega_y + b_3\,\omega_z, \\ \omega_\zeta &= c_1\,\omega_x + c_2\,\omega_y + c_3\,\omega_z\,; \end{aligned}\right\} \tag{86}$$

and conversely,

$$\left.\begin{aligned} \omega_x &= a_1\,\omega_\xi + b_1\,\omega_\eta + c_1\,\omega_\zeta, \\ \omega_y &= a_2\,\omega_\xi + b_2\,\omega_\eta + c_2\,\omega_\zeta, \\ \omega_z &= a_3\,\omega_\xi + b_3\,\omega_\eta + c_3\,\omega_\zeta. \end{aligned}\right\} \tag{87}$$

Also, from the second vertical row of (79), we have

$$\begin{aligned} c_1\frac{db_1}{dt} + c_2\frac{db_2}{dt} + c_3\frac{db_3}{dt} &= (b_2c_3 - b_3c_2)\omega_x + (b_3c_1 - b_1c_3)\omega_y + (b_1c_2 - b_2c_1)\omega_z \\ &= a_1\omega_x + a_2\omega_y + a_3\omega_z \\ &= \omega_\xi, \end{aligned}$$

using the equalities contained in (11), (12), and (13) of Art. 2. By a similar process, equivalents of ω_η and ω_ζ in terms of the t-differentials of the direction-cosines are determined, and we have

$$\left.\begin{aligned} \omega_\xi &= c_1\frac{db_1}{dt} + c_2\frac{db_2}{dt} + c_3\frac{db_3}{dt} = -\left\{b_1\frac{dc_1}{dt} + b_2\frac{dc_2}{dt} + b_3\frac{dc_3}{dt}\right\}, \\ \omega_\eta &= a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt} = -\left\{c_1\frac{da_1}{dt} + c_2\frac{da_2}{dt} + c_3\frac{da_3}{dt}\right\}, \\ \omega_\zeta &= b_1\frac{da_1}{dt} + b_2\frac{da_2}{dt} + b_3\frac{da_3}{dt} = -\left\{a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right\}. \end{aligned}\right\} \tag{88}$$

And the components of the absolute velocity of the particle m at the time t along the moving axes which are fixed in the body may thus be found; let them be v_ξ, v_η, v_ζ; then

$$\begin{aligned} v_\xi &= a_1\frac{dx}{dt} + a_2\frac{dy}{dt} + a_3\frac{dz}{dt} \\ &= \xi\left(a_1\frac{da_1}{dt} + a_2\frac{da_2}{dt} + a_3\frac{da_3}{dt}\right) + \eta\left(a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right) \\ &\qquad + \zeta\left(a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt}\right); \end{aligned}$$

whence, and by similar processes for the other components, we have

$$\left.\begin{aligned} v_\xi &= \zeta\,\omega_\eta - \eta\,\omega_\zeta, \\ v_\eta &= \xi\,\omega_\zeta - \zeta\,\omega_\xi, \\ v_\zeta &= \eta\,\omega_\xi - \xi\,\omega_\eta. \end{aligned}\right\} \tag{89}$$

These equations enable us to determine the position of the ro-

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tation-axis at the time t relatively to the axes fixed in the moving body; for if $v_\xi = v_\eta = v_\zeta = 0$,

$$\frac{\xi}{\omega_\xi} = \frac{\eta}{\omega_\eta} = \frac{\zeta}{\omega_\zeta}; \tag{90}$$

and these are the equations to the line of quiescent particles.

Now, in the general motion of a body, the axial components of the angular velocities are functions of the time, and may be expressed in terms of t; and therefore the position of the rotation-axis will vary from time to time relatively to both systems of axes, and will describe a conical surface the vertex of which is the fixed point. If then we eliminate t from (75), the resulting equation will be that of the conical surface fixed in space; and if we eliminate t from (90), the resultant equation will be the conical surface fixed in the body; and these two conical surfaces will always have a generating line common; which will be the rotation-axis at the time t. These are the cones referred to in Art. 36.

41.] Let us now suppose the body to be free from all constraint: let us as heretofore take a point in or rigidly connected with the body to be the origin of a system of rectangular coordinate axes fixed in it and moving with it; relatively to a system of axes fixed in space, let, at the time t, (x, y, z) be the place of a type-particle m of the body, and let (x_0, y_0, z_0) be the place of the moving origin; and let (ξ, η, ζ) be the place of m relatively to the origin and to the axes which are fixed in and move with the body; then, taking the scheme of direction-cosines of Art. 2, we have

$$\left.\begin{aligned} x &= x_0 + a_1\xi + b_1\eta + c_1\zeta, \\ y &= y_0 + a_2\xi + b_2\eta + c_2\zeta, \\ z &= z_0 + a_3\xi + b_3\eta + c_3\zeta; \end{aligned}\right\} \tag{91}$$

where also we have the inverse system

$$\left.\begin{aligned} \xi &= a_1(x-x_0) + a_2(y-y_0) + a_3(z-z_0), \\ \eta &= b_1(x-x_0) + b_2(y-y_0) + b_3(z-z_0), \\ \zeta &= c_1(x-x_0) + c_2(y-y_0) + c_3(z-z_0). \end{aligned}\right\} \tag{92}$$

As the body moves x_0, y_0, z_0 and the nine direction-cosines evidently vary; but ξ, η, ζ are constant; therefore

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_0}{dt} + \xi\frac{da_1}{dt} + \eta\frac{db_1}{dt} + \zeta\frac{dc_1}{dt}, \\ \frac{dy}{dt} &= \frac{dy_0}{dt} + \xi\frac{da_2}{dt} + \eta\frac{db_2}{dt} + \zeta\frac{dc_2}{dt}, \\ \frac{dz}{dt} &= \frac{dz_0}{dt} + \xi\frac{da_3}{dt} + \eta\frac{db_3}{dt} + \zeta\frac{dc_3}{dt}; \end{aligned}\right\} \tag{93}$$

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which are the components along the fixed axes of the velocity of any particle m, and admit also of expression in the following forms;

$$\frac{dx}{dt} = \frac{dx_0}{dt} + (x-x_0)\left(a_1\frac{da_1}{dt} + b_1\frac{db_1}{dt} + c_1\frac{dc_1}{dt}\right)$$
$$+ (y-y_0)\left(a_2\frac{da_1}{dt} + b_2\frac{db_1}{dt} + c_2\frac{dc_1}{dt}\right)$$
$$+ (z-z_0)\left(a_3\frac{da_1}{dt} + b_3\frac{db_1}{dt} + c_3\frac{dc_1}{dt}\right);$$

$$\left.\begin{aligned} \therefore\quad \frac{dx}{dt} &= \frac{dx_0}{dt} + (z-z_0)\,\omega_y - (y-y_0)\,\omega_z, \\ \text{Also similarly,}\quad & \\ \frac{dy}{dt} &= \frac{dy_0}{dt} + (x-x_0)\,\omega_z - (z-z_0)\,\omega_x, \\ \frac{dz}{dt} &= \frac{dz_0}{dt} + (y-y_0)\,\omega_x - (x-x_0)\,\omega_y. \end{aligned}\right\} \tag{94}$$

Let v_ξ, v_η, v_ζ be the components of the velocity of m, along the moving axes of ξ, η, ζ; so that

$$v_\xi = a_1\frac{dx}{dt} + a_2\frac{dy}{dt} + a_3\frac{dz}{dt}$$
$$= a_1\frac{dx_0}{dt} + a_2\frac{dy_0}{dt} + a_3\frac{dz_0}{dt} + \xi\left(a_1\frac{da_1}{dt} + a_2\frac{da_2}{dt} + a_3\frac{da_3}{dt}\right)$$
$$+ \eta\left(a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right) + \zeta\left(a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt}\right);$$

$$\left.\begin{aligned} \therefore\quad v_\xi &= a_1\frac{dx_0}{dt} + a_2\frac{dy_0}{dt} + a_3\frac{dz_0}{dt} + \zeta\,\omega_\eta - \eta\,\omega_\zeta, \\ v_\eta &= b_1\frac{dx_0}{dt} + b_2\frac{dy_0}{dt} + b_3\frac{dz_0}{dt} + \xi\,\omega_\zeta - \zeta\,\omega_\xi, \\ v_\zeta &= c_1\frac{dx_0}{dt} + c_2\frac{dy_0}{dt} + c_3\frac{dz_0}{dt} + \eta\,\omega_\xi - \xi\,\omega_\eta. \end{aligned}\right\} \tag{95}$$

In these expressions the first three terms of each are the resolved parts along the axes of ξ, η, ζ of the velocity of the moving origin; and the last two terms are the velocities due to the rotation about an axis passing through the moving origin. But the point (x_0, y_0, z_0) is arbitrary; so that we have the following theorem;

An infinitesimal motion of a body in a time dt is compounded of a motion of translation of any particle of it at (x_0, y_0, z_0), and of a motion of rotation of the body about an axis passing through that point.

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If ω is the angular velocity about the rotation-axis passing through (x_0, y_0, z_0),

$$\omega^2 = \omega_\xi^2 + \omega_\eta^2 + \omega_\zeta^2; \tag{96}$$

and the direction-cosines of its rotation-axis are

$$\frac{\omega_\xi}{\omega}, \qquad \frac{\omega_\eta}{\omega}, \qquad \frac{\omega_\zeta}{\omega}. \tag{97}$$

If the displacement of translation lies along the rotation-axis of the angular velocity, the motion is compounded of a sliding along a line and of a rotation about that line: this is indeed the case which we have considered in Art. 36; the sliding axis of rotation being the central axis of the body; the motion of the body is helical; and we may thus find the equations of the central axis.

Let (x_0, y_0, z_0) be any point chosen arbitrarily; of the displacement of translation of which let dx_0, dy_0, dz_0 be the projections on the fixed axes. Let (x, y, z) be a point on the central axis; this point has therefore only the sliding displacement of translation, and the line of its motion is along the rotation-axis of the angular velocity; therefore dx, dy, dz are proportional to ω_x, ω_y, ω_z respectively. Hence from (94),

$$\frac{\dfrac{dx_0}{dt} + (z-z_0)\,\omega_y - (y-y_0)\,\omega_z}{\omega_x} = \frac{\dfrac{dy_0}{dt} + (x-x_0)\,\omega_z - (z-z_0)\,\omega_x}{\omega_y}$$

$$= \frac{\dfrac{dz_0}{dt} + (y-y_0)\,\omega_x - (x-x_0)\,\omega_y}{\omega_z}; \tag{98}$$

from which we have

$$\frac{x - x_0 - \dfrac{\omega_y\,dz_0 - \omega_z\,dy_0}{\omega^2\,dt}}{\omega_x} = \frac{y - y_0 - \dfrac{\omega_z\,dx_0 - \omega_x\,dz_0}{\omega^2\,dt}}{\omega_y}$$

$$= \frac{z - z_0 - \dfrac{\omega_x\,dy_0 - \omega_y\,dx_0}{\omega^2\,dt}}{\omega_z}; \tag{99}$$

which are the equations to the central axis; and are identical with (61), Art. 35, if the origin is taken at the point (x_0, y_0, z_0); in which case dx_0, dy_0, dz_0 become respectively ξ, η, ζ of that Article.

42.] The two systems to which the rotation of a rigid body has been referred in the preceding Articles are related to each other by means of nine direction-cosines, but as six equations of condition are given, only three of the direction-cosines are arbitrary; that is, three independent variables are sufficient for pass-

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ing from one system of rectangular coordinates to another, both of which originate at the same point. I proceed to explain the mode of expressing angular velocities referred to one system in terms of angular velocities referred to another system by means of the formulæ which are investigated in Art. 3; and to give clearness to our thoughts let us consider the systems of axes which are delineated in Fig. 1. O is the common point at which the two systems originate, and from O as a centre let a sphere of radius unity be described, on the surface of which the great circles delineated in the figure are supposed to be. Let $\omega_x, \omega_y, \omega_z, \omega_\xi, \omega_\eta, \omega_\zeta$ be the angular velocities about the axes of $x, y, z, \xi, \eta, \zeta$ respectively. Let the planes of (xy) and of $(\xi\eta)$ intersect in the line ON, and be inclined to each other at the angle θ; so that θ is also the angle between the axes of z and ζ; ON is technically called the line of nodes. Let, in Art. 3, $x\text{ON} = \psi$, $\xi\text{ON} = \phi$; then, as the body moves, θ, ψ, and ϕ vary. The angle θ is technically called the obliquity, and $\frac{d\theta}{dt}$ is the angular velocity of the body about the line ON; $\frac{d\phi}{dt}$ is the angular velocity about the axis of ζ; $\frac{d\psi}{dt}$ is the angular velocity about the axis of z, and indicates the velocity with which ON moves along the plane of (xy); it is called the velocity of precession, the precession being the angle $x\text{ON}$; and the precession is direct or retrograde according as the angle ψ is increasing or diminishing. The angular velocity $\frac{d\theta}{dt}$ is sometimes called the nutation; of these terms however and of the origin of them more will be said hereafter.

Let us express the angular velocities $\omega_\xi, \omega_\eta, \omega_\zeta$ in terms of $\frac{d\theta}{dt}, \frac{d\phi}{dt}$, and $\frac{d\psi}{dt}$; that is, let us resolve the latter along the rotation axes of the former;

$$\begin{aligned}\omega_\xi &= \frac{d\theta}{dt}\cos\xi\,\text{ON} + \frac{d\phi}{dt}\cos\xi\,\text{O}\zeta + \frac{d\psi}{dt}\cos\xi\,\text{O}z \\ &= \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi; \qquad (100)\end{aligned}$$

$$\begin{aligned}\omega_\eta &= \frac{d\theta}{dt}\cos\eta\,\text{ON} + \frac{d\phi}{dt}\cos\eta\,\text{O}\zeta + \frac{d\psi}{dt}\cos\eta\,\text{O}z \\ &= -\frac{d\theta}{dt}\sin\phi + \frac{d\psi}{dt}\sin\theta\cos\phi; \qquad (101)\end{aligned}$$

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$$\omega_\zeta = \frac{d\theta}{dt}\cos\zeta\,\text{ON} + \frac{d\phi}{dt}\cos\zeta\,\text{o}\zeta + \frac{d\psi}{dt}\cos\zeta\,\text{o}z$$

$$= \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta. \tag{102}$$

Therefore by elimination

$$\frac{d\theta}{dt} = \omega_\xi\cos\phi - \omega_\eta\sin\phi. \tag{103}$$

$$\frac{d\psi}{dt} = \frac{\omega_\xi\sin\phi + \omega_\eta\cos\phi}{\sin\theta}. \tag{104}$$

$$\frac{d\phi}{dt} = \omega_\zeta - \frac{\cos\theta}{\sin\theta}(\omega_\xi\sin\phi + \omega_\eta\cos\phi); \tag{105}$$

whereby $\frac{d\theta}{dt}, \frac{d\psi}{dt}, \frac{d\phi}{dt}$ are given in terms of the angular velocities of the body about three axes fixed in and moving with it; and if by integration or otherwise θ, ψ, ϕ can be found in terms of t, these quantities will determine the place of the body at a given time.

If, to fix our thoughts, the moving body is the earth, of which oζ is the polar axis; then the plane of $(\xi\eta)$ is the equator; and, if the plane of (xy) is the ecliptic, the axis of z passes through the pole of the ecliptic, which is fixed in the heavens. In this case θ is the obliquity of the ecliptic, ON passes through the vernal and the autumnal equinoxes and is the line of equinoxes; and ϕ is the longitude of a certain meridian plane, viz. ξ o ζ, measured from the line of equinoxes.

Let this much at present suffice for the cinematics of a moving body; the subject however is far from being exhausted; we shall return to it hereafter in a more general case, and we shall arrive at formulæ of which all the preceding are only particular cases.

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CHAPTER III.

THE DYNAMICS PROPER OF A MATERIAL SYSTEM.

SECTION I.—*D'Alembert's Principle; the equations of motion of a material system.*

43.] WE now come to the dynamics proper of a material system.

A material system is an assemblage of particles dependent on each other by the action of certain forces which have their origin in and pervade the system. These are called internal forces and are generally different in different systems. The system also is such that if one particle or body of it moves by the action of a force external to the system, one or more of the other particles will also move. Thus a material system is always subject to the action of internal forces; and may also be acted on by external forces, in which case the system will move. Let us first consider the former forces; and the constitution of the system of the particles as it depends on the nature of these internal forces.

(1) The system may be a rigid body; then the internal forces are molecular, and of such an intensity that all the particles of the body are at relative rest during the whole motion; and the external forces, whatever are the particles they act on, do not effect any separation of the particles; so that the molecular forces are infinitely great in comparison of them. Such a system is not probably found in nature: all bodies are more or less compressible and elastic, and the particles have a relative motion under the action of external forces. Nevertheless as such a system is imaginable, and is the limit towards which rigid bodies in nature tend, it is necessary to consider it and to discuss its properties.

(2) The system may consist of particles invariably connected by rigid and inextensible rods, which are capable of bearing force of either compression or extension; so that during the motion the particles are at relative rest. This system has dynamically

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the same properties as the former, and differs only in the nature of the internal forces. As in this and the former cases the particles have always the same relative places, these systems are called geometrical.

(3) The several particles of the system may be connected by flexible and extensible strings or rods; in which case the internal forces acting on the particles, by the strings or rods, may vary from time to time as the system moves, and according to the nature of the external forces. We have instances of these systems, when bodies move about pulleys by means of flexible strings, either extensible or inextensible; when a perfect pendulum vibrates with an extensible rod.

(4) The system may consist of particles and of bodies which act on each other by mutual attractions or repulsions; and these may be functions of the several distances of the particles or bodies; so that during the motion of the system its form may not be invariable. The solar system is of this nature, in which we have a series of planets, secondaries, &c. subject to mutual attractions. We have also other systems of the same kind in the motion of liquids; of oscillating flexible cords; of air; of the ethereal medium, &c.; in all which cases the form of the system is continually changing.

These two latter are generally called dynamical systems.

Such are some of the material systems, the motion of which we have now to examine. They consist of material particles, each of which is acted on by certain forces both internal and external. Now if all the forces which act on any one particle are given, the motion of that particle may be determined by the processes of Vol. III. The internal forces however are generally not known, and the determination of them is beyond our power. Let us then consider whether the equations of motion of the aggregate system cannot be constructed without a knowledge of these internal forces, and thus without a knowledge of the motion of the several constituent particles. With this object in view, we must examine the circumstances of motion more exactly, and this we shall do by contrasting it with that of a single particle.

When a single particle moves under the action of a certain external force, the expressed momentum is equal to that impressed on it by this force, whether these are respectively infinitesimal, which is the case when a finite force acts for an infini-

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tesimal time-element; or whether the momentum-increase is finite, as is the case when an instantaneous force acts. These results are consequent on that property of matter which we call Inertia.

In the motion of a material system, the momentum expressed in any particle is not necessarily equal to that impressed on it from an external source, because each particle is influenced or constrained by one or more of the other particles; that is, is acted on by internal forces, as we have just now explained, and thus it is not free for the development of the momentum communicated to it. The expressed momentum may be either greater or less than that impressed by the external forces according to the internal forces which act on it from the other particles. Some other principle therefore beyond that of inertia is necessary for the construction of equations by which the motion of the system may be determined. This has been supplied by D'Alembert. It was first enunciated in a Memoir read before the Academy of Sciences in Paris at the end of the year 1742; and it is now always known as "D'Alembert's Principle." I propose to consider the circumstances which require it in one or two particular cases; because by these it will be better understood. And although I shall take continuously acting finite forces by which momentum-increments are impressed, yet, as what is true of them will also be true of momenta impressed by instantaneous forces, the explanation will be applicable to both kinds of force.

44.] In the first place, a difference usually exists between the momentum-increment impressed on, and that expressed in the motion of a particle of a material system. To shew that this is the case, let us suppose a heavy rigid body to be composed, say, one half of pith, and the other half of gold; and suppose it to fall towards the earth through the air which is a resisting medium. Now gravity acts as an accelerating force equally on the pith and the gold; that is, gravity impresses equal velocity on both; and in an exhausted receiver of an air pump, as we well know, both fall through equal vertical spaces in equal times, and both in equal times acquire equal velocities. The air however by its resistance is a retarding force, and acts with greater effect, cæteris paribus, in diminishing velocity on bodies whose density is more nearly equal to its own density than it does on those whose density is greater. Hence the velocity of the pith

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will be diminished by the action of the air more than the velocity of the gold if both were of equal size and shape and were separate. If the two substances were separated the gold would fall faster than the pith; that is, in other words, in a given time greater velocity would be impressed upon and expressed in the gold than in the pith. Let the two be connected so that the retarding action of the air is equal on both; by reason of the connection all falls together; and has therefore a common velocity; and in neither the pith nor the gold is the momentum expressed equal to that impressed, for the expressed momentum of the gold is less than it would be if it were not attached to the pith; and the expressed momentum of the pith is greater than it would be if it were separate from the gold. Some of the momentum which is impressed on the gold is not expressed in its motion; and that expressed in the pith is in excess of that which is impressed on it. The gold therefore loses momentum as exhibited in that expressed, and the pith gains.

Again, let us take an instance which is similar to that for the solution of which the principle was first devised. Let us suppose a circular horizontal plate to rotate about a vertical axis passing through its centre; and let us suppose it to rotate in the exhausted receiver of an air pump, so that no diminution of velocity takes place by reason of the resistance of the air. At the ends of the vertical axis let pivots be placed in fixed centres, so that the plate continues to rotate about the fixed vertical axis; and let the friction of the pivots be the sole cause of the diminution of the velocity of the plate. Now to the plate let a certain angular velocity be imparted; then if the plate thus rotating were divided into two equal concentric parts, the momentum of the exterior part would be greater than that of the interior, in the ratio indeed of $2^{\frac{3}{2}}-1$ to 1; and therefore if the interior part alone rotated it would be brought to rest by the friction of the pivots much sooner than the exterior part would, if it were connected with the axis, the interior part having been removed. However if the whole forms one rigid body all is simultaneously brought to rest; the exterior in less time than it would be, if it were separated from the interior; and the interior in longer time than it would be if it were alone. In the withdrawal therefore of the momentum of the plate by the friction of the pivots, the diminution of the exterior part is greater, and that of the interior part is less than it would be if each were

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separate from the other. Thus the exterior loses and the interior gains momentum. In neither one part nor the other is the momentum expressed equal to that impressed. Similarly if the plate is divided into concentric rings of infinitesimal breadth, generally the expressed momentum-increment of any ring (I use the term increment algebraically) is not equal to the impressed. It is so doubtless in a certain ring, but all rings external to that lose momentum; and all rings internal to it gain momentum; and, as we shall presently shew, the aggregate of the momentum lost throughout the plate is equal to the aggregate of that which is gained.

45.] What has here been said of rigid bodies, is also generally true of material systems. In the motion of each particle a difference will exist between the momentum impressed by a given external force and that expressed in the motion of the particle; and this difference too exists not only in the intensity of these momenta but also as to their lines of action; the particle m (say) will not move along the line of action of the force which impresses momentum, as it would do if it were free, but it will generally move along some other line; thus the momentum due to the acting force is not expressed in the particle's motion either as to intensity or as to line of action. And the connection of the particle with other particles of the system is the cause of this difference.

On what however is this difference spent? the momentum is impressed; matter is inert and cannot absorb it; it is not expressed in the motion of the particle on which the force acts; it must therefore be expressed elsewhere; and must in the first place produce a strain or an action between that particle and one or more of the other particles of the system. And what is the result of this? It must be that these other particles will gain exactly as much momentum as the original particle m has lost. A similar result will also be true for every other particle of the system; so that the sum of the momenta expressed in all the particles will be equal to the sum of those impressed. Thus, if the momenta are calculated throughout the system, the sum of those which are lost is exactly equal to those which are gained. Hence the differences between the impressed and the expressed momenta, taken throughout the system, are in equilibrium, and satisfy the equations (94) and (95), Art. 57. Vol. III. This theorem of the equality of the impressed and expressed

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momenta, taken through the whole material system in motion, was devised first by D'Alembert, and is now called D'Alembert's Principle. It is enunciated in the following form;

When a material system is in motion, and is acted on by forces which impress momenta, the momenta lost by all the particles of the system are in equilibrium.

In this enunciation the term "momentum lost" is equivalent to the excess of the momentum impressed over that expressed in any particle of the system. The term also is employed algebraically, and includes cases in which the expressed momentum is in excess of that impressed. It will be perceived that in the explanation above, we have fixed our thoughts on a particle in which the impressed momentum is greater than that expressed.

If the system is acted on by finite accelerating forces, so that infinitesimal momentum-increments are impressed in infinitesimal time-elements, the term momentum in the preceding enunciation must be replaced by "momentum-increment."

The mathematical expression of this principle is as follows;

Let $m_1, m_2, m_3 \ldots m_n$ be the particles of which a system is composed; of which let m be the type, and let (x, y, z) be its place at the time t; let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the axial components of the momentum-increment impressed by the external forces on m; and let $\mathrm{I}\cos\alpha$, $\mathrm{I}\cos\beta$, $\mathrm{I}\cos\gamma$ be the axial components of the momentum-increment arising from the internal forces; then the equations of motion of translation of m are

$$\left.\begin{aligned} \mathrm{X} - m\frac{d^2x}{dt^2} &= \mathrm{I}\cos\alpha, \\ \mathrm{Y} - m\frac{d^2y}{dt^2} &= \mathrm{I}\cos\beta, \\ \mathrm{Z} - m\frac{d^2z}{dt^2} &= \mathrm{I}\cos\gamma; \end{aligned}\right\}$$

let the equations of which these are the types be written for every particle of the system; and let them be added; then we have

$$\left.\begin{aligned} \Sigma\left(\mathrm{X} - m\frac{d^2x}{dt^2}\right) &= \Sigma.\mathrm{I}\cos\alpha, \\ \Sigma\left(\mathrm{Y} - m\frac{d^2y}{dt^2}\right) &= \Sigma.\mathrm{I}\cos\beta, \\ \Sigma\left(\mathrm{Z} - m\frac{d^2z}{dt^2}\right) &= \Sigma.\mathrm{I}\cos\gamma; \end{aligned}\right\}$$

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if however the system is such that all the momentum-increments arising from the internal forces neutralize each other, the right hand members of the preceding equations vanish; and we have

$$\left.\begin{aligned} \Sigma.\left(\text{X} - m\frac{d^2x}{dt^2}\right) &= 0, \\ \Sigma.\left(\text{Y} - m\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma.\left(\text{Z} - m\frac{d^2z}{dt^2}\right) &= 0. \end{aligned}\right\}$$

This is one form of the equations of motion which is involved in D'Alembert's Principle.

46.] Before we proceed to the complete and purely mathematical expression of the principle, and to the investigation of the general equations which arise out of it, I will explain two or three simple problems, so that the mode of application may be more exactly apprehended. Certain circumstances must be omitted, because we have not yet deduced from the principle theorems which they require. Hereafter the problems will be treated completely.

Ex. 1. Let m and m', Fig. 17, be two heavy particles attached to the ends of a perfectly flexible and inextensible string, which we will suppose to be without weight. The string with the weights at its ends is suspended over a small pulley which we will assume to be without inertia and to be perfectly smooth. It is required to determine the motion of the particles and the tension of the string.

Let A and A′ be the places of m and m' at the beginning of the time, P and P′ at the time t; OA $= a$, O′A′ $= a'$; OP $= x$, O′P′ $= x'$;

$$\therefore \quad x + x' = a + a'. \tag{1}$$

Let us consider the circumstances at the beginning of the motion. Let us suppose impulsive forces to act on m and m' downwards, and to impress on them velocities u and u'; that is, m and m' would move with velocities u and u' by the action of the impulsive forces, if they were free. Let v and v' be the expressed velocities of each in its constrained state; and let τ and τ' be the tensions of the strings OA and O′A′, when $t = 0$. Then, as the impressed and expressed momenta of m are respectively mu and mv,

$$\tau = mu - mv; \tag{2}$$

and similarly

$$\tau' = m'u' - m'v'. \tag{3}$$

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But since from (1) $\frac{dx}{dt}+\frac{dx'}{dt}=0$; $\therefore\ v+v'=0$, and $v'=-v$.

As the pulley has no inertia, and neither friction nor roughness, $\tau=\tau'$;

$$\therefore\ mu-mv=m'u'+m'v;$$

$$v=\frac{mu-m'u'}{m+m'}; \tag{4}$$

which gives the velocity of descent of m, and of ascent of m', when $t=0$. Also

$$\tau=\tau'=\frac{mm'(u+u')}{m+m'}. \tag{5}$$

Let us next consider the circumstances at the time t; that is, when the bodies are no longer under the action of impulsive forces, but of the continuously accelerating force of gravity. Then the impressed and the expressed momentum-increments of m are respectively mg and $m\frac{d^2x}{dt^2}$; and of m', $m'g$ and $m'\frac{d^2x'}{dt^2}$; also let the tension of the string OP $=$ T, and of the string O'P' $=$ T';

$$\therefore\ \mathrm{T}=mg-m\frac{d^2x}{dt^2}, \tag{6}$$

$$\mathrm{T}'=m'g-m'\frac{d^2x'}{dt^2}; \tag{7}$$

but from (1) $$\frac{d^2x}{dt^2}+\frac{d^2x'}{dt^2}=0.$$

And as the pulley has no inertia and is smooth and free from friction, by D'Alembert's principle, T $=$ T',

$$\therefore\ mg-m\frac{d^2x}{dt^2}=m'g+m'\frac{d^2x}{dt^2}.$$

$$\therefore\ \frac{d^2x}{dt^2}=\frac{m-m'}{m+m'}g=-\frac{d^2x'}{dt^2}; \tag{8}$$

$$\left.\begin{aligned}\therefore\ \frac{dx}{dt}-v&=\frac{m-m'}{m+m'}gt,\\ \frac{dx'}{dt}+v&=-\frac{m-m'}{m+m'}gt;\end{aligned}\right\} \tag{9}$$

$$\left.\begin{aligned}x&=a+vt+\frac{m-m'}{m+m'}\frac{1}{2}gt^2,\\ x'&=a'-vt-\frac{m-m'}{m+m'}\frac{1}{2}gt^2;\end{aligned}\right\} \tag{10}$$

$$\mathrm{T}=\mathrm{T}'=\frac{2mm'g}{m+m'}; \tag{11}$$

so that all the circumstances of motion are determined.

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Ex. 2. Let m and m', Fig. 18, be two heavy particles attached by means of flexible and inextensible strings without weight to a wheel and axle respectively, which are supposed to be without inertia; the initial circumstances being given, it is required to determine the subsequent circumstances of motion.

In Fig. 18, C is the common centre of the wheel and axle; $\text{CO} = c$, $\text{CO}' = c'$ are the radii; A and A′ are the places of m and m' when $t = 0$, P and P′ when $t = t$; $\text{OA} = a$, $\text{O}'\text{A}' = a'$, $\text{OP} = x$, $\text{O}'\text{P}' = x'$.

As x increases by dx, and as x' decreases by dx', let the wheel and axle rotate through an angle $d\theta$; so that

$$dx = c\,d\theta, \qquad dx' = -c'\,d\theta,$$

$$\frac{dx}{dt} = c\frac{d\theta}{dt}, \qquad \frac{dx'}{dt} = -c'\frac{d\theta}{dt}; \tag{12}$$

$$\frac{d^2x}{dt^2} = c\frac{d^2\theta}{dt^2}; \qquad \frac{d^2x'}{dt^2} = -c'\frac{d^2\theta}{dt^2}. \tag{13}$$

Let us consider the circumstances at the beginning of the motion; and let the symbols be the same as those of the preceding example; then

$$\left.\begin{aligned} \tau &= mu - mv, \\ \tau' &= m'u' - m'v'; \end{aligned}\right\} \tag{14}$$

By D'Alembert's principle these tensions are in equilibrium; therefore

$$c\tau = c'\tau' \tag{15}$$

and from (12)

$$\frac{v}{c} = -\frac{v'}{c'};$$

$$\therefore \quad \frac{v}{c} = -\frac{v'}{c'} = \frac{cmu - c'm'u'}{mc^2 + m'c'^2}; \tag{16}$$

$$\frac{\tau}{c'} = \frac{\tau}{c} = \frac{mm'(c'u + cu')}{mc^2 + m'c'^2}; \tag{17}$$

whereby the initial velocities of the particles and the initial tensions of the strings are known.

Let us now consider the circumstances at the time t; and let the symbols be the same as those of the preceding example; then

$$\begin{aligned} \text{T} &= mg - m\frac{d^2x}{dt^2} \\ &= mg - mc\frac{d^2\theta}{dt^2}; \end{aligned} \tag{18}$$

$$\begin{aligned} \text{T}' &= m'g - m'\frac{d^2x'}{dt^2} \\ &= m'g + m'c'\frac{d^2\theta}{dt^2}; \end{aligned} \tag{19}$$

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and as these tensions are in equilibrium by D'Alembert's principle, we have

$$c\mathrm{T} = c'\mathrm{T}'; \tag{20}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{mc - m'c'}{mc^2 + m'c'^2} g; \tag{21}$$

$$\frac{1}{c}\frac{d^2x}{dt^2} = -\frac{1}{c'}\frac{d^2x'}{dt^2} = \frac{mc - m'c'}{mc^2 + m'c'^2} g;$$

$$\therefore \quad \frac{1}{c}\left(\frac{dx}{dt} - v\right) = -\frac{1}{c'}\left(\frac{dx'}{dt} - v'\right) = \frac{mc - m'c'}{mc^2 + m'c'^2} gt;$$

$$\therefore \quad x = a + vt + c\frac{mc - m'c'}{mc^2 + m'c'^2}\frac{gt^2}{2}; \tag{22}$$

$$x' = a' + v't - c'\frac{mc - m'c'}{mc^2 + m'c'^2}\frac{gt^2}{2}; \tag{23}$$

$$c\mathrm{T} = c'\mathrm{T}' = \frac{cc'mm'g(c + c')}{mc^2 + m'c'^2}; \tag{24}$$

whereby all the circumstances of motion in both the initial and the general states are known.

Ex. 3. A heavy chain, flexible and inextensible, homogeneous and smooth, hangs over a small pulley at the common vertex of two smooth inclined planes; it is required to determine the motion of the chain.

Let the two inclined planes, the chain and the pulley, be represented in Fig. 19, each of the inclined planes being supposed to be longer than the length of the chain; so that the chain, as we consider its motion, is on one or the other of the planes. Let O be the common vertex of the two planes; A and A′ the ends of the chain when $t = 0$, P and P′ the ends when $t = t$; OA $= a$, OA′ $= a'$, OP $= x$, OP′ $= x'$; and let α and α' be the angles of inclination of the planes to the horizon; $l =$ the length of the chain; therefore

$$x + x' = a + a' = l. \tag{25}$$

We will suppose the chain to be initially at rest. Let ω be the area of a transverse section, $\rho =$ the density; $\mathrm{T} =$ the tension at the time t;

$$\therefore \quad \mathrm{T} = \omega\rho x\left\{g\sin\alpha - \frac{d^2x}{dt^2}\right\} = \omega\rho x'\left\{g\sin\alpha' - \frac{d^2x'}{dt^2}\right\}; \tag{26}$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{\sin\alpha + \sin\alpha'}{l} gx - g\sin\alpha'; \tag{27}$$

$$\frac{d^2x'}{dt^2} = \frac{\sin\alpha + \sin\alpha'}{l} gx' - g\sin\alpha; \tag{28}$$

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$$\therefore \quad \frac{dx^2}{dt^2} = \frac{g(\sin a + \sin a')}{l}(x^2 - a^2) - 2g \sin a'(x - a); \quad (29)$$

$$\frac{dx'^2}{dt^2} = \frac{g(\sin a + \sin a')}{l}(x'^2 - a'^2) - 2g \sin a(x' - a'); \quad (30)$$

whence the relations between x and t, and between x' and t, may be found; but the form of the equations is too complicated to be of any use. Also

$$\text{T} = \frac{\omega \rho g x x'}{l}(\sin a + \sin a'). \quad (31)$$

If the chain, instead of resting on two inclined planes, hangs over a small pulley without inertia, then, all the other circumstances being the same, $a = a' = 90°$; and the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{2g}{l}x - g, \\ \frac{d^2x'}{dt^2} &= \frac{2g}{l}x' - g. \end{aligned}\right\} \quad (32)$$

47.] The following explanation of D'Alembert's principle is much the same as that which he first gave in the Traité de Dynamique; and as it will thus be stated in a mathematical form, the general equations of motion will be most conveniently deduced from it.

Let P, Fig. 20, be the place of a particle m of a material system. During the infinitesimal time dt let a force act on m which would impress on it, if it were free, a velocity whose line-representative is PA; let the impressed velocity be V; so that mV is the impressed momentum along, and proportional to PA; let v be the velocity of m; that is, let mv be the expressed momentum; and let its line of action be PB; let PC be the line which would complete the parallelogram of which PA is the diagonal, and PB one of the containing sides: then resolving V into the velocity v along PB, and v' along PC, v', which is represented by PC, is the velocity lost; and mv', which is proportional to and acts along PC, is the momentum lost. D'Alembert's principle asserts that all the lost momenta taken throughout the system are in equilibrium. His words are;

Décomposez les mouvements a, b, c ..... imprimeés à chaque corps, chacun en deux autres a, α; b, β; c, γ; ..... qui soient tels que si l'on n'eût imprimeé aux corps que les mouvements $a, b, c, \ldots$ ils eussent pu conserver ces mouvements sans se nuire

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réciproquement; et que si on ne leur eût imprimé que les mouvements $\alpha, \beta, \gamma, \ldots\ldots$, le systême fût demeuré en repos.

Again, produce BP to B′, so that PB′ = PB; then the momentum represented by PC is evidently the resultant of those represented by PA and PB′; hence we have D'Alembert's principle in the following form;

If the expressed momenta of the several particles of a material system are estimated in a direction the contrary of that in which they act, they, together with the impressed momenta when taken through the whole system, will satisfy the conditions of statical equilibrium.

48.] Such is D'Alembert's principle, as to its origin and as to its form of expression; it reduces all the theorems of motion of material systems to those of statical equilibrium; and so it is commonly said that D'Alembert reduced dynamics to statics. The principle does not indeed directly furnish the equations necessary for the solution of the different problems of dynamics; but it teaches the mode by which they are to be deduced from the equations of equilibrium; and thus, if we apply to the "momenta lost," the conditions of statical equilibrium, the dynamical equations will be formed. It is evident too that we may introduce them as pressures into the equation of vertical velocities, and this will hereafter be done. The equations of equilibrium of a system of pressures acting on the several points of a rigid body are investigated in Vol. III. The number of them is six; of which three are of translation and three are of rotation: the momenta lost must satisfy these six conditions.

Firstly, let us suppose the acting forces on the system of particles to be impulsive and instantaneous, so that finite momenta are impressed instantaneously, and the expressed momenta are also instantaneously developed.

Let m be the symbol of a type-particle; (x, y, z) its place at the time t; $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ the axial components of its expressed velocity due to the acting instantaneous forces; $\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z$ the axial components of the velocity impressed on m; so that the differences between the axial components of the impressed and expressed momenta are

$$m\left(\mathrm{v}_x - \frac{dx}{dt}\right), \qquad m\left(\mathrm{v}_y - \frac{dy}{dt}\right), \qquad m\left(\mathrm{v}_z - \frac{dz}{dt}\right). \quad (33)$$

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By D'Alembert's principle these and similar quantities for all the other particles of the system are in equilibrium; therefore the six following equations must be satisfied by them;

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{v}_x - \frac{dx}{dt}\right) &= 0, \\ \Sigma.m\left(\mathrm{v}_y - \frac{dy}{dt}\right) &= 0, \\ \Sigma.m\left(\mathrm{v}_z - \frac{dz}{dt}\right) &= 0; \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{v}_z - \frac{dz}{dt}\right) - z\left(\mathrm{v}_y - \frac{dy}{dt}\right)\right\} &= 0, \\ \Sigma.m\left\{z\left(\mathrm{v}_x - \frac{dx}{dt}\right) - x\left(\mathrm{v}_z - \frac{dz}{dt}\right)\right\} &= 0, \\ \Sigma.m\left\{x\left(\mathrm{v}_y - \frac{dy}{dt}\right) - y\left(\mathrm{v}_x - \frac{dx}{dt}\right)\right\} &= 0; \end{aligned}\right\} \tag{35}$$

whereof the first three are the equations of translation, and the last three are the equations of the moments of the couples which arise from the excess of the expressed over the impressed momenta about the three coordinate axes. The sign of summation extends to, and includes, all the particles of the system; and the expressed velocities are those due to the action of the impressed forces.

Hence, if u, v, w are the axial components of the velocity of m before the instantaneous forces act, and if $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are the axial components after the velocities $\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z$ have been impressed, $\frac{dx}{dt} - u, \frac{dy}{dt} - v, \frac{dz}{dt} - w$ are the expressed axial velocities due to the instantaneous forces; and in equations (34) and (35) $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are to be replaced by these quantities.

Secondly, let us suppose the system of particles to be under the action of finite accelerating forces, so that in infinitesimal time-elements, infinitesimal momentum-increments are impressed upon and expressed in the type-particle m.

Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the axial components of the impressed velocity-increment on m, which is supposed to be at (x, y, z) at the time t; and $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are the axial components of the expressed velocity-increment; so that the differences between the axial

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components of the impressed and the expressed momentum-increments of m are

$$m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right), \qquad m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right), \qquad m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right). \quad (36)$$

By D'Alembert's principle these and similar quantities for all the other particles of the system are in equilibrium; therefore the six following equations must be satisfied by them;

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right) &= 0, \\ \Sigma.m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma.m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right) &= 0; \end{aligned}\right\} \quad (37)$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)-z\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{z\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)-x\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{x\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)-y\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\right\} &= 0; \end{aligned}\right\} \quad (38)$$

the sign of summation includes all the particles of the system.

In these equations the power of the sign of summation should be carefully observed; it includes all the particles of the moving system, whether that system be continuous or discontinuous. Thus, if the several particles are $m_1, m_2, \ldots m_n$; and their places at the time t are (x_1, y_1, z_1), $(x_2, y_2, z_2), \ldots (x_n, y_n, z_n)$, and the impressed velocity-increments are $(\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1)$, $(\mathrm{X}_2, \mathrm{Y}_2, \mathrm{Z}_2), \ldots (\mathrm{X}_n, \mathrm{Y}_n, \mathrm{Z}_n)$; then the first of (37) is the abbreviated form of

$$m_1\left(\mathrm{X}_1-\frac{d^2x_1}{dt^2}\right)+m_2\left(\mathrm{X}_2-\frac{d^2x_2}{dt^2}\right)+\ldots+m_n\left(\mathrm{X}_n-\frac{d^2x_n}{dt^2}\right)=0.$$

Similarly the other five equations are abbreviated forms of analogous expressions.

49.] The equations of motion of a material system may also be expressed in a shorter form. For since D'Alembert's principle enables us to deduce them from the equilibrium which subsists among the "lost momenta;" that equilibrium will be obtained not only from the six equations which correspond to (37) and (38) of the preceding Article, but also from the equation of virtual velocities. The truth of this last equation has been demonstrated in Articles 104 and 395 of Vol. III, but

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since we shall now apply it somewhat extensively, and very generally, as it will include all dynamics, it is necessary to say a few words on its form and the conditions of its exactness. We imagine a material system to be at rest under the action of many forces, which may be external as well as internal to the system: of these forces we take P to be the type, and we suppose it to act on m, which we take to be the type-particle: so that $\Sigma.P$ will be the sum of all the forces which act on all the particles; many of which may act on one and the same particle; and others of which will enter in pairs of equal and opposite forces, when there are mutual tensions or reactions or constraints among the particles of the system. We imagine the system to receive an arbitrary infinitesimal displacement, consistently with its geometrical relations, whereby the points of application of the forces are changed, but neither the intensities nor the directions of the lines of action are altered. Let the displacements of the points of application of the forces be estimated along the lines of action of the forces; and let δp be the infinitesimal displacement of (x, y, z), the point of application of P, thus estimated; then the equation of virtual velocities is

$$\Sigma.P\,\delta p = 0;$$

and this expresses the condition that the forces are in equilibrium.

Let us put into an equation of this form the several quantities which are active in the motion of a material system. As $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are the axial components of the expressed velocity-increments of m, which is at (x, y, z) at the time t, it is evident that the impressed momentum-increments along these axes, which would have their full effect in producing pressure if the system were at rest, must be diminished by $m\frac{d^2x}{dt^2}, m\frac{d^2y}{dt^2}, m\frac{d^2z}{dt^2}$, now that the system moves; and the actual effects will be the excesses of the former over the latter: in the equation therefore of virtual velocities these latter quantities must be affected with negative signs. We shall use the symbol δ to express the variations of the points of application of the forces which are due to the arbitrary geometrical displacement of the system; and we shall still indicate by the symbol d the time-variation of the coordinates and velocities. Let mP be the type momentum-increment acting on the type particle m; and let us suppose δp

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to be the infinitesimal displacement of the point of application of P estimated along its line of action, P tending to remove m from the origin, and δp being positive when the point of application of P is moved in the direction along which P acts; then, if the line of action of a force is along a coordinate axis, say that of x, the variation of the point of application is δx. Now, estimating forces according to these conditions, the equation of virtual velocities is

$$\Sigma . m \, \mathrm{P} \, \delta p - \Sigma . m \left\{ \frac{d^2 x}{dt^2} \delta x + \frac{d^2 y}{dt^2} \delta y + \frac{d^2 z}{dt^2} \delta z \right\} = 0; \quad (39)$$

which is the most general equation of motion of a material system.

$\Sigma . m \, \mathrm{P} \, \delta p$ includes all the forces which act on the several particles of the system, both internal and external; if however two particles are acted on by a force along the line which joins them, and if the distance between these particles is unchanged in the geometrical displacement, this force will disappear in the aggregate; because the geometrical displacements of the two particles estimated along the line of force will be equal and opposite, and therefore the two effects, as they are measured in the preceding equation, will neutralize each other, and will disappear.

50.] Let all the impressed momentum-increments, as they are applied to each particle, be resolved into components parallel to the three coordinate axes; and let X, Y, Z be the axial components of P as it acts on m at (x, y, z); then the infinitesimal displacements of m along the three axes will be δx, δy, δz; which are the same as the displacements of the point of application of the expressed momentum-increments: so that (39) becomes in this case

$$\Sigma . m \left\{ \left(\mathrm{X} - \frac{d^2 x}{dt^2} \right) \delta x + \left(\mathrm{Y} - \frac{d^2 y}{dt^2} \right) \delta y + \left(\mathrm{Z} - \frac{d^2 z}{dt^2} \right) \delta z \right\} = 0; \quad (40)$$

which is another form of the equation of virtual velocities.

Now no restriction has been made as to the kind of displacement of (x, y, z), of which the axial projections are represented by δx, δy, δz; it is only to be consistent with the geometrical relations of the system: let us suppose it therefore to be most general, and to be compounded of motions of translation and of rotation of the whole system. Let the system receive a displacement of translation, so that every particle moves over an equal and parallel space in the direction of the coordinate axes, which

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we will represent severally by δx_0, δy_0, δz_0; and also let the system receive three successive displacements of rotation through the angles $\delta\theta$, $\delta\phi$, $\delta\psi$ about the three coordinate axes: then the total variations of the coordinates of the point (x, y, z) are

$$\left.\begin{aligned} \delta x &= \delta x_0 + z\,\delta\phi - y\,\delta\psi, \\ \delta y &= \delta y_0 + x\,\delta\psi - z\,\delta\theta, \\ \delta z &= \delta z_0 + y\,\delta\theta - x\,\delta\phi; \end{aligned}\right\} \tag{41}$$

and substituting these in (40) we have

$$\begin{aligned} &\delta x_0 \Sigma . m\left(\text{X} - \frac{d^2x}{dt^2}\right) + \delta y_0 \Sigma . m\left(\text{Y} - \frac{d^2y}{dt^2}\right) + \delta z_0 \Sigma . m\left(\text{Z} - \frac{d^2z}{dt^2}\right) \\ &\quad + \delta\theta \Sigma . m\left\{y\left(\text{Z} - \frac{d^2z}{dt^2}\right) - z\left(\text{Y} - \frac{d^2y}{dt^2}\right)\right\} \\ &\quad + \delta\phi \Sigma . m\left\{z\left(\text{X} - \frac{d^2x}{dt^2}\right) - x\left(\text{Z} - \frac{d^2z}{dt^2}\right)\right\} \\ &\quad + \delta\psi \Sigma . m\left\{x\left(\text{Y} - \frac{d^2y}{dt^2}\right) - y\left(\text{X} - \frac{d^2x}{dt^2}\right)\right\} = 0; \end{aligned} \tag{42}$$

as the several variations on which the displacement depends are independent of each other, their coefficients must separately vanish; and hence we have the six equations of motion, viz. (37) and (38) of Art. 48.

By a similar process we may deduce the six equations (34) and (35) for instantaneous forces.

51.] When the system of particles on which the forces act is rigid or invariable in form, that invariability of form will be secured by means of certain equations which the coordinates of the several particles must satisfy. Thus, if the number of particles is n, and if all the distances of these particles from each other are invariable during the motion, $(3n-6)$ equations must be satisfied; for if the distances between three are given, and the distances of every one of the remaining $(n-3)$ from each of these three are given, whereby we have $3n-6$ given distances, the system of particles is invariable in form. Now the position of every particle of the system at any time is determined, when the coordinates of every particle are expressed in terms of t; and as each particle has three coordinates, $3n$ quantities must be expressed in terms of t: these however are subjected to $3n-6$ conditions of relative position: six other conditions therefore are necessary, and are sufficient for the complete solution of the problem; and these are given by the equations (37) and (38) of Art. 48.

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And now let us take a more general case; and let us suppose certain particles of a material system to be constrained to move on certain curves, so that certain relations will be given which the coordinates of the particles must satisfy. Suppose the number of these relations to be k; and let them be $F_1 = 0$, $F_2 = 0$, $\ldots F_k = 0$; then, taking the most general case, and supposing each of these functions to contain all the coordinates of all the points, their total variations are

$$\left.\begin{array}{l}\left(\frac{dF_1}{dx_1}\right)\delta x_1 + \left(\frac{dF_1}{dy_1}\right)\delta y_1 + \ldots + \left(\frac{dF_1}{dy_n}\right)\delta y_n + \left(\frac{dF_1}{dz_n}\right)\delta z_n = 0,\\ \left(\frac{dF_2}{dx_1}\right)\delta x_1 + \left(\frac{dF_2}{dy_1}\right)\delta y_1 + \ldots + \left(\frac{dF_2}{dy_n}\right)\delta y_n + \left(\frac{dF_2}{dz_n}\right)\delta z_n = 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \left(\frac{dF_k}{dx_1}\right)\delta x_1 + \left(\frac{dF_k}{dy_1}\right)\delta y_1 + \ldots + \left(\frac{dF_k}{dy_n}\right)\delta y_n + \left(\frac{dF_k}{dz_n}\right)\delta z_n = 0;\end{array}\right\} \quad (43)$$

and from (40) we have an equation of the form

$$A_1\,\delta x_1 + B_1\,\delta y_1 + C_1\,\delta z_1 + \ldots\ldots + A_n\,\delta x_n + B_n\,\delta y_n + C_n\,\delta z_n = 0. \quad (44)$$

If then we multiply (43) severally by indeterminate multipliers $\lambda_1, \lambda_2, \ldots \lambda_k$, and add all and (44); and if we equate to zero the coefficients of $\delta x_1, \delta y_1, \ldots \delta z_n$, we have

$$\left.\begin{array}{l}\lambda_1\left(\frac{dF_1}{dx_1}\right) + \lambda_2\left(\frac{dF_2}{dx_1}\right) + \ldots + \lambda_k\left(\frac{dF_k}{dx_1}\right) + A_1 = 0,\\ \lambda_1\left(\frac{dF_1}{dy_1}\right) + \lambda_2\left(\frac{dF_2}{dy_1}\right) + \ldots + \lambda_k\left(\frac{dF_k}{dy_1}\right) + B_1 = 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \lambda_1\left(\frac{dF_1}{dz_n}\right) + \lambda_2\left(\frac{dF_2}{dz_n}\right) + \ldots + \lambda_k\left(\frac{dF_k}{dz_n}\right) + C_n = 0;\end{array}\right\} \quad (45)$$

by means of which equations, and of the given equations of condition, $\lambda_1, \lambda_2, \ldots \lambda_k$ may be eliminated, and the coordinates of the particles may be determined.

52.] We will now apply the process of the preceding Article to the motion of a flexible and inextensible string, which has its two ends fixed, under the action of given forces.

Let ds be the length-element of the string at the point (x, y, z); let ω be the area of the transverse section, ρ = the density; so that the mass-element of the string is $\rho\,\omega\,ds$. Let the points (x_1, y_1, z_1), (x_2, y_2, z_2) be the two ends of the string, and let $\rho_1, \omega_1, \rho_2, \omega_2$ be the values of ρ, ω at them respectively. Let X, Y, Z be the axial components of the impressed velocity-increment at the point (x, y, z); and let X_1, Y_1, Z_1, X_2, Y_2, Z_2 be

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the values of these quantities at the two ends of the string which we suppose to be dynamically fixed, but to admit of geometrical variation. Then (40) becomes

$$\int_1^2 \rho\,\omega\,ds\left\{\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\delta x+\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\delta y+\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\delta z\right\}$$
$$+\mathrm{X}_1\delta x_1+\mathrm{Y}_1\delta y_1+\mathrm{Z}_1\delta z_1+\mathrm{X}_2\delta x_2+\mathrm{Y}_2\delta y_2+\mathrm{Z}_2\delta z_2=0; \quad (46)$$

and since the string is of constant length

$$\int_1^2 ds = \text{a constant.} \quad (47)$$

$$\therefore \int_1^2\left\{\frac{dx}{ds}\delta.dx+\frac{dy}{ds}\delta.dy+\frac{dz}{ds}\delta.dz\right\}=\delta.\int_1^2 ds$$
$$=0; \quad (48)$$

and multiplying the quantity under the sign of integration by λ, and adding it to (46), we have

$$\int_1^2\left\{\rho\,\omega\,ds\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\delta x+\lambda\frac{dx}{ds}d.\delta x\right\}+\ldots$$
$$\ldots+\mathrm{X}_1\delta x_1+\mathrm{Y}_1\delta y_1+\ldots\ldots+\mathrm{Z}_2\delta z_2=0; \quad (49)$$

and integrating by parts the second terms of the several members of the upper line, as we have explained in the Calculus of Variations, Vol. II, we have

$$\int_0^2\left\{\rho\,\omega\,ds\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)-d.\lambda\frac{dx}{ds}\right\}\delta x+\ldots\ldots$$
$$+\left[\lambda\frac{dx}{ds}\delta x+\lambda\frac{dy}{ds}\delta y+\lambda\frac{dz}{ds}\delta z\right]_1^2$$
$$+\mathrm{X}_1\delta x_1+\mathrm{Y}_1\delta y_1+\ldots\ldots+\mathrm{Z}_2\delta z_2=0. \quad (50)$$

As no other relation is given between δx, δy, and δz, we have

$$\left.\begin{aligned}\rho\,\omega\,ds\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)-d.\lambda\frac{dx}{ds}&=0,\\ \rho\,\omega\,ds\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)-d.\lambda\frac{dy}{ds}&=0,\\ \rho\,\omega\,ds\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)-d.\lambda\frac{dz}{ds}&=0;\end{aligned}\right\} \quad (51)$$

which are the equations of motion of the cord. The latter terms in (50) give values at the ends of the string; if the ends are independent of each other,

$$\left.\begin{aligned}\left(\lambda_1\frac{dx_1}{ds_1}+\mathrm{X}_1\right)\delta x_1+\left(\lambda_1\frac{dy_1}{ds_1}+\mathrm{Y}_1\right)\delta y_1+\left(\lambda_1\frac{dz_1}{ds_1}+\mathrm{Z}_1\right)\delta z_1&=0,\\ \left(-\lambda_2\frac{dx_2}{ds_2}+\mathrm{X}_2\right)\delta x_2+\left(-\lambda_2\frac{dy_2}{ds_2}+\mathrm{Y}_2\right)\delta y_2+\left(-\lambda_2\frac{dz_2}{ds_2}+\mathrm{Z}_2\right)\delta z_2&=0;\end{aligned}\right\} \quad (52)$$

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and if the ends are fixed

$$\left.\begin{array}{lll} \lambda_1 \dfrac{dx_1}{ds_1} + \mathrm{X}_1 = 0, & \lambda_1 \dfrac{dy_1}{ds_1} + \mathrm{Y}_1 = 0, & \lambda_1 \dfrac{dz_1}{ds_1} + \mathrm{Z}_1 = 0; \\ -\lambda_2 \dfrac{dx_2}{ds_2} + \mathrm{X}_2 = 0, & -\lambda_2 \dfrac{dy_2}{ds_2} + \mathrm{Y}_2 = 0, & -\lambda_2 \dfrac{dz_2}{ds_2} + \mathrm{Z}_2 = 0. \end{array}\right\}$$

The form of the last terms of (51) shews that λ is the tension of the string at the point (x, y, z), and acts along the length-element ds: indeed the equations (51) are only the particular form of (37) when the forces arising from the tension are introduced into them. If we eliminate λ from (51), we shall have two equations in terms of x, y, z, t, which will give the position of the string at any time. I may observe too that λ is evidently the tension, because the mode in which it is introduced shews that it is an internal force acting along ds; and that the variations of its points of action are the same as the variations of the ends of the length-element.

Thus much must at present suffice for this problem; and we will proceed to the demonstration of various general theorems which arise out of the equations of motion.

SECTION 2.—*Independence of the motions of translation of the centre of gravity, and of rotation about an axis passing through it.*

53.] Many general theorems of dynamics are deducible from D'Alembert's principle as expressed in the six equations, either (34) and (35) of instantaneous forces, or (37) and (38) of finite accelerating forces: the investigation of the most general of these will be reserved to a subsequent Chapter of our Treatise; but it is convenient at once to discuss certain theorems relative to the centre of gravity of a material system as well as some other simple properties of motion which follow directly from them. I must first observe on the name "centre of gravity;" it is given to the so called point of a heavy body or system for a reason much too narrow for the properties of it which we shall presently develope; and a term of wider application is required. We shall also apply the term to a system of particles which perhaps may not gravitate at all. In geometry, as it is well known, a similar point is called the centre of mean distances; in dynamics the term "centre of masses" has been proposed; but the inconvenience of a new name is so great that I propose to retain the old one, "cen-

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tre of gravity." The student however must remember that we do not employ the term in its restricted sense, that is, in the statical meaning as applied to heavy bodies under the action of the earth's attraction; but we mean that point $(\bar{x}, \bar{y}, \bar{z})$ of a system of particles which, relatively to all the particles, is defined by the following equations:

$$\left.\begin{aligned} \bar{x}\,\Sigma.m &= \Sigma.mx, \\ \bar{y}\,\Sigma.m &= \Sigma.my, \\ \bar{z}\,\Sigma.m &= \Sigma.mz; \end{aligned}\right\}$$

so that
$$\left.\begin{aligned} \frac{d\bar{x}}{dt}\Sigma.m &= \Sigma.m\frac{dx}{dt}, \\ \frac{d\bar{y}}{dt}\Sigma.m &= \Sigma.m\frac{dy}{dt}, \\ \frac{d\bar{z}}{dt}\Sigma.m &= \Sigma.m\frac{dz}{dt}; \end{aligned}\right\} \quad \left.\begin{aligned} \frac{d^2\bar{x}}{dt^2}\Sigma.m &= \Sigma.m\frac{d^2x}{dt^2}, \\ \frac{d^2\bar{y}}{dt^2}\Sigma.m &= \Sigma.m\frac{d^2y}{dt^2}, \\ \frac{d^2\bar{z}}{dt^2}\Sigma.m &= \Sigma.m\frac{d^2z}{dt^2}; \end{aligned}\right\}$$

so that if the centre of gravity is the origin

$$\Sigma.mx = \Sigma.my = \Sigma.mz = 0;$$

$$\Sigma.m\frac{dx}{dt} = \Sigma.m\frac{dy}{dt} = \Sigma.m\frac{dz}{dt} = 0; \tag{53}$$

$$\Sigma.m\frac{d^2x}{dt^2} = \Sigma.m\frac{d^2y}{dt^2} = \Sigma.m\frac{d^2z}{dt^2} = 0. \tag{54}$$

In the preceding Chapter it has been proved that the general motion of a body may be resolved into a motion of translation of any one point of it chosen arbitrarily, and into a motion of rotation about an axis passing through that arbitrarily chosen point. One simplification indeed of that theorem was pointed out on principles purely cinematical; viz. that if that point was taken on the central axis, the motion of the point would lie along the axis of rotation; and this is true of course, when all the expressed velocities of the body or system due to the acting forces are given, whereby the position of the central axis at the time t may be determined. But here another problem is offered for consideration. When a body or system is in motion, certain velocities are developed during the motion by reason of the inertia of the particles, and hereby certain velocities are impressed on the particles: these vary according to the point which is assumed to be the moving point of translation, and through which the instantaneous rotation-axis of the body passes. We shall shew that these velocities, due to the motion, neutralize each other, when the centre of gravity is taken to be the moving

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point of translation, so that the instantaneous rotation-axis passes through it; and that the motion of translation of the centre of gravity is independent of the motion of rotation of the system about the instantaneous axis passing through it.

54.] Let $(\bar{x}, \bar{y}, \bar{z})$ be the place of the centre of gravity at the time t; (x, y, z) the place of the type-particle m; and let us suppose a system of coordinate axes to originate at the centre of gravity parallel to the original system of reference; and let the place of m relatively to the new axes be (x', y', z'); so that

$$x = \bar{x} + x', \qquad y = \bar{y} + y', \qquad z = \bar{z} + z';$$

$$\therefore \quad \Sigma.mx' = \Sigma.my' = \Sigma.mz' = 0;$$

$$\Sigma.m\frac{dx'}{dt} = \Sigma.m\frac{dy'}{dt} = \Sigma.m\frac{dz'}{dt} = 0:$$

$$\Sigma.m\frac{d^2x'}{dt^2} = \Sigma.m\frac{d^2y'}{dt^2} = \Sigma.m\frac{d^2z'}{dt^2} = 0.$$

Let M = the mass of all the particles of the moving system; then

$$\Sigma.mx = \text{M}\bar{x}, \qquad \Sigma.my = \text{M}\bar{y}, \qquad \Sigma.mz = \text{M}\bar{z};$$

$$\Sigma.m\frac{dx}{dt} = \text{M}\frac{d\bar{x}}{dt}, \qquad \Sigma.m\frac{dy}{dt} = \text{M}\frac{d\bar{y}}{dt}, \qquad \Sigma.m\frac{dz}{dt} = \text{M}\frac{d\bar{z}}{dt};$$

$$\Sigma.m\frac{d^2x}{dt^2} = \text{M}\frac{d^2\bar{x}}{dt^2}, \quad \Sigma.m\frac{d^2y}{dt^2} = \text{M}\frac{d^2\bar{y}}{dt^2}, \quad \Sigma.m\frac{d^2z}{dt^2} = \text{M}\frac{d^2\bar{z}}{dt^2}.$$

Firstly, let us take equations (34) and (35) which refer to instantaneous forces; then

$$\frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt}, \qquad \frac{dy}{dt} = \frac{d\bar{y}}{dt} + \frac{dy'}{dt}, \qquad \frac{dz}{dt} = \frac{d\bar{z}}{dt} + \frac{dz'}{dt'}; \tag{55}$$

and (34) become

$$\left.\begin{aligned} \text{M}\frac{d\bar{x}}{dt} &= \Sigma.m\,\text{v}_x, \\ \text{M}\frac{d\bar{y}}{dt} &= \Sigma.m\,\text{v}_y, \\ \text{M}\frac{d\bar{z}}{dt} &= \Sigma.m\,\text{v}_z; \end{aligned}\right\} \tag{56}$$

which equations are of the same form as those of the motion of a material particle whose mass is M. Whence it appears that

The motion of translation of the centre of gravity of a system of particles under the action of instantaneous forces is the same as if the whole mass were collected into it, and all the impressed momenta were applied at it, each in a line parallel to its own line of action.

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If $\frac{d\bar{s}}{dt}$ is the velocity of the centre of gravity,

$$\text{M}\frac{d\bar{s}}{dt} = \{(\Sigma.m\text{v}_x)^2 + (\Sigma.m\text{v}_y)^2 + (\Sigma.m\text{v}_z)^2\}^{\frac{1}{2}};$$

and the direction-cosines of the path which the centre of gravity takes are given by (56).

Again, taking the first of (35), and making similar substitutions, we have

$$\Sigma.m\left\{(\bar{y}+y')\left(\text{v}_z-\frac{d\bar{z}}{dt}-\frac{dz'}{dt}\right)-(\bar{z}+z')\left(\text{v}_y-\frac{d\bar{y}}{dt}-\frac{dy'}{dt}\right)\right\}=0;$$

which may be expressed as follows;

$$\bar{y}\,\Sigma.m\left(\text{v}_z-\frac{d\bar{z}}{dt}\right)-\bar{y}\,\Sigma.m\frac{dz'}{dt}+\Sigma.my'\left(\text{v}_z-\frac{dz'}{dt}\right)-\frac{d\bar{z}}{dt}\Sigma.my'$$

$$-\bar{z}\,\Sigma.m\left(\text{v}_y-\frac{d\bar{y}}{dt}\right)+\bar{z}\,\Sigma.m\frac{dy'}{dt}-\Sigma.mz'\left(\text{v}_y-\frac{dy'}{dt}\right)+\frac{d\bar{y}}{dt}\Sigma.mz'=0;$$

and thus we have

$$\left.\begin{aligned}
&\Sigma.m\left\{y'\left(\text{v}_z-\frac{dz'}{dt}\right)-z'\left(\text{v}_y-\frac{dy'}{dt}\right)\right\}=0;\\
&\text{similarly from the second and third of (35) we have}\\
&\Sigma.m\left\{z'\left(\text{v}_x-\frac{dx'}{dt}\right)-x'\left(\text{v}_z-\frac{dz'}{dt}\right)\right\}=0,\\
&\Sigma.m\left\{x'\left(\text{v}_y-\frac{dy'}{dt}\right)-y'\left(\text{v}_x-\frac{dx'}{dt}\right)\right\}=0.
\end{aligned}\right\}\qquad(57)$$

These are evidently the equations of the three couples of the lost momenta relatively to the axes of a system originating at the centre of gravity; and the impressed momenta are the same as in the original equations (35): whence we infer that

If the motion of a system of material particles, under the action of impulsive forces, is resolved into a motion of translation of the centre of gravity, and of rotation about an axis passing through that point, the motion of the centre of gravity is the same as if the masses of all the particles were collected at it, and all the impressed momenta were applied at it each in a line parallel to its own line of action; and the motion of rotation about the axis passing through the centre of gravity is the same as if the centre of gravity were a fixed point, and the system rotated about an axis passing through that point under the

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action of the impressed momenta which are actually applied to the several particles of the system.

And the centre of gravity of the system is the only point which has this property; for there is no other point for which

$$\Sigma.m\frac{dx}{dt} = \Sigma.m\frac{dy}{dt} = \Sigma.m\frac{dz}{dt} = 0;$$

so that the terms omitted in (57) should disappear.

55.] Secondly, let us take equations (37) and (38), which D'Alembert's principle gives when the system of particles is subject to finite accelerating forces; then, differentiating (55), we have

$$\frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}, \quad \frac{d^2z}{dt^2} = \frac{d^2\bar{z}}{dt^2} + \frac{d^2z'}{dt^2};$$

so that (37) become

$$\left.\begin{aligned} \text{M}\frac{d^2\bar{x}}{dt^2} &= \Sigma.m\text{X}, \\ \text{M}\frac{d^2\bar{y}}{dt^2} &= \Sigma.m\text{Y}, \\ \text{M}\frac{d^2\bar{z}}{dt^2} &= \Sigma.m\text{Z}; \end{aligned}\right\} \qquad (58)$$

which equations have the same form as those of the motion of a material particle whose mass is M. Whence it appears that

If the whole mass of a material system is collected into its centre of gravity, and the several impressed momentum-increments are applied at it, each in a line parallel to its own line of action, the expressed momentum-increment, and therefore the motion of translation, of the mass thus condensed is that due to all the impressed momentum-increments thereat applied.

Again let us substitute in (38); then we have

$$\Sigma.m\left\{(\bar{y}+y')\left(\text{Z}-\frac{d^2\bar{z}}{dt^2}-\frac{d^2z'}{dt^2}\right)-(\bar{z}+z')\left(\text{Y}-\frac{d^2\bar{y}}{dt^2}-\frac{d^2y'}{dt^2}\right)\right\}=0;$$

and this when expressed at length becomes

$$\bar{y}\,\Sigma.m\left(\text{Z}-\frac{d^2\bar{z}}{dt^2}\right)-\bar{y}\,\Sigma.m\frac{d^2z'}{dt^2}+\Sigma.my'\left(\text{Z}-\frac{d^2z'}{dt^2}\right)-\frac{d^2\bar{z}}{dt^2}\Sigma.my'$$

$$-\bar{z}\,\Sigma.m\left(\text{Y}-\frac{d^2\bar{y}}{dt^2}\right)+\bar{z}\,\Sigma.m\frac{d^2y'}{dt^2}-\Sigma.mz'\left(\text{Y}-\frac{d^2y'}{dt^2}\right)+\frac{d^2\bar{y}}{dt^2}\Sigma.mz'=0;$$

and therefore, omitting terms which vanish by reason of preceding equations and conditions, we have

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$$\left.\begin{aligned} &\Sigma.m\left\{y'\left(\text{Z}-\frac{d^2z'}{dt^2}\right)-z'\left(\text{Y}-\frac{d^2y'}{dt^2}\right)\right\}=0;\\ &\text{similarly from the other two equations of (38) we have}\\ &\Sigma.m\left\{z'\left(\text{X}-\frac{d^2x'}{dt^2}\right)-x'\left(\text{Z}-\frac{d^2z'}{dt^2}\right)\right\}=0,\\ &\Sigma.m\left\{x'\left(\text{Y}-\frac{d^2y'}{dt^2}\right)-y'\left(\text{X}-\frac{d^2x'}{dt^2}\right)\right\}=0. \end{aligned}\right\}\quad(59)$$

But these are evidently, in reference to the system of coordinate axes originating at the centre of gravity, the three equations of the couples which arise from the excess of the impressed over the expressed momentum-increments; and the impressed momentum-increments are at each point the same as in the original equations (38); whence we infer that

If the motion of a system of material particles, under the action of finite accelerating forces, is resolved into a translation of the centre of gravity, and a rotation about an axis passing through that point, the motion of the centre of gravity is the same as if the mass of all the particles were collected at it, and all the impressed momentum-increments were applied at it, each in a line parallel to its own line of action; and the motion of rotation about an axis passing through the centre of gravity is the same as if that centre of gravity were a fixed point, and the system rotated about an axis passing through it under the action of the impressed momentum-increments which are actually applied to the several particles of the system.

56.] Thus, when a body is projected in any direction, and moves under the force of gravity, which acts on all the particles of the body, the centre of gravity of the body describes a parabola in a vertical plane. Again, if a shell is projected and the shell bursts before it meets the earth, by the action of internal forces, these latter forces, being related in equal and opposite pairs, do not appear in the right-hand member of (58), and the centre of gravity of all the broken parts moves in the same parabolic path as before the explosion.

This theorem is also true of the solar system; because it is a material system of the nature explained in Art. 43; so that if the solar system has a proper motion in space by the action of forces external to it, which have either acted once for all, or are finite and continuous, that proper motion will be shewn in the change of place of the centre of gravity of the system; and

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if the path, velocity, &c., of the centre of gravity can be determined by observation, the force to which it is due may also be determined. Now as the mass of the sun is so very much greater than that of the other constituent bodies of its system, and as they are arranged around it, we may, without great error, assume the centre of the sun to be the centre of gravity of its system; and this being so, the result of the calculations of M. F. G. W. Struvè, founded on the studies of Argelander, O. Struvè, and Peters, is that the sun advances annually in space through 154,185,000 miles towards a point in the heavens situated in the constellation Hercules*. This result is arrived at from an estimation of the proper motion of the stars: but our knowledge of these motions is at present far too imperfect for us to decide how far the assigned velocity and direction of the solar motion deviates from exactness; and whether it continues uniform, or whether it shows any symptoms of deflection from rectilinearity. At present, says sir John Herschel, we require more precise and extensive knowledge, before we can hold out a prospect of being one day enabled to trace out an arc of the solar orbit, and to indicate the direction in which the preponderant gravitation of the sidereal firmament is urging the central body of our system.

SECTION 3.—*Principles of the conservation of the motion of the centre of gravity; and of the conservation of moments and of areas. Laplace's invariable plane.*

57.] I PROPOSE now to consider certain theorems which arise out of equations (37) and (38), when the impressed momentum-increments are of certain particular forms; and, firstly, we will take equations (37).

Suppose a material system to have been put into motion by the action of instantaneous forces, so that the axial components of the velocity of its centre of gravity are those given in equations (56); and let us suppose the forces which subsequently act on the system to be such that

$$\Sigma . m \mathrm{X} = \Sigma . m \mathrm{Y} = \Sigma . m \mathrm{Z} = 0; \qquad (60)$$

the meaning of which condition is, that either the system is free from the action of any forces; or the forces are such that the

* See Études d'Astronomie Stellaire. St. Petersburg, 1847.

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momentum-increments impressed by them mutually destroy each other, when all are transferred to the centre of gravity in lines parallel to their own lines of action; then, from (58), we have

$$\frac{d^2\bar{x}}{dt^2} = 0, \qquad \frac{d^2\bar{y}}{dt^2} = 0, \qquad \frac{d^2\bar{z}}{dt^2} = 0; \tag{61}$$

$$\therefore \quad \frac{d\bar{x}}{dt} = \frac{\Sigma.m\mathrm{v}_x}{\mathrm{M}}, \qquad \frac{d\bar{y}}{dt} = \frac{\Sigma.m\mathrm{v}_y}{\mathrm{M}}, \qquad \frac{d\bar{z}}{dt} = \frac{\Sigma.m\mathrm{v}_z}{\mathrm{M}}; \tag{62}$$

$$\bar{x} - a = \frac{\Sigma.m\mathrm{v}_x}{\mathrm{M}}t, \quad \bar{y} - b = \frac{\Sigma.m\mathrm{v}_y}{\mathrm{M}}t, \quad \bar{z} - c = \frac{\Sigma.m\mathrm{v}_z}{\mathrm{M}}t; \tag{63}$$

$$\therefore \quad \frac{\bar{x}-a}{\Sigma.m\mathrm{v}_x} = \frac{\bar{y}-b}{\Sigma.m\mathrm{v}_y} = \frac{\bar{z}-c}{\Sigma.m\mathrm{v}_z}; \tag{64}$$

(a, b, c) being the place of the centre of gravity of the system when $t = 0$.

As (64) are the equations to a straight line, it follows that the centre of gravity moves along a straight line, of which the direction-cosines are proportional to the axial components of its initial velocity; and its velocity is given in Art. 54.

If the centre of gravity is initially at rest, so that

$$\Sigma.m\mathrm{v}_x = \Sigma.m\mathrm{v}_y = \Sigma.m\mathrm{v}_z = 0,$$

it remains at rest during the whole motion of the system.

This theorem is called the principle of the conservation of the motion of the centre of gravity; and by virtue of it, in all cases of motion of a free system of particles, and of a system which is subject to forces which mutually destroy each other, the centre of gravity of the system either remains at rest or moves with a constant velocity along a determinate straight line.

Thus the motion of the centre of gravity of a system of particles is not altered by their mutual collision, whatever is their degree of elasticity, because a reaction always exists equal and opposite to the action. If an explosion takes place in a moving body, whereby it is broken into pieces, the line of motion and the velocity of the centre of gravity of the body are not changed by the explosion; thus the motion of the centre of gravity of the earth is unaltered by earthquakes; volcanic explosions in the moon will not change its motion in space. The motion of the centre of gravity of the solar system is not affected by the mutual and reciprocal action of its several members; it is only changed by the action of forces external to the system.

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58.] Next let us take equations (35), and put them into the form

$$\left.\begin{aligned} \Sigma.m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right) &= \Sigma.m\,(y\,\mathrm{v}_z-z\,\mathrm{v}_y),\\ \Sigma.m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right) &= \Sigma.m\,(z\,\mathrm{v}_x-x\,\mathrm{v}_z),\\ \Sigma.m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) &= \Sigma.m\,(x\,\mathrm{v}_y-y\,\mathrm{v}_x)\,; \end{aligned}\right\} \qquad (65)$$

the left-hand members of which equations are the axial components of the moments of the couples of the expressed momenta of all the particles of the system. And as the right-hand members are the similar quantities for the impressed momenta, the equality of the two is asserted in the equations. If therefore the system of particles moves at any time t with such momenta that the left-hand members of (65) express the axial components of the moments of the couples of the expressed momenta of all the particles, then $\Sigma.m\mathrm{v}_x$, $\Sigma.m\mathrm{v}_y$, $\Sigma.m\mathrm{v}_z$ are the momenta which impressed in a direction contrary to that of the motion will destroy the rotation of the system; and moreover if $\Sigma.m\mathrm{v}_x$, $\Sigma.m\mathrm{v}_y$, $\Sigma.m\mathrm{v}_z$ are subject to the relations (34), the system will be brought to rest.

Now let us take equations (38), and express them in the following form

$$\left.\begin{aligned} \Sigma.m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right) &= \Sigma.m\,(y\,\mathrm{Z}-z\,\mathrm{Y}),\\ \Sigma.m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right) &= \Sigma.m\,(z\,\mathrm{X}-x\,\mathrm{Z}),\\ \Sigma.m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right) &= \Sigma.m\,(x\,\mathrm{Y}-y\,\mathrm{X})\,; \end{aligned}\right\} \qquad (66)$$

and let us suppose the system of particles to have been put into motion.

And now let us suppose the acting forces to be such that the impressed momentum-increments satisfy the equations,

$$\Sigma.m\,(y\,\mathrm{Z}-z\,\mathrm{Y}) = \Sigma.m\,(z\,\mathrm{X}-x\,\mathrm{Z}) = \Sigma.m\,(x\,\mathrm{Y}-y\,\mathrm{X}) = 0. \qquad (67)$$

This is the case;

(1) When, for every particle of the system, $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$; that is, when the system is free of the action of all continuous forces.

Hence also it is true when the origin is a fixed point, because the pressures which it bears satisfy this equation.

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(2) When the members of the system are subject to forces, to each of which an equal and opposite one corresponds. Thus, for example, suppose m' and m'', situated at (x', y', z') and (x'', y'', z'') respectively, to be attracted towards each other by a certain force P, dependent on their distance (r) from each other; then

$$\left.\begin{aligned} m'\mathrm{X}' &= \frac{x''-x'}{r}\mathrm{P}, & m'\mathrm{Y}' &= \frac{y''-y'}{r}\mathrm{P}, & m'\mathrm{Z}' &= \frac{z''-z'}{r}\mathrm{P}, \\ m''\mathrm{X}'' &= -\frac{x''-x'}{r}\mathrm{P}, & m''\mathrm{Y}'' &= -\frac{y''-y'}{r}\mathrm{P}, & m''\mathrm{Z}'' &= -\frac{z''-z'}{r}\mathrm{P}, \end{aligned}\right\} \quad (68)$$

$$\begin{aligned} \therefore\quad & m'(y'\mathrm{Z}'-z'\mathrm{Y}')+m''(y''\mathrm{Z}''-z''\mathrm{Y}'') \\ &= \frac{\mathrm{P}}{r}\{y'(z''-z')-z'(y''-y')-y''(z''-z')+z''(y''-y')\} \\ &= 0; \end{aligned}$$

and similar results are true for every other pair of equal and opposite actions and reactions; and also for the other couples.

(3) When the lines of action of the forces acting on the several particles of the system pass through the origin; because in this case

$$\frac{\mathrm{X}}{x} = \frac{\mathrm{Y}}{y} = \frac{\mathrm{Z}}{z}. \quad (69)$$

(4) When the forces would be in equilibrium, were the system on which they act brought suddenly to rest; because in that case (67) are the conditions of statical equilibrium.

In all these cases (67) are satisfied, and we have

$$\left.\begin{aligned} \Sigma.m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma.m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right) &= 0, \\ \Sigma.m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right) &= 0. \end{aligned}\right\} \quad (70)$$

But since $$\frac{d}{dt}\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right) = y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}; \quad (71)$$

therefore, observing the effect of the summatory symbols, we have

$$\left.\begin{aligned} \Sigma.m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right) &= h_1, \\ \Sigma.m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right) &= h_2, \\ \Sigma.m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) &= h_3; \end{aligned}\right\} \quad (72)$$

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where h_1, h_2, h_3 are certain constants of integration; and, as is evident from (65), are the axial components of the moments of the couples of momenta of all the particles due to the instantaneous forces by which the system of particles is originally put into motion; or, as we may say, they are the axial components of the sum of the moments of all the momenta at any time. Hence

If a system of material particles is put into motion by the action of instantaneous or other forces, and, when the action of these forces ceases, is acted on by forces which satisfy the conditions (67), then, notwithstanding the alteration in the expressed momenta of individual particles, the axial components of the moments of the couples of the expressed momenta of all the particles, at any time t, are constant. Also the moment-axis of this resultant couple of all the expressed momenta is constant, and the direction of the rotation-axis is fixed; that is, these are independent of the time, and remain the same throughout the motion. Thus, if h is the moment-axis of the resultant couple, and α, β, γ are the direction-angles of the rotation-axis,

$$h^2 = h_1^2 + h_2^2 + h_3^2; \tag{73}$$

$$\frac{\cos\alpha}{h_1} = \frac{\cos\beta}{h_2} = \frac{\cos\gamma}{h_3} = \frac{1}{h}. \tag{74}$$

This theorem is called the principle of the conservation of moments, and holds good when the equations (67) are satisfied; and the cases where this occurs are numerous enough to make the theorem of great importance. Thus, it is true when a collision takes place between two or more members of the system, because equal and opposite actions are generated thereby, whatever is the degree of elasticity. It is also true when two or more members become suddenly united; when parts of the system pass from the gaseous to the fluid state, or from the fluid to the solid state; provided that the causes by which such a transmutation takes place produce equal and opposite actions. This is a remarkable case, because the forces may be functions of the time explicitly, but, as they disappear, the principle is true. Thus the moment of the couple of all the momenta of the earth, as well as the direction of its rotation-axis, would remain the same, supposing the earth to be cooled down, without loss of gravitating matter. And the principle is also true when the magnetic or electrical state of two particles or of two members of the system is altered,

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if the change is accompanied by an equal and opposite action. Thus, no alteration is caused either in the length of the day, or in the position of the earth's axis, that is, in the place of the polar star, by earthquakes, volcanic explosions, the rolling of the sea on the shore, the fall of avalanches, the continual friction of the wind against the surface of the earth, &c.; because all these actions are accompanied by equal and opposite reactions, and therefore satisfy the equations (67).

59.] Equations (72) also admit of further interpretation. From the origin let radii vectores be drawn to each of the particles of the system; as the system moves, then, if the origin is assumed not to move during the time dt, each radius describes an infinitesimal sectorial area, which is part of a conical surface; let these sectorial areas be projected on the coordinate planes; from (72) we shall infer that the aggregate of the products of each particle, and the projection of the sectorial area described by its radius vector relatively to each of the coordinates planes, varies as the time.

Let r be the radius vector, drawn from the origin to the place of m, at the time t; let $d\text{A}$ be twice the infinitesimal sectorial area over which r passes in the time dt; and let $d.\text{A}_x$, $d.\text{A}_y$, $d.\text{A}_z$ be the projections of $d\text{A}$ on the planes of (y, z), (z, x), (x, y) respectively; then, as in Art. 307, Vol. III,

$$\left.\begin{aligned} d.\text{A}_x &= y\,dz - z\,dy, \\ d.\text{A}_y &= z\,dx - x\,dz, \\ d.\text{A}_z &= x\,dy - y\,dx; \end{aligned}\right\} \tag{75}$$

so that (72) become

$$\left.\begin{aligned} \Sigma.m\,d.\text{A}_x &= h_1\,dt, \\ \Sigma.m\,d.\text{A}_y &= h_2\,dt, \\ \Sigma.m\,d.\text{A}_z &= h_3\,dt. \end{aligned}\right\} \tag{76}$$

Now as these equations are true for an infinitesimal time dt, and for any point as centre of areas, provided that in case (3) of Art. 58 that point is the source of the central forces; so will they be true for a finite time, if the centre is fixed, or if the centre moves in a straight line; under either of these circumstances we may integrate (76), and we have

$$\left.\begin{aligned} \Sigma.m\,\text{A}_x &= h_1 t, \\ \Sigma.m\,\text{A}_y &= h_2 t, \\ \Sigma.m\,\text{A}_z &= h_3 t; \end{aligned}\right\} \tag{77}$$

the limits of integration being such that the areas and the time

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begin simultaneously. Thus, the sum of the products of the mass of every particle, and the projection of the sectorial area described by its radius vector on each coordinate plane, varies as the time; and for an unit of time is constant throughout the motion.

This theorem is called the principle of the conservation of areas, and is true whenever equations (67) are satisfied.

The signs of the areas are thus far determined by the signs of the right-hand members of (75); they are therefore to be considered positive when the direction of rotation is positive according to the principle of Art. 33. Thus, for instance, for rotation about the axis of z, if the projection of the radius vector on the plane of (x, y) moves from the axis of x towards the axis of y, $d.\text{A}_z = x\,dy - y\,dx$; in which case $d.\text{A}_z$ is positive; and the signs of the other projections are to be taken on an analogous principle. If therefore the motion is retrograde, the areas will have negative signs.

60.] As the sum of the products of the mass of each particle, and its projected sectorial area, varies as the time, or is constant for an unit of time, for each of the coordinate planes, so will it be also for every plane; the sum however of these products varies as the position of the plane on which they are projected varies: it is evident that there is an infinite number of planes, for which the sum vanishes; viz. all those planes, the direction-cosines of whose normals are l, m, n, and which satisfy the condition

$$lh_1 + mh_2 + nh_3 = 0:$$

hence it is evident that all these planes may intersect along the same straight line. And the plane which is perpendicular to this straight line has the peculiarity that the sum of the products of the masses, and of the projected areas, vanishes for all planes perpendicular to it; and, for a given plane, varies as the cosine of the angle at which the planes are inclined to each other. It is evident then that for this plane the sum is a maximum; and the position of it may thus be found:

Let l, m, n be the direction-cosines of the normal of the required plane; and let u be the sum of the products of each mass and its sectorial area projected on the required plane; then the theory of projections of areas gives us

$$\begin{aligned} u &= l\,\Sigma.m\,\text{A}_x + m\,\Sigma.m\,\text{A}_y + n\,\Sigma.m\,\text{A}_z \\ &= (lh_1 + mh_2 + nh_3)\,t; \end{aligned} \tag{78}$$

and

$$1 = l^2 + m^2 + n^2; \tag{79}$$

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therefore $$\text{D}u = 0 = h_1\,dl + h_2\,dm + h_3\,dn,$$
$$0 = l\,dl + m\,dm + n\,dn;$$
$$\therefore\quad \frac{l}{h_1} = \frac{m}{h_2} = \frac{n}{h_3} = \frac{1}{h} = \frac{u}{h^2 t}; \tag{80}$$
$$\therefore\quad u = ht = \{h_1^2 + h_2^2 + h_3^2\}^{\frac{1}{2}} t; \tag{81}$$
$$l = \frac{h_1}{h}, \qquad m = \frac{h_2}{h}, \qquad n = \frac{h_3}{h}; \tag{82}$$

whereby the maximum value of the products is determined, and also the direction-cosines of the plane for which the sum of the products of the masses and the projected sectorial areas is a maximum. And, if that plane passes through the origin, its equation is
$$h_1 x + h_2 y + h_3 z = 0. \tag{83}$$
Hence it appears that the position of it is independent of t; and is therefore the same throughout the motion. For this reason it is called the invariable plane. The preceding equation shews that if at any time the masses of the moving particles, their places with reference to a centre fulfilling the conditions (67), and their velocities are known, then h_1, h_2, h_3 may be calculated, and the position of the invariable plane will be completely determined.

This plane is evidently that also of the maximum couple of the momenta of all the particles at any time t; and the position therefore of the plane of the maximum couple is constant throughout the motion. And whatever is true of the invariable plane and its normal, is true also, mutatis mutandis, of the plane of the maximum couple of moments and its normal.

Hereafter we shall meet with particular cases of this theorem; (1) when one point of a body or of a system of particles is fixed; (2) when two points are fixed so that the system rotates about a fixed axis passing through them.

61.] In the determination of the places and motion of the heavenly bodies astronomers are always subject to the difficulty that they have no fixed planes and no fixed lines to which they can refer them. It is true that they generally take the sun as a fixed centre and the plane of the ecliptic, that is, the plane in which the centre of the earth always is in its motion around the sun, to be a fixed plane. The proper motion of the stars however renders it almost more than probable, that their motions are in a great measure only apparent, and are due to a true proper motion of the sun: and the position of the plane of the ecliptic

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is subject to small variations by the disturbing effects of the moon, planets, and perhaps other members of the solar system.

Astronomers therefore are referring the places and motions of the planets to the sun, which is not a fixed centre, and to the ecliptic, which is a moving plane: herein lies what may be a fruitful source of uncertainty and inaccuracy; inferences from observations, and theory built upon them, are carried over long ages; and it would be of advantage to astronomy if a fixed point and a fixed plane could be determined, to which all observations and calculations could be conveniently referred; or if, the position of the latter being given in direction, the motion, rectilineal or other, of the former were known. Now it has been before observed that probably the sun has a proper motion in space; and that this is rectilineal, so far as our observations at present indicate, and with a known velocity. Thus far then, if the sun is taken as the centre of areas, the principle of areas may be true for the solar system. The forces which act on the solar system are (1) chiefly internal forces of attraction which will disappear in the aggregate of the moving masses of the system; and (2) the external forces acting on the sun and other members of the system from stars and other bodies, some of which are perhaps not visible to us. As the mass of the sun however is so much larger than the masses of all the other bodies of the solar system, we may assume the sun's centre to be the centre of gravity of all the bodies of the system, and the external forces which act on the several members of the system to be applied at it, in accordance with the principle of Art. 55. We may reasonably suppose that these forces produce the sun's proper motion in space, and do not produce any sensible effect on the rotation of the bodies about it; that is, we shall assume these external forces, approximately and sensibly, to be such as satisfy the equations (67).

These forces therefore are such that the theory of the invariable plane is applicable to the solar system; and as its position is the same during the whole motion, being independent of the time, it is a plane to which the places and motions of the members of the system may be advantageously referred. The determination of its place however requires a knowledge of the masses of all the members of the system, and of the elements of their orbits. Approximate values of these are known for the planets and their satellites, but of the masses of the

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comets we are in total ignorance. As the mutual attractions and perturbations of the several planets however are sufficient for the explanation of all these inequalities, it is manifest that hitherto at least the action of the comets on the planetary system is insensible. The comet of 1770 approached so near to the earth, that the periodic time of the comet is calculated to have been increased by 2.046 days; and, if its mass had been equal to that of the earth, it would, according to Laplace, have increased the length of our year by nearly one hour and fifty-six minutes; but Laplace adds, that if an increase of only two seconds had taken place in the length of the year, it would have been detected; and as such an increase has not been detected, it follows that the mass of the comet must be less than $\frac{1}{5000}$dth part of the mass of the earth. The same comet passed through the satellites of Jupiter in the years 1767 and 1779, without producing any effect. Thus, though comets are greatly disturbed by the action of the planets, it does not appear that they produce any sensible effect on the planets by their action. In the determination therefore of the position of the invariable plane of the solar system, their effect is insensible.

If therefore h_1, h_2, h_3 have been determined for the plane of the ecliptic, as that of (x, y), by observation, and θ is the inclination of the invariable plane to that of the ecliptic, and ψ is the longitude of its ascending node, from Art. 4 we have

$$h_1 = \sin\psi \sin\theta, \quad h_2 = -\cos\psi \sin\theta, \quad h_3 = \cos\theta; \tag{84}$$

$$\therefore \quad \cos\theta = h_3, \quad \tan\psi = -\frac{h_1}{h_2}; \tag{85}$$

and thus the position of the invariable plane would be known.

It will be observed that h_1, h_2, h_3 are in (72) the axial components of the moments of the couples of the expressed momenta of all the particles due to an unit of time; and, in (77), are the sums of the products of every particle and the projected sectorial area of its radius vector about the origin in an unit of time. In calculating therefore these quantities for the determination of the position of the invariable plane of the solar system, since the planets and satellites rotate about their own axes, and the satellites revolve about their primaries, we cannot estimate their moments or their sectorial areas, as if they were single particles; but the required quantities must be calculated separately for each

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individual particle. Thus, as the sun rotates, the sectorial area corresponding to each of its elements has to be estimated. As satellites revolve about their primaries, and also rotate about their own axes, these have to be estimated. It would be out of place here to enter on these calculations, although they are of extreme interest, and of great importance in the calculations of accurate astronomy; I can do no more than refer the student to places where the mode of calculation is explained: (1) Laplace, Exposition du Système du Monde, 5me Ed. Paris, 1824, p. 199, lib. IV, ch. II. (2) Poinsot, Équateur du Système Solaire; appended to the Éléments de Statique; 8me Ed. 1842. (3) Poisson, Traité de Mécanique, 2nde Ed. 1833, Vol. II, p. 469. (4) A note, "Du plan invariable du Système du Monde," appended to the 3rd Vol. of Pontecoulant, "Système du Monde," Paris, 1834. The real dynamical things which are invariable, and on which the position of the plane depends, are the momentum-moments; the products of the masses and the sectorial areas are geometrical representatives of them; and the theorem has been stated in the latter form probably because Kepler's Law of Areas becomes hereby generalized.

62.] In the calculation of the moments of the momenta of these several bodies, and systems relative to particular axes, it is convenient to calculate the moments of an individual body or system relative to parallel axes passing through the centre of gravity of that body; and then to increase that quantity by the moment of the momentum of its whole mass condensed into its centre of gravity. The following process proves the theorem:

Let the moment of the momentum, relative to the axis of x, be taken as the type of that relative to each of the other coordinate axes, and indeed to any other axis. Then, taking the equations of transformation given in Art. 54, we have

$$\begin{aligned} h_1 &= \Sigma.m \left\{ y\frac{dz}{dt} - z\frac{dy}{dt} \right\} \\ &= \Sigma.m \left\{ (\bar{y}+y')\left(\frac{d\bar{z}}{dt}+\frac{dz'}{dt}\right) - (\bar{z}+z')\left(\frac{d\bar{y}}{dt}+\frac{dy'}{dt}\right) \right\} \\ &= \text{M} \left(\bar{y}\frac{d\bar{z}}{dt} - \bar{z}\frac{d\bar{y}}{dt}\right) + \Sigma.m \left\{ y'\frac{dz'}{dt} - z'\frac{dy'}{dt} \right\}; \end{aligned} \tag{86}$$

where M is the mass of the body or system. So that h_1, which is the sum of the moments of the momenta of all the particles of M relative to the axis of x, is the sum of the moment of the

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momentum of M condensed into a particle at its centre of gravity, and of the sum of the moments of the momenta of all the particles, relative to an axis passing through the centre of gravity, and parallel to that of x. Similar values are of course true for h_2 and h_3, and consequently for every axis.

SECTION 4.—*The principle of vis viva; Lagrange's principle of least action; Carnot's theorem.*

63.] THE theorems which have been proved in the preceding section of the motion of a material system are only true when certain relations exist between the acting forces, whether internal or external. I proceed now to a theorem which is much more generally true, and which gives one integral of the equations of motion. Let us return to the equation (40) of Art. 50. Whatever is the relative disposition of the several particles of a material system, provided that the relation is independent of the time, that is, provided that the equations of condition $F_1 = 0$, $F_2 = 0$, ... $F_k = 0$, do not contain explicitly the variable t, it is evident that we can always replace the variations δx, δy, δz by the actual spaces dx, dy, dz, which the point of application of a force describes in the time dt. For suppose $F = 0$ to be an equation of condition; then, for the geometrical displacement, the variation of this is

$$\left(\frac{dF}{dx_1}\right)\delta x_1 + \left(\frac{dF}{dy_1}\right)\delta y_1 + \left(\frac{dF}{dz_1}\right)\delta z_1 + \ldots + \left(\frac{dF}{dz_n}\right)\delta z_n = 0; \quad (87)$$

and if F contained t explicitly, there would be no variation of it in this equation, because the shifting of the system is a virtual geometrical displacement.

But if the changes in the coordinates of the points of application of the forces are due to the dynamical forces, then t varies, and we have for the total variation of $F = 0$, if it contains t explicitly,

$$\left(\frac{dF}{dt}\right)dt + \left(\frac{dF}{dx_1}\right)dx_1 + \left(\frac{dF}{dy_1}\right)dy_1 + \ldots + \left(\frac{dF}{dz_n}\right)dz_n = 0; \quad (88)$$

which equation cannot consist with (87), unless $\left(\frac{dF}{dt}\right) = 0$; that is, unless F does not contain the time explicitly.

We will assume that in the general equation of motion, viz.

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(40) of Art. 50, we can replace δx, δy, δz by dx, dy, dz; then the equation becomes

$$\Sigma.m\left\{\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)dx+\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)dy+\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)dz\right\}=0;$$

which may be put into the form

$$\Sigma.m\left\{\frac{d^2x}{dt^2}dx+\frac{d^2y}{dt^2}dy+\frac{d^2z}{dt^2}dz\right\}=\Sigma.m\,(\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz);$$

and if v is the velocity of the particle m at the time t, this gives

$$d.\Sigma.m\,v^2=2\,\Sigma.m\,(\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz); \tag{89}$$

$$\therefore\quad \Sigma.m\,v^2-\Sigma.m\,v_0^2=2\,\Sigma.m\int(\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz);$$

the limits of the integral in the right-hand member corresponding to those in the left-hand member.

Let us consider the meaning of the right-hand member: let F be the velocity-increment impressed on m in an unit of time along the line ds in which m moves at the time t, so that $m\mathrm{F}$ is the impressed momentum-increment in an unit of time. Let X, Y, Z be the axial components of F; then

$$\mathrm{X}\,dx+\mathrm{Y}\,dy+\mathrm{Z}\,dz=\mathrm{F}\,ds;$$

and (89) becomes

$$d.\Sigma.m\,v^2=2\,\Sigma.m\,\mathrm{F}\,ds; \tag{90}$$

that is, the increment of the vires vivæ of all the particles of the system, or, as we call it, of the vis viva of the system, in the time dt, is equal to twice the sum of the products of each particle, its impressed velocity-increment, and the space through which it moves.

This latter is a cumbrous form of expression, and it has been found convenient to introduce a new term; $m\mathrm{F}\,ds$ is called the *work* or the *labouring force* of m*, or the impressed quantity of work of m, and, as it is taken for an infinitesimal time only, it may be called the impressed increment of work, or the element of impressed work due to the labouring force; so that from (90) we have the following theorem;

In the motion of a system of particles of invariable form the infinitesimal increment of the vis viva of the system is equal to twice the increment of the impressed work. If the labouring force acts to increase the vis viva of the system, it is called a motive force; if it acts to diminish the vis viva, it is called a re-

* See Traité de la Mécanique des corps Solides et du Calcul de l'effet des Machines, par G. Coriolis, 2nd Ed., Paris, 1844, page 37.

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tarding force. As we intend to enter on this subject at greater length hereafter, it is unnecessary now to say more upon it.

It must be observed, that in the motion of a system of particles, which are subject to the action of internal forces as well as of external forces, the internal forces do not necessarily disappear in the right-hand member of (89), even if they enter in pairs of equal and opposite forces; for suppose an internal force, say T, to act mutually on m and m' in equal intensity and in opposite directions along the same line of action; the virtual velocity corresponding to T will not disappear in (89), unless the displacement of m' is exactly equal and parallel to that of m: and as the geometrical displacements are the actual dynamical displacements, the displacements of these two particles are not necessarily equal and opposite. To exemplify this fact, in any displacement of the solar system, the theorems of the last two sections are true; the vis viva however of that system is not always the same, even if we neglect the external forces, because a change of it arises from the internal forces producing an alteration in the form of the system. In the motion of a rigid body, and of a system of particles of invariable form, the internal forces will cancel each other. But if there are elastic connections, or springs, and if expansions or contractions arise from such forces, these may not disappear in the equation of vis viva.

64.] Let us suppose $\Sigma . m (\mathrm{X}\, dx + \mathrm{Y}\, dy + \mathrm{Z}\, dz)$ to be an exact differential of a function of $x_1, y_1, z_1, x_2, y_2, z_2 \ldots x_n, y_n, z_n$; so that

$$\Sigma . m (\mathrm{X}\, dx + \mathrm{Y}\, dy + \mathrm{Z}\, dz) = \mathrm{D} . f(x, y, z); \qquad (91)$$

where $f(x, y, z)$ is a function of some or all of the coordinates of the several particles of the system: then, taking the definite integral of (89), we have

$$\Sigma . m v^2 - \Sigma . m v_0^2 = 2 f(x, y, z) - 2 f(x_0, y_0, z_0); \qquad (92)$$

where the quantities with the subscript o are the initial values of the similar quantities without the subscript; and therefore

The increase of the vires vivæ of all the particles of the system, that is, of the vis viva of the whole system, in passing from one position to another, depends on only the two positions of the system, and is independent of the path described by each particle of the system. This is called the principle of vis viva of a system.

Hence, if all the particles resume the positions which they originally had, the sum of the vires vivæ is the same in both cases;

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and consequently, whenever a system in motion resumes a position which it formerly had, the vis viva of the system is the same in both cases.

If, for every particle of the system,

$$\text{X} = \text{Y} = \text{Z} = 0,$$

$$\Sigma . m v^2 = \Sigma . m v_0^2;$$

and the sum of the vires vivæ is the same throughout the motion; this theorem is called the principle of conservation of vires vivæ.

If all the particles of the system are subject to the action of gravity only, then, taking the plane of (x, y) to be horizontal, $\text{X} = \text{Y} = 0$, $\text{Z} = g$, and therefore

$$\Sigma . m v^2 - \Sigma . m v_0^2 = 2 \Sigma . m g (z - z_0). \tag{93}$$

But if $\bar{z}_0$ and $\bar{z}$ are the distances from the plane of (x, y) to the centre of gravity of the system at the times 0 and t respectively; and if M is the mass of the particles,

$$\text{M} \bar{z}_0 = \Sigma . m z_0, \qquad \text{M} \bar{z} = \Sigma . m z;$$

$$\therefore \quad \Sigma . m v^2 - \Sigma . m v_0^2 = 2 \text{M} g (\bar{z} - \bar{z}_0); \tag{94}$$

that is, the increase of vis viva of a heavy system depends only on the vertical distance over which the centre of gravity passes; and therefore the vis viva is the same whenever the centre of gravity passes through a given horizontal plane.

Whenever the system passes through a position in which it would be in equilibrium under the action of the impressed labouring force, if it had no velocity at the time, then, in that position, by the principle of virtual velocities,

$$\Sigma . m (\text{X} dx + \text{Y} dy + \text{Z} dz) = 0. \tag{95}$$

And therefore from (89), $d . \Sigma . m v^2 = 0$; so that the vis viva is either a maximum or a minimum, or is constant; the last being the case when no forces are impressed, and when therefore every position of the body would be a position of equilibrium, if there was no velocity. And therefore conversely the sum of all the vires vivæ is a maximum or a minimum when the system passes through what would be a position of equilibrium if the particles had no velocity. And the vis viva of the system is a maximum or a minimum according as the position of equilibrium is of stable or of unstable equilibrium.

For on referring to the notation and the process of Art. 103 and 104, Vol. III, it appears that half the vis viva of the system

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is the quantity which is therein expressed by u, and is called the central moment of the system; for, from (373) Art. 103, we have

$$du = \Sigma . P(\cos\alpha\, dx + \cos\beta\, dy + \cos\gamma\, dz)$$

$$= \Sigma . m(X\, dx + Y\, dy + Z\, dz) = \frac{1}{2} d . \Sigma . m v^2; \qquad (96)$$

and therefore, as the equilibrium of a system of forces is stable or unstable, according as the central moment is a maximum or a minimum, so the vis viva of the system is a maximum or a minimum, according as the equilibrium-state through which the system passes is a state of stable or of unstable equilibrium. Thus, it appears that in a heavy system of particles, such as in machines &c., the vis viva of the system is the greatest when the centre of gravity has its lowest position, because the equilibrium in that case is stable: and the vis viva is the least when the centre of gravity has the highest position compatible with the constraints of the machine. This also may be deduced directly from (94).

65.] Let us now consider under what circumstances

$$\Sigma . m(X\, dx + Y\, dy + Z\, dz)$$

is an exact differential.

It is to be observed, that this cannot be the case if X, Y, or Z contains t explicitly. Of course the general conditions for any one particle m are those given in (15), Art. 383, Vol. III. It is however an exact differential so far as any particle m is acted on by a central force whose centre is fixed at (a, b, c); and which is a function of the distance r between the centre and (x, y, z) the place of m. Thus, let P be the central force $= f(r)$, say; then

$$X = \frac{x-a}{r} f(r), \quad Y = \frac{y-b}{r} f(r), \quad Z = \frac{z-c}{r} f(r);$$

$$r^2 = (x-a)^2 + (y-b)^2 + (z-c)^2;$$

$$\therefore \quad r\, dr = (x-a)\, dx + (y-b)\, dy + (z-c)\, dz;$$

$$\therefore \quad m(X\, dx + Y\, dy + Z\, dz) = m f(r)\, dr; \qquad (97)$$

which is an exact differential, and the change of vis viva due to it

$$= 2m \int_{r_0}^{r} f(r)\, dr.$$

So far therefore as the labouring forces, which act on the particles of the system, are central forces with fixed centres, the impressed quantity of work is an exact differential. And if $f(r)$ is positive, so that the force is repulsive, and if dr is positive, then the vis viva is increased; and if the particles approach,

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in which case dr is negative, the vis viva is diminished; similarly, if the central force is attractive, the vis viva is increased or diminished according as the attracted particle approaches to or recedes from the centre of attraction.

The impressed quantity of work is also an exact differential, so far as any two particles of the system are attracted towards or repelled from each other by a force which varies as the mass of each, and is a function of the distance between them. Let the two particles be m and m', and let their places at the time t be (x, y, z) (x', y', z'), and let the distance between them be r; let $\text{P} = f(r)$ be the attractive or repulsive force of an unit particle acting upon them from one to the other; then

$$\left.\begin{aligned} \text{X} &= m'\frac{x-x'}{r}f(r), & \text{Y} &= m'\frac{y-y'}{r}f(r), & \text{Z} &= m'\frac{z-z'}{r}f(r); \\ \text{X}' &= -m\frac{x-x'}{r}f(r), & \text{Y}' &= -m\frac{y-y'}{r}f(r), & \text{Z}' &= -m\frac{z-z'}{r}f(r). \end{aligned}\right\} \quad (98)$$

Also $\qquad r^2 = (x-x')^2+(y-y')^2+(z-z')^2;$

$$\therefore \quad r\,dr = (x-x')(dx-dx')+(y-y')(dy-dy')+(z-z')(dz-dz'); \quad (99)$$

therefore thus far

$$m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz) + m'(\text{X}'dx' + \text{Y}'dy' + \text{Z}'dz')$$

$$= \frac{mm'}{r}\{(x-x')(dx-dx')+(y-y')(dy-dy')+(z-z')(dz-dz')\}f(r)$$

$$= mm'f(r)\,dr;$$

which is an exact differential. And the change in the vis viva due to this impressed labouring force

$$= 2\,mm'\int_{r_0}^{r} f(r)\,dr. \quad (100)$$

And here, as in the former case, if the force is repulsive, the vis viva of the particles is increased or diminished according as the distance between them is increased or diminished; and if the force is attractive, the vis viva is increased or diminished according as the distance is diminished or increased.

Thus, if a system of particles, gaseous or solid, receives an increase of heat, whereby repulsive forces are brought into action, the particles are two and two repelled further from each other, and there is an increase of vis viva. If, on the other hand, heat is withdrawn, the particles are drawn nearer together, and a diminution takes place. Hence also if in a system of moving particles an explosion takes place, so that some of the particles are removed farther from each other, an increase of vis viva takes place.

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If a particle is constrained to move on a smooth surface whose equation is $\text{F} = 0$, no change of vis viva is due to the reaction of the surface. For let R be the reaction of the surface; and let $\text{U}, \text{V}, \text{W}$, be the $x-$, $y-$, $z-$, partial differential coefficients of F; and let $\text{Q}^2 = \text{U}^2 + \text{V}^2 + \text{W}^2$; then, so far as the reaction enters into the impressed quantity of work,

$$m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz) = \frac{\text{R}}{\text{Q}}\{\text{U}\,dx + \text{V}\,dy + \text{W}\,dz\}$$
$$= 0;$$

so that the vis viva is not changed by this reaction. If however the surface on which m is constrained to move is rough, so that there is friction, the above condition is not satisfied, and a loss of vis viva takes place. Hence, if a collision or a sliding occurs amongst the particles of a moving system, a loss of vis viva ensues.

Also, if a particle m moves freely on a given line, and $\text{X}, \text{Y}, \text{Z}$ are functions of the coordinates of its place (x, y, z) at the time t; so far $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz$ is an exact differential. Because x, y, z may be all expressed in terms of one variable ϕ, as in the equations to the helix, Art. 347, Vol. I, 2nd Ed., and thus, if F expresses a function of ϕ,

$$\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz = \text{F}\,d\phi;$$

which is integrable.

Also, if a particle m moves freely on a given surface, and $\text{X}, \text{Y}, \text{Z}$ are functions of the coordinates of its place (x, y, z), then $\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz$ will be an exact differential if one condition is satisfied, viz. that which is given in (20), Art. 383, Vol. III. This however may more shortly be obtained as follows: Let x, y, z be expressed in terms of two variables θ and ϕ; then, if F_1 and F_2 are functions of θ and ϕ,

$$\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz = \text{F}_1\,d\theta + \text{F}_2\,d\phi;$$

which is an exact differential, if

$$\left(\frac{d\text{F}_1}{d\phi}\right) = \left(\frac{d\text{F}_2}{d\theta}\right).$$

66.] The vires vivæ of a system of particles may also be conveniently expressed by the vis viva of the whole mass collected at its centre of gravity, and the sum of the vires vivæ of all the particles relatively to the centre of gravity.

Let the centre of gravity be $(\bar{x}, \bar{y}, \bar{z})$, and let the place of m, relatively to a system of coordinates originating at the centre of

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gravity and parallel to the original system, be (x', y', z'). So that

$$x = \bar{x} + x', \qquad y = \bar{y} + y', \qquad z = \bar{z} + z';$$

$$\therefore \quad \frac{dx}{dt} = \frac{d\bar{x}}{dt} + \frac{dx'}{dt}, \quad \frac{dy}{dt} = \frac{d\bar{y}}{dt} + \frac{dy'}{dt}, \quad \frac{dz}{dt} = \frac{d\bar{z}}{dt} + \frac{dz'}{dt}; \quad (101)$$

$$\therefore \quad v^2 = \left(\frac{d\bar{x}}{dt}\right)^2 + \left(\frac{d\bar{y}}{dt}\right)^2 + \left(\frac{d\bar{z}}{dt}\right)^2 + 2\left(\frac{d\bar{x}}{dt}\frac{dx'}{dt} + \frac{d\bar{y}}{dt}\frac{dy'}{dt} + \frac{d\bar{z}}{dt}\frac{dz'}{dt}\right)$$
$$+ \left(\frac{dx'}{dt}\right)^2 + \left(\frac{dy'}{dt}\right)^2 + \left(\frac{dz'}{dt}\right)^2; \quad (102)$$

therefore, if M = the mass of all the particles, by reason of (53), Art. 53,

$$\therefore \quad \Sigma.m v^2 = \Sigma.m \bar{v}^2 + \Sigma.m v'^2,$$
$$= \text{M}\bar{v}^2 + \Sigma.m v'^2, \quad (103)$$

where $\bar{v}$ is the velocity of the centre of gravity; that is, the vis viva of a system, relatively to a given origin, is equal to the sum of the vis viva of the whole mass collected at its centre of gravity, and the vis viva of the system relatively to the centre of gravity.

Hence also, if the element of the quantity of work can be expressed in the form of an exact differential, say $\text{D}.f(x, y, z)$; then

$$\Sigma.m v'^2 - \Sigma.m v_0'^2 + \text{M}(\bar{v}^2 - \bar{v}_0^2) = 2f(x, y, z) - 2f(x_0, y_0, z_0). \quad (104)$$

67.] If the principle of vis viva is true of a system of particles, so that

$$\Sigma.m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz) = \text{D}.f(x, y, z);$$

then the equations of motion (37), Art. 48, are such that $\Sigma.m\int v\,ds$ is a minimum for the system as it passes from one position to another; in other words, the principle of least action, which has been proved in Art. 384, Vol. III, to be true of a single particle m, is also true of a system of particles.

Since $v\,ds = v^2\,dt$, $\Sigma.m\int v\,ds = \Sigma.\int m v^2\,dt$; thus, the principle asserts that the sum of the vires vivæ of all the particles of the system which accumulates during the time occupied by the system in passing from one position to another, is under the law of connexion of the impressed and expressed momentum-increments, given in (37) Art. 48, less than it would be for any other law: $\Sigma.\int m v^2\,dt$ is called *the action* of the system; and the theorem is called the principle of least action, and in this extended form is due to Lagrange.

Let the limits of integration, which correspond to the two

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positions of the system, be expressed as in Vol. II; then, if u is the action of all the particles,

$$u = \Sigma . m \int_0^1 v\, ds. \tag{105}$$

Let δ be the symbol of variation; then, by the calculus of variations,

$$\delta u = \Sigma . m \int_0^1 (ds\, \delta v + v\, \delta . ds). \tag{106}$$

Now
$$\delta v = \frac{X}{v} \delta x + \frac{Y}{v} \delta y + \frac{Z}{v} \delta z, \tag{107}$$

and
$$\delta . ds = \frac{dx}{ds} \delta . dx + \frac{dy}{ds} \delta . dy + \frac{dz}{ds} \delta . dz; \tag{108}$$

so that

$$\delta u = \Sigma . m \int_0^1 \left\{ \frac{ds}{v} (X \delta x + Y \delta y + Z \delta z) + \frac{v}{ds} (dx\, d.\delta x + dy\, d.\delta y + dz\, d.\delta z) \right\}; \tag{109}$$

$$= \Sigma . m \left[v \left(\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z \right) \right]_0^1$$

$$+ \Sigma . m \int_0^1 \left\{ \left(\frac{X ds}{v} - d . \frac{v dx}{ds} \right) \delta x + \left(\frac{Y ds}{v} - d . \frac{v dy}{ds} \right) \delta y + \left(\frac{Z ds}{v} - d . \frac{v dz}{ds} \right) \delta z \right\}; \tag{110}$$

since the limits of the integral are given and are fixed, the first part vanishes of itself. And the second also vanishes identically; because

$$\Sigma . m \left(\frac{X ds}{v} - d . \frac{v dx}{ds} \right) = \Sigma . m \left(X\, dt - d . \frac{dx}{dt} \right)$$

$$= \Sigma . m \left(X - \frac{d^2 x}{dt^2} \right) dt$$

$$= 0.$$

Similarly, each of the other terms in the latter part of (110) vanishes; therefore $\delta u = 0$; and u is either a maximum or a minimum, or is constant. Although it may be a maximum and a constant in certain cases, yet it will generally be a minimum.

Since $\Sigma . m \int_0^1 v\, ds = \Sigma . \int_0^1 m v^2\, dt$, and this quantity is a minimum, the principle will be more correctly called, "the principle of least vis viva:" and we may then say, that under the existing laws of motion as expressed by the equations of motion, the vis viva acquired by the system during the time of its passage from one position to another is less than it would be under any other law of connexion between momenta impressed and momenta expressed. It is necessary that the first and last positions of the system should be given, because we have assumed the variations of the coordinates which correspond to them to vanish.

This principle of least action is useless as a method of so-

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lution of dynamical problems; because, assuming it to be true from a priori or other reasoning, it gives only the equations of motion (37), Art. 48, which are derived more satisfactorily from D'Alembert's principle; and if the variations of x, y, and z had been taken in their most general forms, which are (41), Art. 50, being due to not only a motion of translation, but also to that of rotation, we should from the principle infer the equations (38) as well as (37) of Art. 48. It is merely then a formula which includes them. The other principles, however, which we have proved in the preceding Articles, are more useful; under certain circumstances, they give us actual integrals of the equations of motion: thus, if the impressed element of work is an exact differential, the equation of vis viva is a first integral, and that from which the time may be found by a single integration. So, if no forces act on the system, or only internal forces which have equal and opposite ones, the principles of conservation of the centre of gravity and of the conservation of moments give integrals of the equations of motion.

I may however observe, that we shall generally find it the most convenient, as it is the most philosophical method, to state the equations of motion in their general forms, and to integrate them directly with the introduction of those limiting values, which are given by the conditions of the problem.

68.] Let us next consider what changes the vis viva of a system undergoes under the action of impulsive forces: and here we are arrested by a difficulty. D'Alembert's principle is applicable to such forces, which act for an infinitesimal time, and in that time impress very great velocities, because the points of the system on which they act are not sensibly displaced relatively to each other; and thus the principle of virtual velocities is applicable also to the momenta lost; whence we have

$$\Sigma.m\left\{\left(\mathrm{v}_x-\frac{dx}{dt}\right)\delta x+\left(\mathrm{v}_y-\frac{dy}{dt}\right)\delta y+\left(\mathrm{v}_z-\frac{dz}{dt}\right)\delta z\right\}=0. \quad (111)$$

If the type-particle m is moving with a velocity whose axial components are u, v, w before the instantaneous force acted on it; and with a velocity whose axial components are $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ after the force has acted, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ in (111) must be replaced by

$$\frac{dx}{dt}-u, \qquad \frac{dy}{dt}-v, \qquad \frac{dz}{dt}-w.$$

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If however a collision takes place between bodies which are more or less elastic, a change of figure takes place; and the relative position of the particles is changed; so that the conditions under which the preceding equations of motion have been found are not fulfilled. The space however through which a particle is displaced is, it may be thought, so small that we may neglect it when taken absolutely; but, as the time in which the displacement takes place is also infinitesimal, the velocity continuously varies, and the change of it cannot be neglected: thus, there will be a difference of velocities of two contiguous particles; and this is inconsistent with the immediate application of D'Alembert's principle. If however the bodies are hard and inelastic, so that no compression takes place, the particles move with the same velocity, and are at relative rest, and the principle applies: also, in the case of elastic or imperfectly elastic bodies, the bodies move with the same velocity when the compression is a maximum, and before the restitution of the figure has begun to take place. At this instant the principle of D'Alembert supplies the equations of motion. Again, when an explosion takes place, the particles in contact move with the same velocity when the explosive forces begin to act, and at that instant again the principle of D'Alembert gives the equations of motion.

I propose then to investigate the following problem: A system of material particles is in motion, and is subject to impulsive forces, either amongst each other or against fixed obstacles, of the nature of collision or explosion; what change of vis viva of the system takes place by virtue of the action of these instantaneous forces?

Firstly, let us take the case of impact and collision of the members of the system against each other or against fixed obstacles And let us take the equation of motion (111) at the instant when the compression is a maximum, because then the particles in contact are at relative rest, and in the case of impact against a fixed obstacle are at absolute rest. In these cases v_x, v_y, v_z disappear from the equation, because any momentum arising from the mutual action of the particles of the system is always accompanied by an equal and opposite reaction. Now let $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ be the axial components of the velocity v of m before collision takes place; and $\frac{dx'}{dt}, \frac{dy'}{dt}, \frac{dz'}{dt}$ the axial components of the

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velocity v' of m, when compression is a maximum; also let the actual spaces described by m along the coordinate axes, at the instant when the compression is a maximum, be the particular values of the virtual velocities δx, δy, δz; so that (111) becomes

$$\Sigma.m\left\{\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)dx'+\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)dy'+\left(\frac{dz}{dt}-\frac{dz'}{dt}\right)dz'\right\}=0; \quad (112)$$

$$\Sigma.m\left\{\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\frac{dz}{dt}\frac{dz'}{dt}\right\}=\Sigma.m\left\{\left(\frac{dx'}{dt}\right)^2+\left(\frac{dy'}{dt}\right)^2+\left(\frac{dz'}{dt}\right)^2\right\}$$
$$=\Sigma.mv'^2. \quad (113)$$

Also we have identically

$$\Sigma.m\left\{\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)^2+\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)^2+\left(\frac{dz}{dt}-\frac{dz'}{dt'}\right)^2\right\}$$
$$=\Sigma.mv^2-2\Sigma.m\left\{\frac{dx}{dt}\frac{dx'}{dt}+\frac{dy}{dt}\frac{dy'}{dt}+\frac{dz}{dt}\frac{dz'}{dt}\right\}+\Sigma.mv'^2; \quad (114)$$

therefore (113) becomes

$$\Sigma.mv^2-\Sigma.mv'^2=\Sigma.m\left\{\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)^2+\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)^2+\left(\frac{dz}{dt}-\frac{dz'}{dt}\right)^2\right\}$$
$$=\Sigma.mw^2 \text{ (say)}. \quad (115)$$

Thus the loss of vis viva during compression is the vis viva of all the particles, were each to move with a velocity equal to the excess of that at the beginning of collision over that at the end of compression: this quantity we have expressed by $\Sigma.mw^2$; and as it is a positive quantity the compression causes a loss of vis viva.

If the particles of the system are either inelastic or perfectly hard, there is no force of restitution of figure, and the velocity, at the instant when the compression is a maximum, is the permanent velocity of the system: we have

$$\Sigma.mv'^2=\Sigma.mv^2-\Sigma.mw^2.$$

This is Carnot's theorem, having been given by him first in his "Essai sur les Machines en général," Basle, 1797. See also "Principes fondamentaux de l'Equilibre et du Mouvement," Paris, 1803.

Secondly, let us suppose the instantaneous forces, by which the change of vis viva of the system takes place, to be forces of explosion; then, using the same notation as before, the virtual velocities may be replaced by dx, dy, dz, the actual displacements along the axes of the place of m, during the instant dt, before the explosion took place; then, (112) becomes in this case

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$$\Sigma.m\left\{\left(\frac{dx}{dt}-\frac{dx'}{dt}\right)dx+\left(\frac{dy}{dt}-\frac{dy'}{dt}\right)dy+\left(\frac{dz}{dt}-\frac{dz'}{dt}\right)dz\right\}=0;$$

and following the same process, we have

$$\Sigma.m\,v'^2=\Sigma.m\,v^2+\Sigma.m\,w^2; \tag{116}$$

that is, vis viva is gained by the explosion.

It is indeed evident, that if the particles of the system are at rest before the explosion, the separation of the particles must be accompanied with an increase of vis viva.

In the case now of the collision of bodies, the former of the two cases is that which occurs until the compression is a maximum, and the latter is that which occurs during the restitution of the figure. Vis viva is lost during compression, and is gained during restitution. If the bodies are perfectly elastic, so that they recover the same form that they had before collision, the increase of vis viva during the restitution will be doubtless equal to that lost during compression; and thus the vis viva of the system is the same before and after collision. If they are perfectly inelastic there is a loss during compression, and there is no gain, because there is no restitution. And if the bodies are only partially elastic, as is the case with all substances we are acquainted with, the recovery of vis viva during the restitution of the figure is not as large as the loss during compression; and consequently there is a loss of vis viva.

We have supposed the bodies to be smooth, so that the vis viva is not affected by friction. If however they are rough, and the impact is oblique in the cases of collision, terms will enter into equation (111) corresponding to friction, and our results will require modification accordingly.

SECTION 5.—*Gauss' theorem of least constraint.*

69.] IN Art. 105, Vol. III, a statical theorem is given which is there deduced from the principle of virtual velocities. It is however only a particular form of a very general theorem which includes all dynamics. It is useless for the direct solution of dynamical problems, but in the same way as the principle of least action is useless: yet, as it comprises the equations of motion given in (37) and (38) of Art. 48, and gives a new meaning to them, it deserves attention. If we could either assume its

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truth, or prove it to be true by general reasoning, we might deduce from it all the equations of motion: but it is better to take an opposite course; to state the principle, and then to shew that it is deducible from the equations of motion.

The theorem is due to Gauss, and is now called "Gauss' principle of least constraint:" it was first given in Crelle's Journal, Vol. IV, 1829; and a French translation of the memoir is inserted as a note to the 2nd volume of the Mécanique Analytique of Lagrange, edited by M. Bertrand, Paris, 1855. A full explanation of it with examples is given in "Zeitschrift für Mathematik und Physik von Schlömilch und Witzschel," III. Band, Leipzig, 1858, by Dr. Hermann Scheffler. The following is the enunciation of the theorem: If a system of material particles is in motion, under the action of given finite accelerating forces, the sum of the products of each particle and of the square of the distance between its place at the end of dt, and the place which it would have under the action of the given forces, and in the same initial circumstances, if it were free, is a minimum.

If therefore we measure constraint by the square of the distance between the actual place of m, and the place which it would have if it were under the action of the same forces and were a single unconstrained particle, then the theorem is, that the sum of the products of each particle and its constraint is a minimum.

Let the particles of the system be, as heretofore, $m_1, m_2, \ldots m_n$; of which let m be the type; at the time t let (x, y, z) be its place; let u, v, w be the axial components of the velocity of m, and let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the axial components of the impressed velocity-increments. Then, at the time $t+dt$, the coordinates of m are respectively

$$x+udt+\frac{1}{2}\frac{du}{dt}dt^2,\quad y+vdt+\frac{1}{2}\frac{dv}{dt}dt^2,\quad z+wdt+\frac{1}{2}\frac{dw}{dt}dt^2; \tag{117}$$

but if m were unconstrained, and moved with the same initial circumstances, the coordinates of its place would be

$$\left.\begin{aligned} &x+u\,dt+\frac{1}{2}\mathrm{X}\,dt^2,\\ &y+v\,dt+\frac{1}{2}\mathrm{Y}\,dt^2,\\ &z+w\,dt+\frac{1}{2}\mathrm{Z}\,dt^2;\end{aligned}\right\} \tag{118}$$

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Let us suppose $(x+\xi,\ y+\eta,\ z+\zeta)$ to be any other place which it were possible for m to take; and let U be the sum of the products of every particle, and the square of the distance between the possible place and the place which the particle would have if it were unconstrained; so that

$$\mathrm{U} = \Sigma.m\left\{(\xi - u\,dt - \frac{\mathrm{X}}{2}dt^2)^2 + (\eta - v\,dt - \frac{\mathrm{Y}}{2}dt^2)^2 + (\zeta - w\,dt - \frac{\mathrm{Z}}{2}dt^2)^2\right\}. \quad (119)$$

Then Gauss' theorem consists in the assertion that U is a minimum when the possible place is that given by the coordinates (117).

Let us differentiate (119); then, if the total differential vanishes,

$$\mathrm{D}\mathrm{U} = 0 = 2\Sigma.m(\xi - u\,dt - \frac{\mathrm{X}}{2}dt^2)d\xi + 2\Sigma.m(\eta - v\,dt - \frac{\mathrm{Y}}{2}dt^2)d\eta$$
$$+ 2\Sigma.m(\zeta - w\,dt - \frac{\mathrm{Z}}{2}dt^2)d\zeta; \quad (120)$$

and as $d\xi$, $d\eta$, $d\zeta$ are independent of each other, the coefficient of each must separately vanish; therefore

$$\left.\begin{aligned} \Sigma.m\,(\xi - u\,dt - \frac{\mathrm{X}}{2}dt^2) &= 0, \\ \Sigma.m\,(\eta - v\,dt - \frac{\mathrm{Y}}{2}dt^2) &= 0, \\ \Sigma.m\,(\zeta - w\,dt - \frac{\mathrm{Z}}{2}dt^2) &= 0. \end{aligned}\right\} \quad (121)$$

But by the equations (37), Art. 48,

$$\left.\begin{aligned} \Sigma.m\,\mathrm{X} &= \Sigma.m\frac{d^2x}{dt^2} = \Sigma.m\frac{du}{dt}, \\ \Sigma.m\,\mathrm{Y} &= \Sigma.m\frac{d^2y}{dt^2} = \Sigma.m\frac{dv}{dt}, \\ \Sigma.m\,\mathrm{Z} &= \Sigma.m\frac{d^2z}{dt^2} = \Sigma.m\frac{dw}{dt}; \end{aligned}\right\} \quad (122)$$

so that (121) become

$$\left.\begin{aligned} \Sigma.m\,(\xi - u\,dt - \frac{1}{2}\frac{du}{dt}dt^2) &= 0, \\ \Sigma.m\,(\eta - v\,dt - \frac{1}{2}\frac{dv}{dt}dt^2) &= 0, \\ \Sigma.m\,(\zeta - w\,dt - \frac{1}{2}\frac{dw}{dt}dt^2) &= 0; \end{aligned}\right\} \quad (123)$$

and therefore it is necessary that the possible place should coin-

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cide with the actual place of which the coordinates are given in (117).

Since U, in (119), is the sum of a series of positive quantities, it is evident that it does not admit of a maximum; neither generally is it a constant: it must therefore be a minimum; so that the sum of the products of each particle and its constraint is a minimum.

And that it is a minimum, we may also thus demonstrate. Differentiating again (120) we have,

$$\mathrm{D}^2\mathrm{U} = 2\,\Sigma.m\,d\xi^2 + 2\,\Sigma.m\left(\xi - u\,dt - \frac{\mathrm{X}}{2}dt^2\right)d^2\xi + \ldots + \ldots$$

$$= 2\,\Sigma.m\,(d\xi^2 + d\eta^2 + d\zeta^2),$$

by reason of (121); and this is necessarily a positive quantity; so that U is a minimum for the particular values of ξ, η, ζ given in (123).

The minimum value of U is evidently

$$\mathrm{U} = \frac{1}{4}\,\Sigma.m\left\{\left(\mathrm{X} - \frac{du}{dt}\right)^2 + \left(\mathrm{Y} - \frac{dv}{dt}\right)^2 + \left(\mathrm{Z} - \frac{dw}{dt}\right)^2\right\}dt^4. \quad (124)$$

In statics m has no velocity, so that $u = v = w = 0$; and the theorem takes the following form:

If a system of particles of invariable form is in equilibrium under the action of given pressures, and is disturbed; the sum of the products of each particle and the square of the distance between the original place, and the place which it would have in the displacement, if it had been free, is a minimum.

Let us apply this principle to the following example. Two inelastic particles, m and m', impinge on each other; it is required to find their common velocity after impact.

Let v and v' be the velocities of m and m' respectively before impact; and let V be the common velocity after impact: then, if U is the sum of the products of each mass and the square of the distance between the place which it has actually, and the place which it would have if it were unconstrained at the end of dt,

$$\mathrm{U} = m\,(v - \mathrm{V})^2\,dt^2 + m'\,(\mathrm{V} - v')^2\,dt^2;$$

$$-2m\,(v - \mathrm{V})\,dt^2 + 2\,m'\,(\mathrm{V} - v')\,dt^2 = 0,$$

if

$$m\,(v - \mathrm{V}) = m'\,(\mathrm{V} - v');$$

$$\therefore \quad \mathrm{V} = \frac{mv + m'v'}{m + m'}.$$

It is hardly necessary to observe the close analogy which exists between the principle of least constraint and the method of least

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squares. As that method gives the most likely value to quantities determined by observation, which are subject to small accidental errors, so, if we apply to dynamics similar considerations, this theorem of least constraint would assign the most plausible position to a system of constrained particles. We have however deduced this theorem from the ordinary laws of motion, and from D'Alembert's principle; from laws that are incontestably and otherwise true; so that the places which would be assigned to the particles as the most plausible by the method of least squares are proved to be their actual places.

Notwithstanding the disparaging remarks which are not unfrequently made on the principle of least action, and which Lagrange thought it worth while to reply to; and although similar remarks may be made on this theorem, inasmuch as it is, like that of least action, the expression of a metaphysical idea, yet it commends itself to every mathematician on account of its elegance and its comprehensiveness. The end of all science is a knowledge of laws which govern phænomena: and therefore a theorem which includes all mechanical phænomena, and gives a new meaning to them, and indicates that there is no waste of power, is not to be discarded as useless. In the solution of particular problems we may indeed apply less general laws: but the more general law cannot fail of exciting curiosity and of creating a desire to know the details which it contains.

SECTION 6.—*Newton's principle of similitude.*

70.] As in the present chapter we are investigating those theorems and principles which follow immediately from the equations of motion of Art. 48, and which will be, each in its own degree, applied in the following Chapters; so shall we here introduce a principle of similitude which is given by Newton in Book II, Section VII, Prop. XXXII, of the Principia; and of which a proof and some applications are given by M. Bertrand in Cahier XXXII of the Journal de l'Ecole Polytechnique, Paris, 1848.

The problem which the principle of similitude solves is this: A system of material particles of a certain form is in motion under the action of certain forces; we have a new system exactly similar, and either larger or smaller: in what proportions are

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the masses, their expressed velocities, the time of motion, and the impressed momenta or the acting forces to be changed, that is, either increased or diminished, so that the motion of the new system may be similar to that of the old system? In other words, a machine "works" on a given scale; in what proportion are its parts to be changed, that it should "work" on another given scale? A model succeeds; a machine made after the model breaks into pieces: what is the cause of this? Here is a distinguishing point between geometry and mechanics; whatever in geometry is true of a triangle on a small scale, is equally true of a triangle on a large scale: in mechanics it is not so; If a large machine is made with all its parts geometrically proportional to all the parts of a small one, the "working" of the large one cannot be inferred from that of the small one. Now the proportion in which the parts are to be changed is rightly given by Newton in the proposition above mentioned, and of which the enunciation is;

Si corporum systemata duo similia ex æquali particularum numero constent et particulæ correspondentes similes sint et proportionales, singulæ in uno systemate singulis in altero, et similiter sitæ inter se, ac datam habeant rationem densitatis ad invicem, et inter se temporibus proportionalibus similiter moveri incipiant (ea inter se quæ in uno sunt systemate, et ea inter se quæ sunt in altero), et si non tangant se mutuo quæ in eodem sunt systemate, nisi in momentis reflexionum, neque attrahant, vel fugent se mutuo nisi viribus acceleratricibus quæ sint ut particularum correspondentium diametri inversi et quadrata velocitatum directi, dico quod systematum particulæ illæ pergent inter se temporibus proportionalibus similiter moveri.

According to this theorem, to any system of particles of a certain form, the number of similar systems is infinite. A proportionality however must exist between five elements of the two systems, each to each; the lengths, the times, the velocities, the acting forces, and the masses; instead of between the lengths only, as the principle of similitude in geometry requires.

Let us take the general equation of motion (40), Art. 50, as given by the principle of virtual velocities; and modify it so far as to make X, Y, Z the impressed momentum-increments; then we have

$$\mathrm{Z}.\left\{\left(\mathrm{X}-m\frac{d^2x}{dt^2}\right)\delta x+\left(\mathrm{Y}-m\frac{d^2y}{dt^2}\right)\delta y+\left(\mathrm{Z}-m\frac{d^2z}{dt^2}\right)\delta z\right\}=0; \quad (125)$$

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and this equation, as we have shewn in that Article, includes all possible cases of motion. Suppose now that all the circumstances of motion of the system are deduced from this equation and that the place of every particle at the time t is given in terms of t. Also, let us suppose that we have a second system of particles of a form similar to the first; so that when the motion begins, the position of the particles in one system is similar and similarly situated to that of those in the other; and let both systems be subject to similar constraints, so that both have similar equations of condition; and let the equation of motion which applies to this latter system be

$$\Sigma.\left\{\left(\mathrm{X}'-m'\frac{d^2x'}{dt'^2}\right)\delta x'+\left(\mathrm{Y}'-m'\frac{d^2y'}{dt'^2}\right)\delta y'+\left(\mathrm{Z}'-m'\frac{d^2z'}{dt'^2}\right)\delta z'\right\}=0; \quad (126)$$

Let

$$\left.\begin{aligned} r &= \text{the ratio of the linear similitude in the two systems,}\\ k &= \text{the ratio of the masses of corresponding particles,}\\ \rho &= \text{the ratio of the corresponding impressed momenta,}\\ n &= \text{the ratio of the times in each,}\\ \sigma &= \text{the ratio of the velocities of corresponding particles in each;}\end{aligned}\right\} \quad (127)$$

so that

$$\left.\begin{aligned} &\frac{x'}{x}=\frac{y'}{y}=\frac{z'}{z}=r=\frac{\delta x'}{\delta x}=\frac{\delta y'}{\delta y}=\frac{\delta z'}{\delta z};\\ &\frac{\mathrm{X}'}{\mathrm{X}}=\frac{\mathrm{Y}'}{\mathrm{Y}}=\frac{\mathrm{Z}'}{\mathrm{Z}}=\rho;\\ &m'=km, \qquad t'=nt, \qquad \frac{ds'}{dt'}=\sigma\frac{ds}{dt};\end{aligned}\right\} \quad (128)$$

then it is evident that, if these quantities are substituted in (126), the result is identical with (125) if

$$\rho=\frac{kr}{n^2}; \quad (129)$$

in which equation any three quantities being given, we can determine the fourth. Thus, for instance, if all the elements of a system are changed in the ratios given in (127), the time varies directly as the square root of the linear distances, directly as the square root of the masses, and inversely as the square root of the acting force. Newton's principle of similitude consists in equation (129).

Also
$$\sigma=\frac{ds'}{dt'}\frac{dt}{ds}$$
$$=\frac{r}{n}=\left(\frac{\rho r}{k}\right)^{\frac{1}{2}}. \quad (130)$$

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Thus, in all questions of Dynamics, if the motion or "working" of any system is to be inferred from that of a similar system: if the linear distances, the times, and the masses are increased in the ratios $r : 1$, $n : 1$, $k : 1$ respectively; then the velocities, and the impressed momenta or acting forces, must be increased in the ratios respectively of $r : n$ and of $kr : n^2$. Let us take the following case of this: A large locomotive works; what conditions are necessary that a small similar locomotive should also work?

Let the ratio of the linear dimensions of the parts in the small machine be to that in the larger machine as $r : 1$, where r is a proper fraction; then the ratio of the masses is as $r^3 : 1$; so that $k = r^3$; also, since gravity acts in both cases, and since the ratio of the weights of the several parts are as the masses, therefore the ratio of the forces is as $r^3 : 1$; and this is to be constant with all the forces, so that $\rho = r^3$: thus, from (129), $n = r^{\frac{1}{2}}$; and, from (130), $\sigma = r^{\frac{1}{2}}$: thus the times and the velocities will both be in the ratio $r^{\frac{1}{2}} : 1$. Let us consider other forces: the pressure of steam on the piston will be diminished in the ratio of $r^3 : 1$. As the resistance of the air varies as the square of the velocity and as the area of the surface, it is diminished in the ratio of $r^3 : 1$. Sliding frictions, being proportional to the pressures, are diminished in the same ratio. Rolling frictions, which are found to vary directly as the pressures, and inversely as the radii of the rolling wheels, will be diminished in the ratio of $r^2 : 1$. Thus, in a small machine, this ratio may be very great.

I subjoin two simple examples of this principle of similitude; others will be found in the Memoir of M. Bertrand, which has already been alluded to.

Ex. 1. Two equal particles at rest are attracted towards a centre of force, which varies directly as the distance; to prove that, whatever are their initial distances, they arrive at the centre simultaneously, and that their distances at any instant from the centre are proportional to their initial distances.

Let x_0 and x'_0 be the initial distances of the particles from the centre of force; so that $x'_0 = rx_0$: also, if μx is the attractive force, then $\mu x'_0 = \rho \mu x_0$; therefore $\rho = r$; also $k = 1$; so that (129) gives $n = 1$; therefore the times are equal in the two cases and both arrive simultaneously at the centre of force. And as the systems are similar at first, they are similar through-

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out the motion; and thus, if x and x' are the distances of the particles from the centre of force at the time t,

$$\frac{x}{x_0} = \frac{x'}{x_0'};$$

which is the second part of the theorem.

Ex. 2. Two simple pendulums, with equal weights, whose lengths are l and l' respectively, are moved from their vertical position through equal angles, and are under the action of the gravities g and g' respectively; it is required to compare the times of oscillation.

These are evidently two similar systems: $l' = rl$, $g' = \rho g$, $k = 1$.

$$\therefore \quad n^2 = \frac{r}{\rho} = \frac{l'g}{lg'};$$

therefore, if τ and τ' are the times of oscillation,

$$\tau : \tau' :: \left(\frac{l}{g}\right)^{\frac{1}{2}} : \left(\frac{l'}{g'}\right)^{\frac{1}{2}};$$

which is the well known result.

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CHAPTER IV.

THE EQUATIONS OF MOTION OF A RIGID BODY EXPRESSED IN TERMS OF ANGULAR VELOCITIES AND THEIR INCREMENTS. PRINCIPAL AXES AND MOMENTS OF INERTIA.

SECTION 1.—*The transformation of the equations of motion.*

71.] THE most general motion of a system of material particles of invariable form may be, as we have already proved, resolved into a motion of translation of any point, and a motion of rotation about an axis passing through that point. Generally the position and direction of that axis undergoes a continual change, and the axis may be considered to be constant during only an infinitesimal time-element dt; for it is only in a few cases that the axis is fixed during the whole motion.

From the nature of angular velocities, which have been explained in Chapter II, it is evident that they admit of increase and decrease, either continuously or discontinuously; and, in the general motion of a body, there will generally be a continuous variation of angular velocity, whether the rotation-axis is permanently fixed, or has the same position for only an infinitesimal time-element; and the angular velocity may either increase or decrease. Now this change of angular velocity cannot take place without the action of some force of which it is the effect: a rotating body can no more increase or diminish its angular velocity, than a material particle increase or diminish its linear velocity: this fact is involved in the law of inertia of matter: whenever a change takes place, some force acts to produce that change; and the relation between the change of angular velocity and the producing forces will be the subject of our inquiry in the present Chapter. We shall demonstrate the relation indirectly at first, by a transformation of the preceding equations of motion: but we shall introduce direct proofs as occasion arises in the course of our inquiry. Thus, while the process of transformation will enable us to conduct our treatise in a systematic form, the direct proofs will remove all intermediate operations, shew the close dependence of our results on first principles, and thus enable us to view the relations as they are in themselves. Thus, as I conceive, we have the advantage of both the

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analytical and the synthetical processes, of which such admirable examples are given respectively in the Mécanique Analytique of Lagrange, and the Nouvelle Théorie de Rotation of Poinsot.

72.] As, for the most part, we consider changes of angular velocities during an infinitesimal time only, the position of the rotation-axis, whether in the body or in space, will be unchanged during that time; and, if the rotation-axis is unchanged during a finite time, the change in angular velocity will be found by integration. In all cases therefore, during the change of the angular velocity, we shall suppose the position of the rotation-axis to be fixed.

If, by the action of an impulsive force, the angular velocity of a body is abruptly changed, or if a body at rest receives a finite angular velocity, we consider only the whole velocity which is communicated to be the effect of the force: we do not inquire into the law of communication, which would assign the rate at which, in successive time-elements, the communication took place, but as the whole process is completed in an infinitesimal time, we take the whole at once.

If however a finite force acts, whereby the angular velocity of the body about the given axis continuously varies, then there are two cases to be considered, according as equal or unequal angular velocities are communicated (or abstracted) in equal time-elements; these two cases corresponding to those of a constant and of a variable force in the linear motion of material particles respectively. Let us first take the case in which equal angular velocities are impressed in equal times. Let Φ be the angular velocity impressed in an unit of time; and let ω be the angular velocity impressed and also expressed in t units of time: therefore

$$\omega = \Phi t. \tag{1}$$

As equal angular velocities are impressed in equal times, and as Φ is the measure of them, Φ is called a constant angular force.

If the body moves with an angular velocity Ω, before the force Φ acts, and if ω is its angular velocity, when Φ has acted for t units of time,

$$\omega = \Omega + \Phi t; \tag{2}$$

and if Φ acts in a direction contrary to that of Ω,

$$\omega = \Omega - \Phi t. \tag{3}$$

Now from Art. 22, equation (4), $\dfrac{d\theta}{dt} = \omega$; so that, if θ is the

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angle through which the body rotates in the time t, and if $\theta = 0$, when $t = 0$, then generally from (2)

$$\frac{d\theta}{dt} = \Omega + \Phi t;$$

$$\therefore \quad \theta = \Omega t + \frac{1}{2}\Phi t^2; \tag{4}$$

which gives the angle through which the body rotates in the time t under the action of a constant angular force Φ.

Next let us suppose unequal angular velocities to be impressed in equal time-elements; then the force is called a variable angular force. Let us however suppose it to be such at the time t, that an angular velocity Φ would be impressed by it in an unit of time, if the force were constant during that unit of time; and to be such at the time $t + dt$, that an angular velocity $\Phi + d\Phi$ would be impressed by it under the same supposition as to constancy; let ω be the angular velocity at the time t, and $\omega + d\omega$ at the time $t + dt$; then, if e is the symbol for a proper fraction, $\Phi + e\,d\Phi$ will express the *mean* or average value of the impressed angular velocity due to an unit of time during dt; that is, during dt, $\Phi + e d\Phi$ is the mean constant angular force; and as $d\omega$ is the angular velocity expressed in dt, we have from (2)

$$d\omega = (\Phi + e\,d\Phi)\,dt;$$

and neglecting $d\Phi \times dt$, which is an infinitesimal of the second order,

$$d\omega = \Phi\,dt; \tag{5}$$

$$\therefore \quad \Phi = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\theta}{dt}$$

$$= \frac{d^2\theta}{dt^2}, \tag{6}$$

if t is equicrescent; and this supposition we shall make throughout the treatise, unless it is expressly stated that t is not equicrescent.

Hence, if Φ is given in terms of either θ or t, we can deduce from (6) by means of two integrations the relation between θ and t, and thus determine the angle through which the body rotates in the time t.

As $d\omega$ is an angular velocity, although it is infinitesimal, it is capable of resolution and composition according to the laws which have been investigated in Chapter II. This observation is important.

Let thus much suffice for angular velocity-increments; we return to the equations for rotatory motion which have been found

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in Art. 48, with the purpose of expressing them in terms of angular velocities, and angular velocity-increments.

73.] Let us first take equations (35), Art. 48; and replace the linear velocities in them in terms of the angular velocities about the three coordinate axes, these angular velocities being due to the acting forces.

Let us take *any* point of the body for the origin; and let three rectangular axes fixed in space originate at it: the origin we may consider to be fixed, while we calculate the rotation about an axis through it. We will assume the body to be initially at rest. Let α, β, γ be the direction-angles of the rotation-axis; let Ω be the angular velocity due to the acting forces; let Ω_x, Ω_y, Ω_z be the axial components of Ω; then

$$\left.\begin{aligned} \Omega_x = \Omega\cos\alpha, \qquad \Omega_y = \Omega\cos\beta, \qquad \Omega_z = \Omega\cos\gamma;\\ \Omega^2 = \Omega_x^2 + \Omega_y^2 + \Omega_z^2. \end{aligned}\right\} \tag{7}$$

Let L, M, N be the moments of the axial components of the couple of the impressed momenta; so that

$$\left.\begin{aligned} \mathrm{L} &= \Sigma.m\,(y\,\mathrm{v}_z - z\,\mathrm{v}_y),\\ \mathrm{M} &= \Sigma.m\,(z\,\mathrm{v}_x - x\,\mathrm{v}_z),\\ \mathrm{N} &= \Sigma.m\,(x\,\mathrm{v}_y - y\,\mathrm{v}_x); \end{aligned}\right\} \tag{8}$$

then from (35), Art. 48, we have

$$\left.\begin{aligned} \Sigma.m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) &= \mathrm{L},\\ \Sigma.m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) &= \mathrm{M},\\ \Sigma.m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) &= \mathrm{N}. \end{aligned}\right\} \tag{9}$$

Now, by (72) Art. 38, we have the following values for the axial components of the linear velocity of m, which is due to the angular velocity Ω,

$$\left.\begin{aligned} \frac{dx}{dt} &= z\,\Omega_y - y\,\Omega_z = \Omega\,(z\cos\beta - y\cos\gamma),\\ \frac{dy}{dt} &= x\,\Omega_z - z\,\Omega_x = \Omega\,(x\cos\gamma - z\cos\alpha),\\ \frac{dz}{dt} &= y\,\Omega_x - x\,\Omega_y = \Omega\,(y\cos\alpha - x\cos\beta); \end{aligned}\right\} \tag{10}$$

then substituting in (9), we have

$$\left.\begin{aligned} \Omega\{\ \cos\alpha\,\Sigma.m(y^2+z^2) - \cos\beta\,\Sigma.mxy - \cos\gamma\,\Sigma.mzx\} &= \mathrm{L},\\ \Omega\{-\cos\alpha\,\Sigma.mxy + \cos\beta\,\Sigma.m(z^2+x^2) - \cos\gamma\,\Sigma.myz\} &= \mathrm{M},\\ \Omega\{-\cos\alpha\,\Sigma.mzx - \cos\beta\,\Sigma.myz + \cos\gamma\,\Sigma.m(x^2+y^2)\} &= \mathrm{N}; \end{aligned}\right\} \tag{11}$$

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Ω, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ having being placed outside the summatory symbols, because they are the same for all particles of the body. And these three equations are, in terms of the resultant angular velocity and the direction-angles of the rotation-axis, the equivalents of (35), Art. 48; and by these the angular velocity Ω and the position of the instantaneous rotation-axis are to be determined.

Let us multiply them severally by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and add; then

$$\Omega\,\Sigma.m\,\{(y^2+z^2)(\cos\alpha)^2+(z^2+x^2)(\cos\beta)^2+(x^2+y^2)(\cos\gamma)^2$$
$$-2yz\cos\beta\cos\gamma-2zx\cos\gamma\cos\alpha-2xy\cos\alpha\cos\beta\}$$
$$= \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (12)$$

which may be expressed as follows;

$$\Omega\,\Sigma.m\,\{(z\cos\beta-y\cos\gamma)^2+(x\cos\gamma-z\cos\alpha)^2+(y\cos\alpha-x\cos\beta)^2\}$$
$$= \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (13)$$

and if r is the perpendicular distance from (x, y, z), the place of m, to the rotation-axis,

$$r^2=(z\cos\beta-y\cos\gamma)^2+(x\cos\gamma-z\cos\alpha)^2+(y\cos\alpha-x\cos\beta)^2; \quad (14)$$

so that (13) becomes

$$\Omega.\Sigma.mr^2 = \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (15)$$

$$\therefore\quad \Omega = \frac{\text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma}{\Sigma.mr^2}; \quad (16)$$

which gives the angular velocity about the instantaneous axis.

74.] The right-hand member of this equation requires accurate and close examination. The numerator of the fraction is the moment of the couple of the impressed momenta of all the particles about the rotation-axis; for L, M, N are the moments of the axial components of the couples of the impressed momenta, and the numerator is the sum of the parts of those axial components which are effective about the rotation-axis. The denominator is the sum of the products of every moving particle and the square of its distance from the rotation-axis: and in the case of a continuous body it becomes the integral of r^2dm, the integration extending over and including all the mass-elements of the body. This quantity is called the *moment of inertia* of the body or of the moving system, relatively to the particular rotation-axis, and the geometrical definition of it is that just given. It appears also from (15) that it is the factor by which the angular velocity Ω is multiplied, and thus equated to the

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moment of the couple of the impressed momenta about the rotation-axis. This last is the dynamical definition of it.

The name "moment of inertia" has been given for the following reason. Let us compare (15) with the fundamental theory of Art. 210, Vol. III, of the motion of *translation* of a material particle m, which is acted on by an impulsive force. It appears that if v is the expressed velocity of m, and if x is the momentum impressed by the instantaneous force, then

$$m\mathrm{v} = \mathrm{x}; \tag{17}$$

so that m, which symbolises the mass, is the factor by which v is multiplied, and so equated to the impressed momentum; and as in (15) $\Sigma . m r^2$ is the factor by which n is multiplied, and thus equated to the moment of the impressed momentum, so the old mechanicians compared the m in (17) with the $\Sigma . m r^2$ in (15); and as they were wont to say that a body's inertia was proportional to or identical with its mass, so, by an analogy somewhat rough, did they call $\Sigma . m r^2$ the moment of inertia. It seems difficult to demonstrate the correctness of the term; but as it is undesirable to introduce a new name, except by urgent necessity, I shall retain the old one, and call $\Sigma . m r^2$ the moment of inertia of the body or system of particles relatively to the rotation-axis. The determination of this quantity is evidently the first step in the solution of a problem which depends on the equation (16); and is otherwise of great importance. Hereafter many properties of moments of inertia will be investigated, and I shall calculate the moments of inertia of bodies and moving systems in many particular cases.

Sometimes the moment of inertia is expressed in the following manner: Let M be the mass of the moving system, and let us suppose the whole system to be condensed into a particle of mass M, at a distance k from the rotation-axis, so that the moment of inertia of the system thus condensed may be the same relatively to the axis as that of the moving system: then, as the moment of inertia of M in this imaginary and condensed state is $\mathrm{M}\, k^2$, so by our assumption,

$$\mathrm{M}\, k^2 = \Sigma . r^2\, dm; \tag{18}$$

k is called the radius of gyration of the body relatively to the particular rotation-axis.

Hence, if a continuous body is referred to three rectangular axes in space, and if ρ is the density of the particle at (x, y, z),

$$dm = \rho\, dx\, dy\, dz, \tag{19}$$

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and the moments of inertia of the body relatively to the three coordinate axes of x, y, and z are severally

$$\iiint \rho(y^2+z^2)\,dx\,dy\,dz,\ \iiint \rho(z^2+x^2)\,dx\,dy\,dz,\ \iiint \rho(x^2+y^2)\,dx\,dy\,dz;$$

the integrals being definite, and including all the elements of the body.

75.] As (16) is the fundamental equation of rotation of a body under the action of instantaneous forces, it is worth while to deduce it immediately from the first principles of motion.

For the sake of simplicity, let us take the rotation-axis to be the coordinate axis of z, and suppose the line of action of the impressed momentum to be in a plane perpendicular to this axis. Let $m\,\text{V}$ be the momentum impressed on m at the place (x, y, z); of which let N be the moment of the couple about the axis of z; let r be the distance of m from the axis of z; and let Ω be the expressed angular velocity, and v the expressed velocity due to the instantaneous force; so that

$$v = \frac{ds}{dt} = r\,\Omega; \tag{20}$$

hence the expressed momentum is $m\dfrac{ds}{dt} = m r\,\Omega$, of which the moment, relatively to the rotation-axis, is $m r^2 \Omega$; so that the excess of the moment of the couple of the impressed momentum over that of the expressed momentum in the case of the particle m is

$$\text{N} - m r^2 \Omega;$$

and as by D'Alembert's principle these taken throughout the moving mass are in equilibrium, we have

$$\Sigma.\text{N} - \Sigma.m\,r^2\,\Omega = 0. \tag{21}$$

As Ω is the same for all mass-elements, it may be placed outside the sign of summation; also let G be the moment of the couple of the impressed momenta of all the particles, then we have

$$\Omega\,.\,\Sigma.m\,r^2 = \text{G}; \tag{22}$$

$$\therefore\quad \Omega = \frac{\text{G}}{\Sigma.m\,r^2}; \tag{23}$$

which is the same equation as (16); for if the axis of z is the rotation axis, then in (16) $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$; and we have

$$\Omega = \frac{\text{N}}{\Sigma.m\,r^2}; \tag{24}$$

where N is the moment of the couple of the impressed momenta about the rotation-axis, and is the same as G in (23).

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Equations (11) are so close on the first principles of motion, as explained in Art. 38, and of the measure of couples, that further explanation is unnecessary.

76.] The direction-cosines of the instantaneous-axis are proportional to $\Omega_x, \Omega_y, \Omega_z$; and these latter quantities may be thus found:

Let us once for all make the following abbreviations; let

$$\Sigma.m(y^2+z^2) = \text{A}, \quad \Sigma.m(z^2+x^2) = \text{B}, \quad \Sigma.m(x^2+y^2) = \text{C}; \quad (25)$$

$$\Sigma.mx^2 = \text{A}', \quad \Sigma.my^2 = \text{B}', \quad \Sigma.mz^2 = \text{C}'; \quad (26)$$

$$\Sigma.myz = \text{D}, \quad \Sigma.mzx = \text{E}, \quad \Sigma.mxy = \text{F}. \quad (27)$$

These nine quantities are of great importance in the following investigations, and the substitutions which are here made will be continued throughout the treatise.

A, B, C are the sums of the products of each particle of the moving mass and the square of its distance from the axes of x, y, z respectively: in other words, A, B, C are the moments of inertia of the moving system relatively to the axes of x, y, z respectively.

A′, B, C′ are severally the sum of the products of each particle and the square of the x-, y-, z- coordinate of its place.

D, E, F will have full explanation in the following section, although (27) evidently exhibit their meaning.

I may observe, that

$$\text{A} = \text{B}'+\text{C}', \quad \text{B} = \text{C}'+\text{A}', \quad \text{C} = \text{A}'+\text{B}'; \quad (28)$$

$$\text{A}' = \frac{\text{B}+\text{C}-\text{A}}{2}, \quad \text{B}' = \frac{\text{C}+\text{A}-\text{B}}{2}, \quad \text{C}' = \frac{\text{A}+\text{B}-\text{C}}{2}; \quad (29)$$

whereby A, B, C are severally determined in terms of A′, B′, C′; and A′, B′, C′ in terms of A, B, C. Also

$$\text{A}+\text{A}' = \text{B}+\text{B}' = \text{C}+\text{C}' = \text{A}'+\text{B}'+\text{C}'. \quad (30)$$

Now, using in (11) these abbreviating symbols, we have, by means of (7),

$$\left.\begin{aligned} \text{A}\,\Omega_x - \text{F}\,\Omega_y - \text{E}\,\Omega_z &= \text{L}, \\ -\text{F}\,\Omega_x + \text{B}\,\Omega_y - \text{D}\,\Omega_z &= \text{M}, \\ -\text{E}\,\Omega_x - \text{D}\,\Omega_y + \text{C}\,\Omega_z &= \text{N}; \end{aligned}\right\} \quad (31)$$

whence

$$\left.\begin{aligned} \Omega_x &= \frac{\text{L}(\text{BC}-\text{D}^2) - \text{M}(\text{DE}+\text{CF}) - \text{N}(\text{DF}+\text{BE})}{\text{ABC} - \text{AD}^2 - \text{BE}^2 - \text{CF}^2 - 2\,\text{DEF}}, \\ \Omega_y &= \frac{-\text{L}(\text{ED}+\text{CF}) + \text{M}(\text{CA}-\text{E}^2) - \text{N}(\text{EF}+\text{AD})}{\text{ABC} - \text{AD}^2 - \text{BE}^2 - \text{CF}^2 - 2\,\text{DEF}}, \\ \Omega_z &= \frac{-\text{L}(\text{FD}+\text{BE}) - \text{M}(\text{FE}+\text{AD}) + \text{N}(\text{AB}-\text{F}^2)}{\text{ABC} - \text{AD}^2 - \text{BE}^2 - \text{CF}^2 - 2\,\text{DEF}}; \end{aligned}\right\} \quad (32)$$

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from which, and from (7), α, β, γ may be determined. The equations to the instantaneous-axis are

$$\frac{x}{\Omega_x} = \frac{y}{\Omega_y} = \frac{z}{\Omega_z}. \tag{33}$$

We shall hereafter have geometrical interpretations of these results.

77.] Next let us consider equations (38), Art. 48. Let any point of the body be the origin; and at it let three rectangular coordinate axes fixed in space originate; and let us consider the body at the time t, and during dt, so that the rotation-axis may be considered fixed during that time. Let α, β, γ be the direction-angles of the rotation-axis, and let ω be the angular velocity about it at the time t; of which let ω_x, ω_y, ω_z be the axial-components; so that

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2; \tag{34}$$

$$\omega_x = \omega \cos \alpha, \qquad \omega_y = \omega \cos \beta, \qquad \omega_z = \omega \cos \gamma. \tag{35}$$

Let the moments of the axial-components of the couples of the impressed momentum-increments at the time t be L, M, N; so that from (38), Art. 48, we have

$$\left.\begin{aligned} \Sigma . m\,(y\,\mathrm{Z} - z\,\mathrm{Y}) &= \mathrm{L}, \\ \Sigma . m\,(z\,\mathrm{X} - x\,\mathrm{Z}) &= \mathrm{M}, \\ \Sigma . m\,(x\,\mathrm{Y} - y\,\mathrm{X}) &= \mathrm{N}; \end{aligned}\right\} \tag{36}$$

and

$$\left.\begin{aligned} \Sigma . m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= \mathrm{L}, \\ \Sigma . m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= \mathrm{M}, \\ \Sigma . m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= \mathrm{N}. \end{aligned}\right\} \tag{37}$$

Now, by (72), Art. 38, for the linear velocity of m at (x, y, z), which is due to the angular velocity ω, we have

$$\left.\begin{aligned} \frac{dx}{dt} &= \omega\,(z \cos \beta - y \cos \gamma), \\ \frac{dy}{dt} &= \omega\,(x \cos \gamma - z \cos \alpha), \\ \frac{dz}{dt} &= \omega\,(y \cos \alpha - x \cos \beta); \end{aligned}\right\} \tag{38}$$

and therefore for the increment of the linear velocity, which is due to the increment of angular velocity, we have

$$\frac{d^2x}{dt^2} = \frac{d\omega}{dt}\,(z \cos \beta - y \cos \gamma) + \omega\left(\cos \beta \frac{dz}{dt} - \cos \gamma \frac{dy}{dt}\right); \tag{39}$$

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$$\left.\begin{aligned}
\therefore\ \frac{d^2x}{dt^2} &= \frac{d\omega}{dt}(z\cos\beta - y\cos\gamma) + \omega^2\cos a(x\cos a + y\cos\beta + z\cos\gamma) - \omega^2 x. \\
&\text{Similarly,} \\
\frac{d^2y}{dt^2} &= \frac{d\omega}{dt}(x\cos\gamma - z\cos a) + \omega^2\cos\beta(x\cos a + y\cos\beta + z\cos\gamma) - \omega^2 y, \\
\frac{d^2z}{dt^2} &= \frac{d\omega}{dt}(y\cos a - x\cos\beta) + \omega^2\cos\gamma(x\cos a + y\cos\beta + z\cos\gamma) - \omega^2 z.
\end{aligned}\right\} \quad (40)$$

Therefore, substituting in (37), and placing ω and $\frac{d\omega}{dt}$ outside the sign of summation, as they, as well as the direction-cosines of the rotation-axis, are the same for all the particles, we have

$$\frac{d\omega}{dt}\cos a\,\Sigma.m(x^2+y^2+z^2) - \frac{d\omega}{dt}\Sigma.mx(x\cos a + y\cos\beta + z\cos\gamma)$$
$$+\omega^2\Sigma.m\{(x\cos a + y\cos\beta + z\cos\gamma)(y\cos\gamma - z\cos\beta)\} = \text{L}; \quad (41)$$

$$\frac{d\omega}{dt}\cos\beta\,\Sigma.m(x^2+y^2+z^2) - \frac{d\omega}{dt}\Sigma.my(x\cos a + y\cos\beta + z\cos\gamma)$$
$$+\omega^2\Sigma.m\{(x\cos a + y\cos\beta + z\cos\gamma)(z\cos a - x\cos\gamma)\} = \text{M}; \quad (42)$$

$$\frac{d\omega}{dt}\cos\gamma\,\Sigma.m(x^2+y^2+z^2) - \frac{d\omega}{dt}\Sigma.mz(x\cos a + y\cos\beta + z\cos\gamma)$$
$$+\omega^2\Sigma.m\{(x\cos a + y\cos\beta + z\cos\gamma)(x\cos\beta - y\cos a)\} = \text{N}. \quad (43)$$

The complete solution of the problem requires that ω, $\cos a$, $\cos\beta$, $\cos\gamma$ should be expressed in terms of t; L, M, and N being functions of these five quantities: now, as a relation exists between a, β, γ, the preceding equations contain only three independent quantities which are to be expressed in terms of t; for this purpose the number of the equations is sufficient; but as they generally do not admit of integration, we can apply them only to particular cases, and have recourse to such artifices as a particular problem suggests.

78.] Let us multiply (41), (42), and (43), severally, by $\cos a$, $\cos\beta$, $\cos\gamma$, and let us add them; then

$$\frac{d\omega}{dt}\Sigma.m\{x^2+y^2+z^2-(x\cos a + y\cos\beta + z\cos\gamma)^2\}$$
$$= \text{L}\cos a + \text{M}\cos\beta + \text{N}\cos\gamma; \quad (44)$$

but if r is the perpendicular distance from (x, y, z), the place of m at the time t, on the rotation-axis,

$$r^2 = x^2+y^2+z^2-(x\cos a + y\cos\beta + z\cos\gamma)^2; \quad (45)$$

so that (44) becomes

$$\frac{d\omega}{dt}\Sigma.mr^2 = \text{L}\cos a + \text{M}\cos\beta + \text{N}\cos\gamma; \quad (46)$$

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$$\therefore \quad \frac{d\omega}{dt} = \frac{\text{L} \cos \alpha + \text{M} \cos \beta + \text{N} \cos \gamma}{\Sigma . m r^2}; \tag{47}$$

which gives the angular velocity-increment about the rotation-axis which is due to the impressed momentum-increments.

Now this equation, like (16), requires careful attention; it is that from which, by integration, the increase or diminution of the angular velocity in a finite time is to be found. The numerator of the right-hand member is the moment of the couple of the impressed momentum-increments of all the particles relatively to the rotation-axis; for L, M, N are the axial components of the moments of these couples; and $\text{L} \cos \alpha + \text{M} \cos \beta + \text{N} \cos \gamma$ is the sum of the parts of these axial components which are effective about the rotation-axis. The denominator is the moment of inertia of the body or moving mass relatively to the rotation-axis; and the remarks made in Art. 74 are applicable equally to that and this case.

79.] If the rotation-axis of the body has the same position during the whole motion, either because two or more points in it are fixed, or because it bears a certain relation to the particles of the moving mass, then α, β, γ are constant, and are known, and the numerator of (47) is given at the time t; and, if the integration can be performed, the angular velocity will be determined. If, however, the position of the rotation-axis changes continuously from time to time, so that it can be considered fixed only for an infinitesimal time-element dt, then α, β, γ are functions of t, and equation (47) cannot generally be integrated as it stands. In this case we must return to equations (41), (42), (43); in them let us replace $\omega \cos \alpha$, $\omega \cos \beta$, $\omega \cos \gamma$, severally, by ω_x, ω_y, ω_z, and use the abbreviating symbols of Art. 76; then,

$$\text{A} \frac{d\omega_x}{dt} + (\text{C} - \text{B})\, \omega_y \omega_z - \text{D}\, (\omega_y^2 - \omega_z^2) - \text{E} \left(\frac{d\omega_z}{dt} + \omega_x \omega_y \right) - \text{F} \left(\frac{d\omega_y}{dt} - \omega_x \omega_z \right) = \text{L}; \tag{48}$$

$$\text{B} \frac{d\omega_y}{dt} + (\text{A} - \text{C})\, \omega_z \omega_x - \text{E}\, (\omega_z^2 - \omega_x^2) - \text{F} \left(\frac{d\omega_x}{dt} + \omega_y \omega_z \right) - \text{D} \left(\frac{d\omega_z}{dt} - \omega_y \omega_x \right) = \text{M}; \tag{49}$$

$$\text{C} \frac{d\omega_z}{dt} + (\text{B} - \text{A})\, \omega_x \omega_y - \text{F}\, (\omega_x^2 - \omega_y^2) - \text{D} \left(\frac{d\omega_y}{dt} + \omega_z \omega_x \right) - \text{E} \left(\frac{d\omega_x}{dt} - \omega_z \omega_y \right) = \text{N}; \tag{50}$$

from which three equations ω_x, ω_y, ω_z are to be determined in terms of t: the integration however is beyond our power, except in a very few special cases, which we shall consider hereafter.

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80.] As these last are the fundamental equations of rotation of a solid body, or of a material system of invariable form, and will be employed in all our subsequent investigations, they require close examination. We have arrived at them by transformations from expressions involving velocities of translation into those involving angular velocities. I will now shew that they may be found more directly by D'Alembert's principle: and, in the course of the inquiry, we shall dissect the equations, and shew the independent origin of their several terms; and I shall also exhibit other properties of these equations of rotating rigid systems besides those of the preceding pages.

As the particles of the moving system are in a state of relative rest, the moments of the forces acting on them, relatively to every and any axial line, satisfy the conditions of statical equilibrium; and thus, by D'Alembert's principle, the moments of the tensions or strains which arise from the excess of the impressed over the expressed momentum-increments must satisfy the laws of equilibrium when they are taken throughout the whole system. In reference to any axis for any one particle m we have the following moments: (1) the moment of the impressed momentum-increments; (2) the moment of the expressed momentum-increment; (3) the moment of the centrifugal force which is due to the motion of the body about the instantaneous axis at the time t; and the moment of the tension, which is effective at m, is the excess of the moment of the impressed momentum-increment and of the centrifugal force over the moment of the expressed momentum-increment; and the moment of all these tensions vanishes for every axis.

Let us employ the same notation as heretofore; m is the type-particle of the system; (x, y, z) is its place at the time t; ω is the angular velocity at the time t about the instantaneous axis, of which ω_x, ω_y, ω_z are the axial components; α, β, γ are the direction-angles of the instantaneous axis; r is the perpendicular distance from (x, y, z) on the instantaneous axis; L, M, N are the axial components of the moments of the couples of the impressed momentum-increments on all the particles of the system.

81.] Let X′, Y′, Z′ be the axial components of the expressed momentum-increments of all the particles of the system due to the increments of the angular velocities at the time t; and let P′ be the resultant of these; then, from Art. 38, we have

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$$\left.\begin{aligned} \Sigma.m\left(z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}\right) &= \mathrm{X}', \\ \Sigma.m\left(x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}\right) &= \mathrm{Y}', \\ \Sigma.m\left(y\frac{d\omega_x}{dt}-x\frac{d\omega_y}{dt}\right) &= \mathrm{Z}'; \end{aligned}\right\} \tag{51}$$

and
$$\mathrm{P}'^2 = \mathrm{X}'^2+\mathrm{Y}'^2+\mathrm{Z}'^2. \tag{52}$$

If the origin moves, P' is proportional to the expressed velocity-increment of it, which is due to the increments of the angular velocities at the time t; and if the origin is absolutely fixed, it is the increment of pressure on it during the time dt.

Let L', M', N' be the axial components of the moments of the couples which arise from these expressed momentum-increments; and let G' be the resultant moment of these; then, as the axial components of the expressed momentum-increment of m are severally,

$$m\left(z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}\right),\quad m\left(x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}\right),\quad m\left(y\frac{d\omega_x}{dt}-x\frac{d\omega_y}{dt}\right), \tag{53}$$

so the axial components of the moment of the couple of this expressed-momentum of m are respectively

$$\left.\begin{aligned} &\frac{d\omega_x}{dt}m(y^2+z^2)-\frac{d\omega_y}{dt}mxy-\frac{d\omega_z}{dt}mxz, \\ &\frac{d\omega_y}{dt}m(z^2+x^2)-\frac{d\omega_z}{dt}myz-\frac{d\omega_x}{dt}myx, \\ &\frac{d\omega_z}{dt}m(x^2+y^2)-\frac{d\omega_x}{dt}mzx-\frac{d\omega_y}{dt}mzy; \end{aligned}\right\} \tag{54}$$

and taking the aggregates of these for all the particles of the system, and using the abridging symbols of Art. 76, we have

$$\left.\begin{aligned} \mathrm{A}\frac{d\omega_x}{dt}-\mathrm{F}\frac{d\omega_y}{dt}-\mathrm{E}\frac{d\omega_z}{dt} &= \mathrm{L}', \\ \mathrm{B}\frac{d\omega_y}{dt}-\mathrm{D}\frac{d\omega_z}{dt}-\mathrm{F}\frac{d\omega_x}{dt} &= \mathrm{M}', \\ \mathrm{C}\frac{d\omega_z}{dt}-\mathrm{E}\frac{d\omega_x}{dt}-\mathrm{D}\frac{d\omega_y}{dt} &= \mathrm{N}'; \end{aligned}\right\} \tag{55}$$

$$\mathrm{G}'^2 = \mathrm{L}'^2+\mathrm{M}'^2+\mathrm{N}'^2; \tag{56}$$

which are indeed the same expressions as (31), Art. 76, and give the moment of the couple of the expressed momentum-increments of the system.

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82.] Let X'', Y'', Z'' be the axial components of the momentum-increments of all the particles which arise from the centrifugal forces; and let P'' be their resultant. As ω is the angular velocity of the system about the instantaneous axis at the time t, and as r is the perpendicular distance from (x, y, z) the place of m on that axis, $m\omega^2 r$ is the centrifugal force of m, the line of action of which lies along r. Now as r is drawn from (x, y, z) at right angles to (α, β, γ) which passes through the origin, the direction-cosines of r are

$$\frac{x-\cos\alpha(x\cos\alpha+y\cos\beta+z\cos\gamma)}{r}, \quad \frac{y-\cos\beta(x\cos\alpha+y\cos\beta+z\cos\gamma)}{r},$$

$$\frac{z-\cos\gamma(x\cos\alpha+y\cos\beta+z\cos\gamma)}{r}; \quad (57)$$

and therefore the axial components of the momentum-increments of m, due to the centrifugal force, are respectively

$$\left.\begin{aligned} &m\omega^2\{x-\cos\alpha(x\cos\alpha+y\cos\beta+z\cos\gamma)\},\\ &m\omega^2\{y-\cos\beta(x\cos\alpha+y\cos\beta+z\cos\gamma)\},\\ &m\omega^2\{z-\cos\gamma(x\cos\alpha+y\cos\beta+z\cos\gamma)\},\end{aligned}\right\} \quad (58)$$

the tendency of these forces being to increase x, y, z as t increases: and, taking the aggregate of these for all the particles of the system,

$$\left.\begin{aligned} &\Sigma.m\omega^2\{x-\cos\alpha(x\cos\alpha+y\cos\beta+z\cos\gamma)\} = X'',\\ &\Sigma.m\omega^2\{y-\cos\beta(x\cos\alpha+y\cos\beta+z\cos\gamma)\} = Y'',\\ &\Sigma.m\omega^2\{z-\cos\gamma(x\cos\alpha+y\cos\beta+z\cos\gamma)\} = Z'';\end{aligned}\right\} \quad (59)$$

and $$P''^2 = X''^2+Y''^2+Z''^2. \quad (60)$$

If the origin moves, P'' is proportional to the impressed velocity-increment of it, which is due to the centrifugal forces of all the particles; and if the origin is absolutely fixed, P'' produces a pressure on it.

Let L'', M'', N'' be the axial components of the moments of the couples which arise from these centrifugal forces; and let G'' be the moment of the resultant of them; then the axial components of the moment of the couple which arises from the centrifugal force of m are, by reason of (58),

$$\left.\begin{aligned} &m\omega^2\{(x\cos\alpha+y\cos\beta+z\cos\gamma)(z\cos\beta-y\cos\gamma)\},\\ &m\omega^2\{(x\cos\alpha+y\cos\beta+z\cos\gamma)(x\cos\gamma-z\cos\alpha)\},\\ &m\omega^2\{(x\cos\alpha+y\cos\beta+z\cos\gamma)(y\cos\alpha-x\cos\beta)\};\end{aligned}\right\} \quad (61)$$

then taking the aggregates of these for all the particles of the system, and using the abridging symbols of Art. 76, we have

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$$\left.\begin{aligned} D(\omega_y^2-\omega_z^2)+E\omega_x\omega_y-F\omega_x\omega_z+(B-C)\omega_y\omega_z &= L'', \\ E(\omega_z^2-\omega_x^2)+F\omega_y\omega_z-D\omega_y\omega_x+(C-A)\omega_z\omega_x &= M'', \\ F(\omega_x^2-\omega_y^2)+D\omega_z\omega_x-E\omega_z\omega_y+(A-B)\omega_x\omega_y &= N''; \end{aligned}\right\} \quad (62)$$

and
$$G''^2 = L''^2 + M''^2 + N''^2; \quad (63)$$

Thus (60) gives the pressure at the origin which is due to the centrifugal forces, and (63) gives the moment of the couple which arises from them.

In reference to (59) and to (62) I must observe that

$$\left.\begin{aligned} X''\cos\alpha+Y''\cos\beta+Z''\cos\gamma &= 0, \\ L''\cos\alpha+M''\cos\beta+N''\cos\gamma &= 0; \end{aligned}\right\}$$

so that the instantaneous axis of rotation is perpendicular to both the line of action of the resultant of translation, and to the axis of the couple, which arise from the centrifugal forces of all the particles.

83.] Now, as we have already observed, the moment of the couple of the expressed momentum-increment is equal to the moment of the couple of the impressed momentum-increment together with that of the couple which arises from the centrifugal forces: and this equality is true relatively to any rotation-axis, so that for the coordinate-axes of x, y, z we have respectively
$$L' = L + L'', \quad M' = M + M'', \quad N' = N + N''; \quad (64)$$
and therefore we have

$$\left.\begin{aligned} A\frac{d\omega_x}{dt}+(C-B)\omega_y\omega_z-D(\omega_y^2-\omega_z^2)-E\left(\frac{d\omega_z}{dt}+\omega_x\omega_y\right)-F\left(\frac{d\omega_y}{dt}-\omega_x\omega_z\right) &= L, \\ B\frac{d\omega_y}{dt}+(A-C)\omega_z\omega_x-E(\omega_z^2-\omega_x^2)-F\left(\frac{d\omega_x}{dt}+\omega_y\omega_z\right)-D\left(\frac{d\omega_z}{dt}-\omega_y\omega_x\right) &= M, \\ C\frac{d\omega_z}{dt}+(B-A)\omega_x\omega_y-F(\omega_x^2-\omega_y^2)-D\left(\frac{d\omega_y}{dt}+\omega_z\omega_x\right)-E\left(\frac{d\omega_x}{dt}-\omega_z\omega_y\right) &= N; \end{aligned}\right\} (65)$$

which are the same equations as (48), (49), and (50). By the preceding process therefore they are, as it were, dissected, and the meaning and origin of the several terms are traced out; and those which arise from the expressed momentum-increments are distinguished from those which arise from the centrifugal forces.

If the system is at rest when the forces begin to act, then $\omega = 0$, and $L'' = M'' = N'' = 0$; and the equations are reduced to forms identical with (31), Art. 76, where however $\frac{d\omega_x}{dt}, \frac{d\omega_y}{dt}, \frac{d\omega_z}{dt}$ take the places of $\Omega_x, \Omega_y, \Omega_z$.

In the course also of this inquiry we have arrived at a more

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complete explanation of those parts of the equations of translation of the origin which arise from the angular velocities. The origin, through which the rotation-axis passes, is considered not to change its place during the time dt, so that the velocity and the velocity-increment of m are due to the rotation only. Now, if we recall to mind the process by which the original equations of statical equilibrium are formed, it will be found that terms are introduced at the origin equal to and having the same line of action with those which are effective at m; and that these produce a pressure of translation on the origin. In this case they produce of course a motion of translation of the origin; the axial components of the expressed velocity-increment of which, as due to these angular velocities, are given in (40). Here we have these expressions dissected into three, viz. X', Y', Z' which correspond to the expressed angular velocity-increments; and three, viz. X'', Y'', Z'', which correspond to the centrifugal forces: thus

$$\mathrm{X}'-\mathrm{X}'', \qquad \mathrm{Y}'-\mathrm{Y}'', \qquad \mathrm{Z}'-\mathrm{Z}'' \tag{66}$$

are the axial components of the momentum-increments of the whole mass collected at the origin, due to the angular velocities of all the particles: and replacing X', X'' &c. by their values, we have

$$\left.\begin{aligned}
&\Sigma.m\left(z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}\right)+\omega^2\,\Sigma.m\,\{\cos\alpha\,(x\cos\alpha+y\cos\beta+z\cos\gamma)-x\},\\
&\Sigma.m\left(x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}\right)+\omega^2\,\Sigma.m\,\{\cos\beta\,(x\cos\alpha+y\cos\beta+z\cos\gamma)-y\},\\
&\Sigma.m\left(y\frac{d\omega_x}{dt}-x\frac{d\omega_y}{dt}\right)+\omega^2\,\Sigma.m\,\{\cos\gamma\,(x\cos\alpha+y\cos\beta+z\cos\gamma)-z\};
\end{aligned}\right\}\tag{67}$$

and these are the same expressions as are deducible from (40). They, if the origin is fixed, are the axial components of the increase of the pressure on the origin due to the angular velocity which is effective during the time dt.

84.] In all cases the equations (31) and (65) admit of great simplification. It will have been observed that the equations of translation of a system of material particles, viz. (34) and (37), Art. 48, are much simplified if the centre of gravity is taken for the origin, as we have shewn in section 2 of the preceding chapter; equations (56) and (58) in Arts. 54 and 55 are more simple than (34) and (37) of Art. 48. It does not however appear thus far that any simplification is hereby introduced into the forms of the equations of motion of rotation; (57) and (59),

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in Arts. 54 and 55, are exactly the same in form as (35) and (38) of Art. 48. Neither does it appear that any change of axes will generally introduce a further simplification into the equations of motion of translation; it may do so in a particular case, because $\Sigma.m\text{X}$, $\Sigma.m\text{Y}$, $\Sigma.m\text{Z}$ may then take simple forms. In the equations of rotation however it is otherwise. Consider the equations (31), which are equivalent to (35), and equations (48), (49), and (50), which are equivalent to (38), of Art. 48; they contain the quantities $\Sigma.mx^2$, $\Sigma.my^2$, $\Sigma.mz^2$, $\Sigma.myz$, $\Sigma.mzx$, $\Sigma.mxy$; and these are dependent on the position of the coordinate axes relative to and in the body. They will be determined by the ordinary processes of summation, and of integration if the moving mass is a continuous body. Now thus far the position of the coordinate axes, to which the moving system is referred, has not been determined; it is fixed neither in the body nor in space. Henceforward we shall suppose a system to be fixed in the body and to move with it, and to have a particular position relatively to the body, which we shall determine with the view of simplifying the preceding equations (31), (32), and (65). By this method we shall investigate the angular velocities of the body about three axes fixed in the moving body; and we can thence determine the angular velocities about three axes fixed in space by means of the equations (87), Art. 40; and $\frac{d\theta}{dt}, \frac{d\phi}{dt}, \frac{d\psi}{dt}$ may be determined by means of (103), (104), (105) of Art. 42. By either process the position of the body in space at the time t will be determined.

85.] Let us examine the coefficients in (31) and in (65) of the angular velocities and of their t-differentials; and let us suppose the moving mass to have volume of three dimensions. Whatever is the system of coordinate axes, it is evident that they cannot be such that generally either $\Sigma.mx^2 = 0$, or $\Sigma.my^2 = 0$, or $\Sigma.mz^2 = 0$; because each of these expressions is the sum of the products of the mass-element and of a quantity which is necessarily positive. Thus, A, B, C, A′, B′, C′, defined as they are in (25) and (26) of Art. 76, are always positive quantities for masses whose volume is of three dimensions: in plates of infinitesimal thickness, if the surface of the plate is taken for the plane of (x, y), $z = 0$ for every element; and therefore $\Sigma.mz^2 = 0$: and in straight wires or rods, of which the transverse section is an infinitesimal area, if the axis of x lies along the rod, $y = z = 0$

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for every mass-element, and consequently $\Sigma.my^2 = \Sigma.mz^2 = 0$; in all other cases A, B, C and A′, B′, C′ are positive quantities. In reference however to D, E, F, which are the symbols for $\Sigma.myz$, $\Sigma.mzx$, $\Sigma.mxy$ respectively, the coordinate-axes may have a position such that

$$\Sigma.myz = \Sigma.mzx = \Sigma.mxy = 0, \tag{68}$$

or that one or two of them may be zero; because in the series, the sums of which are represented by these abridging symbols, some of the terms may be negative and others may be positive, so that the result of the whole may be zero.

Thus, for instance, let us suppose an elliptical plate of infinitesimal thickness to be referred to the centre as origin, and to a system of rectangular coordinate-axes, of which the axes of x and y are coincident with the major and minor axes of the ellipse, and that of z is perpendicular to the plane of the elliptic plate. Then, as $z = 0$ for all the mass-elements of the plate, it is evident that

$$\Sigma.myz = \Sigma.mzx = 0;$$

and since for an element of the plate at (x, y) there is always an equal element at $(-x, y)$, it is plain that $\Sigma.mxy = 0$.

This last result may also thus be found. Let $Y = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$, and let $\rho =$ the density, $\tau =$ the thickness of the plate; then

$$\Sigma.mxy = \rho\tau\int_{-a}^{a}\int_{-Y}^{Y} xy\,dy\,dx$$
$$= 0.$$

We will now prove that a system of rectangular coordinate-axes, fulfilling the conditions (68) exists at every point of a body or system of particles. Such a system is called a system of principal axes relatively to or at that point. The geometrical definition of them is, that they satisfy the conditions (68): in the following sections however several mechanical properties of them will be demonstrated.

SECTION 2.—*Principal axes, and their properties.*

86.] Let us consider a body or a moving mass in reference to a point of it which we take as the origin; and at it let two systems of rectangular coordinates originate; one of which (x, y, z) shall be fixed absolutely; and the other of (ξ, η, ζ) is fixed in the body; and the position of which is to be determined, if it is possible, so that

$$\Sigma.m\eta\zeta = 0; \quad \Sigma.m\zeta\xi = 0; \quad \Sigma.m\xi\eta = 0. \tag{69}$$

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Let these two systems be related by the direction-cosines of the scheme (1), Art. 2. Then, as the systems are rectangular, the nine direction-cosines are subject to six conditions (4) and (6), or (5) and (7), of Art. 2: and as three other conditions are given in (69), we have sufficient data for the determination of the nine direction-cosines.

Substituting in (69) the values of ξ, η, ζ given in (3), Art. 2, and replacing $\Sigma.mx^2$, $\Sigma.my^2$, $\Sigma.mz^2$, $\Sigma.myz$, $\Sigma.mzx$, $\Sigma.mxy$ by their symbols, we have

$$\left.\begin{aligned}
&\mathrm{A}'b_1c_1+\mathrm{B}'b_2c_2+\mathrm{C}'b_3c_3\\
&\qquad+\mathrm{D}(b_2c_3+b_3c_2)+\mathrm{E}(b_3c_1+b_1c_3)+\mathrm{F}(b_1c_2+b_2c_1)=0,\\
&\mathrm{A}'c_1a_1+\mathrm{B}'c_2a_2+\mathrm{C}'c_3a_3\\
&\qquad+\mathrm{D}(c_2a_3+c_3a_2)+\mathrm{E}(c_3a_1+c_1a_3)+\mathrm{F}(c_1a_2+c_2a_1)=0,\\
&\mathrm{A}'a_1b_1+\mathrm{B}'a_2b_2+\mathrm{C}'a_3b_3\\
&\qquad+\mathrm{D}(a_2b_3+a_3b_2)+\mathrm{E}(a_3b_1+a_1b_3)+\mathrm{F}(a_1b_2+a_2b_1)=0;
\end{aligned}\right\}\quad(70)$$

Now these equations are in form identical with (34), Art. 6, and are subject to the same conditions, viz. (5) and (7), Art. 2, so that we have

$$\frac{\mathrm{A}'a_1+\mathrm{F}a_2+\mathrm{E}a_3}{a_1}=\frac{\mathrm{F}a_1+\mathrm{B}'a_2+\mathrm{D}a_3}{a_2}=\frac{\mathrm{E}a_1+\mathrm{D}a_2+\mathrm{C}'a_3}{a_3};\qquad(71)$$

$$\begin{aligned}
&=\frac{\mathrm{A}'a_1^2+\mathrm{B}'a_2^2+\mathrm{C}'a_3^2+2\mathrm{D}a_2a_3+2\mathrm{E}a_3a_1+2\mathrm{F}a_1a_2}{a_1^2+a_2^2+a_3^2}\\
&=\mathrm{A}'a_1^2+\mathrm{B}'a_2^2+\mathrm{C}'a_3^2+2\mathrm{D}a_2a_3+2\mathrm{E}a_3a_1+2\mathrm{F}a_1a_2;\quad(72)\\
&=\Sigma.m(a_1x+a_2y+a_3z)^2\\
&=\Sigma.m\xi^2\\
&=\mathrm{A}''\text{ (say)}.\qquad(73)
\end{aligned}$$

By a similar process we may obtain the following:

$$\frac{\mathrm{A}'b_1+\mathrm{F}b_2+\mathrm{E}b_3}{b_1}=\frac{\mathrm{F}b_1+\mathrm{B}'b_2+\mathrm{D}b_3}{b_2}=\frac{\mathrm{E}b_1+\mathrm{D}b_2+\mathrm{C}'b_3}{b_3}=\Sigma.m\eta^2=\mathrm{B}''\text{ (say)};\quad(74)$$

$$\frac{\mathrm{A}'c_1+\mathrm{F}c_2+\mathrm{E}c_3}{c_1}=\frac{\mathrm{F}c_1+\mathrm{B}'c_2+\mathrm{D}c_3}{c_2}=\frac{\mathrm{E}c_1+\mathrm{D}c_2+\mathrm{C}'c_3}{c_3}=\Sigma.m\zeta^2=\mathrm{C}''\text{ (say)}.\quad(75)$$

As these last three equations are of precisely the same form, let us, as in Art. 6, take a type-expression of all; and assume K to be the type of A'', B'', C''; so that the discriminating cubic will take the forms

$$(\mathrm{A}'-\mathrm{K})(\mathrm{B}'-\mathrm{K})(\mathrm{C}'-\mathrm{K})-\mathrm{D}^2(\mathrm{A}'-\mathrm{K})-\mathrm{E}^2(\mathrm{B}'-\mathrm{K})-\mathrm{F}^2(\mathrm{C}'-\mathrm{K})+2\mathrm{DEF}=0;\quad(76)$$

and

$$\frac{\mathrm{EF}}{\mathrm{D}(\mathrm{K}-\mathrm{A}')+\mathrm{EF}}+\frac{\mathrm{FD}}{\mathrm{E}(\mathrm{K}-\mathrm{B}')+\mathrm{FD}}+\frac{\mathrm{DE}}{\mathrm{F}(\mathrm{K}-\mathrm{C}')+\mathrm{DE}}-1=0;\quad(77)$$

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each of which equations has three real roots, as we have shewn in Art. 7, which are A'', B'', C''; and these quantities are therefore known functions of A', B', C', D, E, F, and will henceforth be treated as such.

Also the direction-cosines of the three principal axes of ξ, η, ζ are given by either of the following formulæ: Let us take the axis of ξ, say; and let a_1, a_2, a_3, A'' correspond to it; then equations (51) and (54), Art. 7, give

$$\frac{a_1^2}{(B'-A'')(C'-A'')-D^2} = \frac{a_2^2}{(C'-A'')(A'-A'')-E^2} = \frac{a_3^2}{(A'-A'')(B'-A'')-F^2}; \quad (78)$$

and

$$a_1\{EF-D(A'-A'')\} = a_2\{FD-E(B'-A'')\} = a_3\{DE-F(C'-A'')\}; \quad (79)$$

and similar forms are true for b_1, b_2, b_3, and for c_1, c_2, c_3, in terms of B'' and C'' respectively.

It appears then that at every point of a body, and of a system of material particles, a system of coordinate-axes in terms of ξ, η, ζ exists, so that, if the body is referred to it,

$$\Sigma.m\eta\zeta = \Sigma.m\zeta\xi = \Sigma.m\xi\eta = 0;$$

and this system at any point is generally unique, and is called the system of principal axes at that point. Thus the term Principal Axis properly belongs to an axis which is one of a system of three axes. But we find it convenient to apply the term to an axis fulfilling any two of the three conditions (69). Thus, if the axis of x is such that $\Sigma.mxy = \Sigma.mxz = 0$, the axis of x is called a principal axis; and if $\Sigma.mzx = \Sigma.mzy = 0$, the axis of z is a principal axis. Hence, if any two of the rectangular coordinate-axes are principal, the third is also principal. Also the three planes which are perpendicular to the three principal axes are called principal planes.

87.] Now all these results admit of a geometrical interpretation by means of the ellipsoid. From (73), which we will take to be the type of (74) and (75) also, since the form is the same in all, we have

$$A'a_1^2 + B'a_2^2 + C'a_3^2 + 2Da_2a_3 + 2Ea_3a_1 + 2Fa_1a_2 = A''. \quad (80)$$

Along the axis (a_1, a_2, a_3), which is the axis of ξ, say, take a length r, and let its extremity be (x, y, z); so that, in reference to the system of axes fixed in space,

$$\frac{x}{a_1} = \frac{y}{a_2} = \frac{z}{a_3} = r; \quad (81)$$

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and therefore, from (81), we have

$$\text{A}'x^2+\text{B}'y^2+\text{C}'z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy-\text{A}''r^2=0. \qquad (82)$$

As r is indeterminate thus far, let it be such that $\text{A}''r^2=\mu'$; where μ' is an undetermined constant, which we may assume to be unity, if such an assumption is convenient; then

$$r^2=\frac{\mu'}{\text{A}''};$$

and (82) becomes

$$\text{A}'x^2+\text{B}'y^2+\text{C}'z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy-\mu'=0; \qquad (83)$$

which is the equation to an ellipsoid, because A', B', C' are all positive quantities, of which the origin is the centre, and the radius vector, corresponding to the direction-cosines (a_1, a_2, a_3), is that along which the axis of ξ lies; and the length of the corresponding radius vector is $\left(\frac{\mu'}{\text{A}''}\right)^{\frac{1}{2}}$.

By a similar process we may obtain the same equation of an ellipsoid from (74) and (75), if we take the lengths of the central radii vectores, which lie along the axes of η, ζ respectively, equal to $\left(\frac{\mu'}{\text{B}''}\right)^{\frac{1}{2}}$ and $\left(\frac{\mu'}{\text{C}''}\right)^{\frac{1}{2}}$; so that (83) represents an ellipsoid, the lengths of three of whose central radii vectores, which are at right angles to each other, are given: this ellipsoid is called the ellipsoid of principal axes.

And these three radii are the geometrical principal axes of the ellipsoid; for if we apply to (83) the processes for determining the lengths and the position of the principal axes of an ellipsoid, which have been developed in Arts. 6 and 7, the equations for the direction-cosines, given in (51) and (54) of Art. 7, are the same as (78) and (79), by which the position of our principal axes is determined; and the coefficients of ξ^2, η^2, and ζ^2 in the reduced equation are A'', B'', and C'', which are determined by (73), (74), (75); so that the equation to the ellipsoid, referred to the axes of ξ, η, ζ, is

$$\text{A}''\xi^2+\text{B}''\eta^2+\text{C}''\zeta^2-\mu'=0. \qquad (84)$$

Hence it appears that at every point of a body or system of particles as a centre, an ellipsoid, whose equation is (83), may be described, the principal axes of which are the principal axes of the body relatively to that point. And if for the body, relatively to the centre and the principal axes of the ellipsoid,

$$\text{A}''=\Sigma.m\xi^2, \qquad \text{B}''=\Sigma.m\eta^2, \qquad \text{C}''=\Sigma.m\zeta^2,$$

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the principal axes of the ellipsoid are inversely proportional to the square roots of A'', B'', C'' respectively. Thus the form of the ellipsoid, as well as the position of its principal axes, depends on the configuration of the system of particles relatively to the point which is the centre of the ellipsoid.

In our subsequent investigations in this subject we shall assume

$$\left.\begin{aligned} &\Sigma.m\xi^2 > \Sigma.m\eta^2 > \Sigma.m\zeta^2, \\ \text{that is,}\quad &A'' > B'' > C''; \end{aligned}\right\} \tag{85}$$

so that the ξ- axis, and the ζ- axis, of the ellipsoid are respectively the least and the greatest of all the axes.

88.] Let us shortly examine the particular forms which the ellipsoid (84) and the position of the principal axes take, corresponding to singular values of the roots of the cubic (76) or (77). The analytical criteria of the conditions it is unnecessary to specify, as they are precisely the same as those which have been determined in Art. 10.

(1) Let two roots be equal; say $A'' = B''$; then the equation (84) represents a prolate spheroid whose axis of revolution is the coordinate-axis of ζ; as C'' is definite, c_1, c_2, c_3 are also determinate, and the axis of ζ is a principal axis; the direction-cosines of the axes of ξ and η, which are the other two principal axes, are indeterminate, and any pair of rectangular axes in the plane of (ξ, η) is a pair of principal axes, and with the axis of ζ completes the system. If $B'' = C''$, the ellipsoid becomes an oblate spheroid; the axis of revolution of which is the axis of ξ, and is the determinate principal axis; and any two axes in the plane of (η, ζ), which are perpendicular to each other, will complete the system of principal axes.

(2) Let all the roots of the cubic be equal; then

$$A'' = B'' = C'' = K, \text{ say};$$

and equation (84) represents a sphere whose equation is

$$\xi^2 + \eta^2 + \zeta^2 = \frac{\mu'}{K}; \tag{86}$$

and every three axes passing through the given point at right angles to each other will form a system of principal axes. In this case

$$\Sigma.mx^2 = \Sigma.my^2 = \Sigma.mz^2 = \Sigma.m\xi^2 = \Sigma.m\eta^2 = \Sigma.m\zeta^2; \tag{87}$$

and all the nine direction-cosines are indeterminate.

Hence, for every point of a system of particles, there is always

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one set of principal axes; and, for certain points of certain bodies, every system of rectangular axes originating thereat may be a principal system.

The ellipsoid (84) will also assume particular forms if one or two of the quantities A'', B'', C'' vanish, that is, if the body is a plate or a straight wire; but these cases are so evident that it is unnecessary to explain them at length.

89.] Let us apply the preceding processes to a particular example. Let us take a cube, each of whose sides $= a$; and let the origin be at one of the angles; and let the axes of x, y, z lie along the edges of the cube: it is required to find the position of the principal axes. Let ρ be the density of the cube; then

$$A' = \Sigma . m x^2 = \int_0^a \int_0^a \int_0^a \rho x^2 dz\, dy\, dx = \frac{\rho a^5}{3}$$
$$= B' = C',$$

as the symmetry indicates.

$$D = \Sigma . m yz = \int_0^a \int_0^a \int_0^a \rho yz\, dz\, dy\, dx = \frac{\rho a^5}{4}$$
$$= E = F;$$

so that the cubic equation (76) becomes

$$K^3 - \rho a^5 K^2 + \frac{7}{48} \rho^2 a^{10} K - \frac{5}{864} \rho^3 a^{15} = 0;$$

the roots of which are $\frac{5\rho a^5}{6}$, $\frac{\rho a^5}{12}$, $\frac{\rho a^5}{12}$; two therefore are equal; let these be B'', C''; so that

$$A'' = \frac{5\rho a^5}{6}, \qquad B'' = C'' = \frac{\rho a^5}{12}.$$

Hence from (78),

$$a_1^2 = a_2^2 = a_3^2 = \frac{1}{3},$$

$$b_1^2 = b_2^2 = b_3^2 = \frac{0}{0},$$

$$c_1^2 = c_2^2 = c_3^2 = \frac{0}{0};$$

so that the axis of ξ is the diagonal of the cube; and the position of the other principal axes is indeterminate; and therefore any two lines perpendicular to each other, and in the plane passing through the angle of the cube and perpendicular to the diagonal, will complete a system of principal axes.

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Thus the equation to the ellipsoid (83) is

$$\frac{\rho a^5}{3}(x^2+y^2+z^2)+\frac{\rho a^5}{2}(yz+zx+xy)-\mu'=0,$$

and the equation to the reduced ellipsoid (84) is

$$\frac{5\rho a^5}{6}\xi^2+\frac{\rho a^5}{12}(\eta^2+\zeta^2)=\mu,$$

which represents an oblate spheroid, whose axis of revolution is the axis of ξ.

If it is required to determine the position of the principal axes at the centre of the cube; then, if $2a$ = the length of a side of the cube,

$$\text{A}'=\text{B}'=\text{C}'=\frac{8\rho a^5}{3};$$

$$\text{D}=\text{E}=\text{F}=0;$$

$$\text{A}''=\text{B}''=\text{C}''=\frac{8\rho a^5}{3};$$

therefore the position of each axis is indeterminate; and any rectangular system originating at the centre of the cube is a system of principal axes; and the ellipsoid (83) becomes a sphere.

90.] It is evident from the preceding general investigation that the position of the principal axes of a body, relatively to a given point, depends on the values of the definite integrals which are expressed by the symbols A′, B′, C′, D, E, F; and therefore on the symmetry of the body relatively to the origin and to the axes of x, y, z: thus, for a solid of revolution bounded by a plane perpendicular to the axis of revolution, for any point on the axis of revolution that axis is evidently one of the principal axes, and the other two are indeterminate in the plane perpendicular to the axis of revolution. Of a sphere, relatively to the centre, every system of rectangular axes is principal. Of an ellipsoid, relatively to the centre, the principal system is unique; and the principal axes coincide with the geometrical principal axes of the solid. Similarly the principal system can often be inferred by general reasoning from the symmetry of the body or system of particles.

If two principal axes relatively to a given point are given, the third is also given. If however one principal axis is given, the other two are at right angles to each other in a plane perpendicular to the given principal axis, and may be determined by the following process.

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Let the given principal axis be the axis of z; so that

$$\Sigma.mxz = \Sigma.myz = 0. \tag{88}$$

Let the new axis of ξ, which is to be principal, be inclined at an angle ϕ to the axis of x; so that

$$\left.\begin{aligned} \xi &= y \sin\phi + x\cos\phi, \\ \eta &= y\cos\phi - x\sin\phi; \end{aligned}\right\} \tag{89}$$

therefore

$$\Sigma.m\xi\eta = \{(\cos\phi)^2 - (\sin\phi)^2\}\Sigma.mxy + \sin\phi\cos\phi\,\Sigma.m(y^2-x^2); \tag{90}$$

and, as this is to vanish, we have,

$$\tan 2\phi = \frac{2\Sigma.mxy}{\Sigma.m(x^2-y^2)} = \frac{2F}{A'-B'}; \tag{91}$$

and the equations to the two principal axes are

$$(\xi^2-\eta^2)\Sigma.mxy + \xi\eta\,\Sigma.m(y^2-x^2) = 0. \tag{92}$$

As by this process z is unchanged,

$$\left.\begin{aligned} \Sigma.m\xi\zeta &= \Sigma.mz(y\sin\phi + x\sin\phi) = 0, \\ \Sigma.m\eta\zeta &= \Sigma.mz(y\cos\phi - x\cos\phi) = 0; \end{aligned}\right\} \tag{93}$$

which, by reason of (88), are true for all values of ϕ: if therefore (88) are true for any pair of rectangular axes in the plane of (x, y), they are true for every pair of rectangular axes in that plane; and indeed for any pair of lines in that plane.

91.] The following are examples of the process for determining the position of two principal axes at a given point when the other principal axis is given.

Ex. 1. To determine the principal axes of a rectangular plate of infinitesimal thickness relatively to the point of intersection of the two diagonals of the plate.

In this case, as in all problems of plane plates of infinitesimal thickness, an axis which passes through the origin and is perpendicular to the plate is one of the principal axes: and if it is taken to be the axis of z, and the plate to be the plane of (x, y), $\Sigma.mxz = \Sigma.myz = 0$, because $z = 0$ for all the elements of the plate.

Let $2a$ and $2b$ be the sides of the plate, τ = the thickness, ρ = the density;

$$\Sigma.mxy = \int_{-a}^{a}\int_{-b}^{b} \rho\tau\, xy\, dy\, dx = 0;$$

$$\begin{aligned} \Sigma.m(x^2-y^2) &= \int_{-a}^{a}\int_{-b}^{b} \rho\tau\,(x^2-y^2)\, dy\, dx \\ &= \frac{4\rho\tau ab}{3}(a^2-b^2); \end{aligned}$$

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therefore $\tan 2\phi = 0$; and $\phi = 0$, $\phi = 90°$; so that the other two principal axes are parallel to the sides of the plate. If the rectangle is a square, $b = a$; in which case $\tan 2\phi = \frac{0}{0}$; and ϕ is indeterminate, so that every pair of rectangular axes in the plane of the plate, together with the given axis, constitutes a principal system.

Ex. 2. To determine the principal axes of a triangular plate through one of its angles.

Let O, an angle of the triangle, be the origin; $OA = a$, $OB = b$; and let the axes of x and y lie along these sides; then the equation to the base is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

let $Y = \frac{b}{a}(a - x)$; and let ω be the angle of the triangle at O;

then
$$F = \int_0^a \int_0^Y \rho\tau (x + y\cos\omega)\, y\, (\sin\omega)^2\, dy\, dx$$
$$= \frac{\rho\tau (\sin\omega)^2 ab^2}{24}(a + 2b\cos\omega);$$
$$A' - B' = \int_0^a \int_0^Y \rho\tau \{(x + y\cos\omega)^2 - (y\sin\omega)^2\}\, dy\, dx \sin\omega$$
$$= \frac{\rho\tau \sin\omega\, ab}{12}(a^2 + ab\cos\omega + b^2\cos 2\omega);$$
$$\therefore \quad \tan 2\phi = \frac{b\sin\omega\,(a + 2b\cos\omega)}{a^2 + ab\cos\omega + b^2\cos 2\omega};$$

if $b = a$, $2\phi = \omega$; in which case the triangle is isosceles, and of the principal axes in its plane one bisects the vertical angle, and the other is perpendicular to the bisecting line.

Ex. 3. To determine the position of the principal axes in the plane of a thin elliptic plate relatively to a point whose place, relatively to the centre and principal axes of the ellipse, is (α, β).

Let the equation to be ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; and let $Y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$: then the ellipse is referred to its centre of gravity as origin, and to its principal axes as coordinate-axes; so that

$$\Sigma . mx = 0, \quad \Sigma . my = 0; \quad \Sigma . mxy = 0.$$

Then
$$F = \int_{-a}^{a} \int_{-Y}^{Y} \rho\tau (x - \alpha)(y - \beta)\, dy\, dx$$
$$= \int_{-a}^{a} \int_{-Y}^{Y} \rho\tau \alpha\beta\, dy\, dx$$
$$= \pi\rho\tau ab\alpha\beta.$$

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$$\begin{aligned} A' - B' &= \int_{-a}^{a}\int_{-Y}^{Y} \rho\tau\{(x-\alpha)^2-(y-\beta)^2\}\,dy\,dx \\ &= \int_{-a}^{a}\int_{-Y}^{Y} \rho\tau\{x^2-y^2+\alpha^2-\beta^2\}\,dy\,dx \\ &= \pi\rho\tau ab\left\{\frac{a^2-b^2}{4}+\alpha^2-\beta^2\right\}; \end{aligned}$$

$$\therefore \quad \tan 2\phi = \frac{8\alpha\beta}{a^2-b^2+4(\alpha^2-\beta^2)};$$

whereby we have the position of the two principal axes in the plane of the plate, and these, with the axis perpendicular to the plate, form the complete system of principal axes.

92.] Many other properties of principal axes will arise incidentally in the following section, and will be there demonstrated. We may therefore now return to the equations of motion, and make those simplifications in them with which the theory of principal axes supplies us.

Let the rotation of the body or system of particles be referred to a system of axes fixed in the body and moving with it; and let this system be that of principal axes, so that

$$\Sigma.myz = \Sigma.mzx = \Sigma.mxy = 0, \quad \text{or} \quad D = E = F = 0.$$

Firstly, let us take the case of a body, originally at rest, acted on by instantaneous forces; and, for the sake of distinctness, let the axial components of the expressed angular velocities, due to the instantaneous forces about the principal axes, be symbolized by Ω_1, Ω_2, Ω_3; and let us reserve Ω_x, Ω_y, Ω_z for the axial components of the expressed angular velocities when the system of axes is not principal; then equations (32), Art. 76, become

$$\Omega_1 = \frac{L}{A}, \qquad \Omega_2 = \frac{M}{B}, \qquad \Omega_3 = \frac{N}{C}; \tag{94}$$

where A, B, C are the moments of inertia of the body about the principal axes; and L, M, N are the moments of the axial components of the couples of the impressed momenta about the corresponding axes.

Secondly, let us take the case of a body rotating under the action of continuous forces; and here again let ω_1, ω_2, ω_3 represent the axial components of the expressed angular velocity relatively to the principal axes; and let ω_x, ω_y, ω_z be the axial components of the expressed angular velocity when the axes are not principal; then, since $D = E = F = 0$, equations (65), Art. 83, become

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$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + (C-B)\,\omega_2\,\omega_3 &= L, \\ B\frac{d\omega_2}{dt} + (A-C)\,\omega_3\,\omega_1 &= M, \\ C\frac{d\omega_3}{dt} + (B-A)\,\omega_1\,\omega_2 &= N; \end{aligned}\right\} \qquad (95)$$

which are evidently much simpler than (65), and are equally general; they were investigated first by Euler, and are now commonly called Euler's Equations of Rotation. No way of integrating them in the form in which they stand is known at present. Particular forms of them will be discussed in the following Chapters; and we shall then employ such artifices of abbreviation and of interpretation as the particular problem suggests.

As (95) are of great importance, let us examine the origin of the several terms: let us take the first of the three; for what is true of that is also true, mutatis mutandis, of the other two. On referring to Art. 81 it will be seen that $A\frac{d\omega_1}{dt}$ is the moment of the axial component of the couple which arises from the expressed momentum-increments of all the particles, the other terms in (55) disappearing because the coordinate axes are principal. And from (62) it appears that $(B-C)\,\omega_2\,\omega_3$ is the moment of the axial component of all the couples which arise from the centrifugal forces of the system; hence, by D'Alembert's principle, the three equations (95) are formed.

If (95) could be integrated, ω_1, ω_2, ω_3 would be expressed in terms of t; and thence by equations (103), (104), (105), Art. 42, θ, ϕ, and ψ could be determined; and the position of the principal axes, and therefore the position of the body, as the principal axes are fixed in it, would also be determined.

93.] Heretofore the point at which the coordinate axes, whether principal or other, originate, has been arbitrary: let us consider whether any simplification will be introduced into the results if we take the origin to be the centre of gravity of the moving system or body; that is, we may suppose the point which has motion of translation to be the centre of gravity, and the axis of rotation to pass through it; and we shall also suppose the coordinate axes fixed in the body, and originating at the centre of gravity, to be principal axes; and these axes we

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shall henceforth call central principal axes. The geometrical definition of such an origin and such axes is,

$$\left.\begin{aligned} \Sigma.mx = \Sigma.my = \Sigma.mz = 0, \\ \Sigma.myz = \Sigma.mzx = \Sigma.mxy = 0. \end{aligned}\right\}$$

Let us first consider the axial components of the expressed momentum-increments of all the particles which arise from the angular velocity-increments. Then, on referring to Art. 81 wherein the effects have been investigated, from (51) and (55) we have

$$X' = Y' = Z' = 0; \tag{96}$$

$$\therefore \quad P' = 0.$$

$$L' = A\frac{d\omega_1}{dt}, \qquad M' = B\frac{d\omega_2}{dt}, \qquad N' = C\frac{d\omega_3}{dt}; \tag{97}$$

so that the momentum-increments are as to translation in equilibrium at the centre of gravity, neither accelerating its motion if it is moving, nor producing a pressure if it is fixed; and the moments of the axial components of the couple arising from these expressed angular velocity-increments are given in (97), and are respectively the first terms of (95).

Next let us take the axial components which arise from the centrifugal forces which are discussed in Art. 82. From (59) we have

$$X'' = Y'' = Z'' = 0; \tag{98}$$

$$\therefore \quad P'' = 0;$$

so that all the centrifugal forces as to translation balance at the centre of gravity; and from (62)

$$L'' = (B-C)\omega_2\omega_3, \quad M'' = (C-A)\omega_3\omega_1, \quad N'' = (A-B)\omega_1\omega_2. \tag{99}$$

Thus it appears that relatively to the centre of gravity as the origin, the forces of translation which are due to the angular velocity-increments and to the centrifugal forces are each separately in equilibrium; but that the equations of rotation are the same as before.

Suppose however that the axis of rotation is a principal axis, say the axis of x; then $\omega_2 = \omega_3 = 0$; and

$$L' = A\frac{d\omega_1}{dt}, \qquad M' = 0, \qquad N' = 0;$$

$$L'' = 0, \qquad M'' = 0, \qquad N'' = 0;$$

so that the centrifugal forces balance themselves as to rotatory effects on the body; producing no change either of the angular velocity of the body or of the position of the axis of rotation;

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and, as (98) shew, they balance each other as to effect of translation.

Hence if a body rotates about a central principal axis, the centrifugal forces which are thereby generated are in equilibrium, and thus do not cause any change of rotation or of the position of the rotation-axis.

Hence also, if no force acts on the body, its equation of motion is

$$A\frac{d\omega_1}{dt} = 0;$$

$$\therefore \quad \omega_1 = \text{a constant.}$$

Thus the angular velocity is constant, and the position of the rotation-axis is the same throughout the motion. For this reason the central principal axes are called the *permanent axes* of the body or of the system; they are also sometimes called the *natural* axes.

It is evident from what has been said that a permanent axis of a body at a certain point may be defined as that line about which if the body revolves, the centrifugal forces generated by the rotation are either in equilibrium, or have a simple resultant passing through the point.

94.] The central principal axes are the only axes which possess this property, that the centrifugal forces balance each other about them both as to translation and as to rotation; for this object it is necessary that, see Art. 82,

$$X'' = Y'' = Z'' = 0, \tag{100}$$

$$L'' = M'' = N'' = 0; \tag{101}$$

from these last three we have, from (62), Art. 82, remembering that $A + A' = B + B' = C + C'$;

$$\left.\begin{aligned}
&(E\cos\alpha + D\cos\beta + C'\cos\gamma)\cos\beta - (F\cos\alpha + B'\cos\beta + D\cos\gamma)\cos\gamma = 0,\\
&(A'\cos\alpha + F\cos\beta + E\cos\gamma)\cos\gamma - (E\cos\alpha + D\cos\beta + C'\cos\gamma)\cos\alpha = 0,\\
&(F\cos\alpha + B'\cos\beta + D\cos\gamma)\cos\alpha - (A'\cos\alpha + F\cos\beta + E\cos\gamma)\cos\beta = 0;
\end{aligned}\right\} \tag{102}$$

so that

$$\frac{A'\cos\alpha + F\cos\beta + E\cos\gamma}{\cos\alpha} = \frac{F\cos\alpha + B'\cos\beta + D\cos\gamma}{\cos\beta} = \frac{E\cos\alpha + D\cos\beta + C'\cos\gamma}{\cos\gamma}; \tag{103}$$

which are the same equations as (71) by which the principal axes are determined; the principal axes therefore are permanent axes so far as the conditions (101) indicate; that is, corresponding to principal axes the centrifugal forces either are in equilibrium, or have a single resultant passing through the origin.

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As to (100) let us replace x, y, z in (59) by $\bar{x}+x', \bar{y}+y', \bar{z}+z'$, wherein the centre of gravity is $(\bar{x}, \bar{y}, \bar{z})$; then, since

$$\Sigma.mx' = \Sigma.my' = \Sigma.mz' = 0,$$

(100) become

$$\left.\begin{aligned} \bar{x}-\cos\alpha(\bar{x}\cos\alpha+\bar{y}\cos\beta+\bar{z}\cos\gamma) &= 0, \\ \bar{y}-\cos\beta(\bar{x}\cos\alpha+\bar{y}\cos\beta+\bar{z}\cos\gamma) &= 0, \\ \bar{z}-\cos\gamma(\bar{x}\cos\alpha+\bar{y}\cos\beta+\bar{z}\cos\gamma) &= 0; \end{aligned}\right\} \qquad (104)$$

and therefore

$$\frac{\bar{x}}{\cos\alpha} = \frac{\bar{y}}{\cos\beta} = \frac{\bar{z}}{\cos\gamma}; \qquad (105)$$

so that the rotation-axis, that is, the principal axis at the point, must pass through the centre of gravity of the body. Thus (105) are to be true for each principal axis at the point; that is, for each of three different values of (α, β, γ); and this is possible only when $\bar{x} = \bar{y} = \bar{z} = 0$; only when the point is the centre of gravity, in which case either of the three central principal axes is a permanent axis. If however a principal axis at a point passes through the centre of gravity, that axis is a permanent axis for that point.

95.] A remarkable application of this theory of permanent axes has been made by M. Foucault to the proof of the rotation of the earth about its polar axis. He presented it to the Academy of Sciences in Paris in the month of September, 1852. He has devised a machine which he calls a Gyroscope, and of which a drawing is given in Fig. 21. I will describe it as it is originally in its position of rest. ABA'B' is a metallic ring suspended by a wire SA from a point S which is fixed to the earth; and at A' is a small pivot working in a small hole, by which the motion of the ring about the vertical line SAA' is kept steady but is not retarded: BB' is the horizontal diameter of the vertical ring; and at B and B' are small holes capable of receiving small pivots or axles; BCB' is a horizontal metallic ring, of which BB' is the diameter, and O is the centre, at B and B' are pivots which work in the beforementioned small holes, so that BCB' is capable of rotating about the horizontal diameter BB'. Across the horizontal ring BCB' another axis COC' is placed, at right angles to, and bisecting, BOB', and capable of rotating about pivots at C and C'; on this axis is fixed a heavy metallic disc DD', whose centre is at O, and which is consequently capable of rotation about the axis COC', and the greater part of the matter of the disc is arranged in a ring as near to the circumference as possible, so as to increase the centrifugal

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force of the disc. This is the arrangement of the several parts of the Gyroscope; and it is evident from the arrangement that the disc is capable of rotation about any axis, so that whatever are the forces which act upon it, it can take the axis which they require, and is indeed in construction identical with Bohnenberger's machine, except that the central mass is a disc instead of a sphere. It is evident also that the centre of gravity of the whole machine is at O, and that all axes of rotation pass through O, and thus gravity does not produce any change of position in the rotation-axis or in the velocity of the disc. Adjustment-screws are placed in various parts of the machine, so that the conditions required may be fulfilled as nearly as possible.

Now the disc DD′ is taken out of the ring BCB′, and has a very rapid rotation given to it by a machine properly contrived for that purpose. While it has this rapid rotation it is replaced in the horizontal ring BCB′; and as the axis COC′ is manifestly one of the principal axes of the ring through its centre of gravity, it is a permanent axis, and as the disc is not under the action of any forces, whether external or centrifugal, whereby its velocity or the position of its rotation-axis may be changed, its axis COC′ keeps an invariable position in space. But what apparent effect is produced by this invariable position? Let us suppose the Gyroscope to be at the north pole: then the earth rotates about the axis SAS′, and to an observer the axis of COC′ will retain its horizontal position, and will have a motion in azimuth, in the same direction as the fixed stars appear to have in passing through 360° in 24 hours, if the rotation of the disc can be kept up as long: if the Gyroscope is at the equator, no such apparent effect will take place, because the axis will have only a parallel displacement of itself in space. In any other latitude the polar axis of the earth will be inclined to the principal axis COC′ of the disc; and as this, being a permanent axis, retains its direction in space, it has an apparent motion about the polar axis; the ring BCB′ will revolve slowly about the axis BB′, and the ring AA′ will also revolve slowly about the vertical axis AA′; and these rotations may be observed by means of microscopes properly placed for the purpose. Indeed the earth truly rotates about the line COC′ which has an invariable position, and that rotation is shewn by the apparent motion of the line. We shall hereafter come to the mathematical calculation of these quantities.

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SECTION 5.—*Moments of inertia, and the distribution in space of principal axes.*

96.] In the present section I propose to examine more closely the theory and properties of moments of inertia, of which definitions have been given in Art. 74.

Moment of inertia is (geometrically) the sum of the products of every particle of a body or of a system of particles, and the square of its distance from the rotation-axis. Thus, if m is a particle of a moving system, and r is the perpendicular distance from the place of m on the rotation-axis, $\Sigma.mr^2$ is the moment of inertia, the summation including all the moving particles, and becoming integration if the moving system is a continuous body. We shall however find it convenient to use the symbol $\Sigma.mr^2$ in all cases, and it is to be observed that this symbol includes integration in the cases wherein the moving system is a continuous mass.

Let us in the first place investigate the moment of inertia in its most general form. Let the origin be taken on the rotation-axis, of which relatively to a system of axes fixed in the body let the direction-angles be (α, β, γ); let (x, y, z) be the place of m; and let r be the perpendicular distance from m on the rotation-axis; so that

$$r^2 = (y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2; \quad (106)$$

$$\Sigma.mr^2 = \Sigma.m(y^2+z^2)(\cos\alpha)^2 + \Sigma.m(z^2+x^2)(\cos\beta)^2 + \Sigma.m(x^2+y^2)(\cos\gamma)^2$$
$$-2\Sigma.myz\cos\beta\cos\gamma - 2\Sigma.mzx\cos\gamma\cos\alpha - 2\Sigma.mxy\cos\alpha\cos\beta. \quad (107)$$

Let H be the general symbol for the moment of inertia; then, using the symbols of Art. 76,

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2$$
$$-2\text{D}\cos\beta\cos\gamma - 2\text{E}\cos\gamma\cos\alpha - 2\text{F}\cos\alpha\cos\beta; \quad (108)$$

and thus if A, B, C, D, E, F are determined for any body and for a particular system of rectangular axes, the moment of inertia of the body for any axis may be found by means of this equation.

If the system of axes to which the body is referred is a principal system at the point, then $\text{D} = \text{E} = \text{F} = 0$; and (108) becomes

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2; \quad (109)$$

where A, B, C are the moments of inertia of the body relatively to the three principal axes of x, y, z respectively, and are for that reason called the principal moments of inertia.

In terms of A′, B′, C′, (109) becomes

$$\text{H} = \text{A}'(\sin\alpha)^2 + \text{B}'(\sin\beta)^2 + \text{C}'(\sin\gamma)^2. \quad (110)$$

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97.] As an example of (108) let us investigate the moment of inertia of a rectangular parallelepipedon about an axis passing through one of its angles.

Let the sides of the parallelepipedon, which meet at the angle through which the rotation-axis passes, be a, b, c; and let the coordinate axes lie along these sides respectively; let ρ be the density of the volume element at (x, y, z); so that the mass-element $= \rho dx dy dz$. Then

$$\text{A} = \int_0^a \int_0^b \int_0^c \rho(y^2+z^2)\,dz\,dy\,dx = \frac{\rho abc(b^2+c^2)}{3};$$

$$\text{B} = \frac{\rho abc(c^2+a^2)}{3}, \qquad \text{C} = \frac{\rho abc(a^2+b^2)}{3};$$

$$\text{D} = \int_0^a \int_0^b \int_0^c \rho yz\,dz\,dy\,dx = \frac{\rho ab^2c^2}{4};$$

$$\text{E} = \frac{\rho bc^2a^2}{4}; \qquad \text{F} = \frac{\rho ca^2b^2}{4};$$

and the moment of inertia about the line (α, β, γ)

$$= \rho abc \left\{ \frac{b^2+c^2}{3}(\cos\alpha)^2 + \frac{c^2+a^2}{3}(\cos\beta)^2 + \frac{a^2+b^2}{3}(\cos\gamma)^2 \right.$$
$$\left. - \frac{bc}{2}\cos\beta\cos\gamma - \frac{ca}{2}\cos\gamma\cos\alpha - \frac{ab}{2}\cos\alpha\cos\beta \right\};$$

and if k is the corresponding radius of gyration,

$$k^2 = \frac{b^2+c^2}{3}(\cos\alpha)^2 + \frac{c^2+a^2}{3}(\cos\beta)^2 + \frac{a^2+b^2}{3}(\cos\gamma)^2$$
$$- \frac{bc}{2}\cos\beta\cos\gamma - \frac{ca}{2}\cos\gamma\cos\alpha - \frac{ab}{2}\cos\alpha\cos\beta.$$

Similarly the moment of inertia of a cube about a diagonal $= \frac{\rho a^5}{6}$; and about one of its edges $= \frac{2\rho a^5}{3}$.

Other examples of the determination of moments of inertia will be given in the following section of this chapter.

98.] The following process gives a geometrical interpretation of (108), and consequently of (109).

Along the rotation-axis from the origin let a length ρ be taken, of which let the end be (x, y, z); then

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = \rho; \tag{111}$$

and (107) becomes

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 - 2\text{D}yz - 2\text{E}zx - 2\text{F}xy = \rho^2 \Sigma . mr^2; \tag{112}$$

Let $\qquad \rho^2 \Sigma . mr^2 = \mu; \quad$ so that $\Sigma . mr^2 = \frac{\mu}{\rho^2}, \qquad (113)$

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where μ is a constant quantity at present undetermined, and may be unity if such a value is convenient; then (112) becomes

$$\mathrm{A}x^2+\mathrm{B}y^2+\mathrm{C}z^2-2\mathrm{D}yz-2\mathrm{E}zx-2\mathrm{F}xy-\mu=0. \qquad (114)$$

By the assumption made in (113) it will be observed that the moment of inertia about any axis ρ varies as the square of the reciprocal of ρ.

As A, B, C are quantities necessarily positive, (114) is the equation to an ellipsoid, whose centre is at the origin, and of which ρ is a central radius vector; and which is such that whatever radius vector is the rotation-axis, the moment of inertia of the body relatively to that axis varies as the square of the reciprocal of the central radius vector. For this reason the ellipsoid is called the momental ellipsoid. As μ is at present undetermined the actual size of the ellipsoid is not fixed: however, according as μ varies, all the corresponding ellipsoids are concentric and are similar, and that corresponding to any particular value of μ will suffice for our present purpose. Let us imagine therefore the ellipsoid, whose equation is (114), to be described with the given point as centre; then that ellipsoid, by means of its central radii vectores, indicates the law of variation of the several moments of inertia of the moving system which correspond to the radii vectores as rotation-axes, the moment of inertia relatively to any one being proportional to the square of the reciprocal of that radius vector.

The momental ellipsoid is evidently concentric with the ellipsoid of principal axes, equation (83); it is also coaxal with it. For, by Art. 6, the equations for determining the principal axes of (114) are

$$\frac{\mathrm{A}a_1-\mathrm{F}a_2-\mathrm{E}a_3}{a_1}=\frac{-\mathrm{F}a_1+\mathrm{B}a_2-\mathrm{D}a_3}{a_2}=\frac{-\mathrm{E}a_1-\mathrm{D}a_2+\mathrm{C}a_3}{a_3}$$
$$=\mathrm{H}_a, \qquad (115)$$

if H_a is the moment of inertia about the axis (a_1, a_2, a_3) of ξ, so that $\mathrm{H}_a=\Sigma.m(\eta^2+\zeta^2)$. Now replacing A, B, C severally by $\mathrm{B}'+\mathrm{C}'$, $\mathrm{C}'+\mathrm{A}'$, $\mathrm{A}'+\mathrm{B}'$, and subtracting each term of the equalities (115) from $\mathrm{A}'+\mathrm{B}'+\mathrm{C}'$, we have

$$\frac{\mathrm{A}'a_1+\mathrm{F}a_2+\mathrm{E}a_3}{a_1}=\frac{\mathrm{F}a_1+\mathrm{B}'a_2+\mathrm{D}a_3}{a_2}=\frac{\mathrm{E}a_1+\mathrm{D}a_2+\mathrm{C}'a_3}{a_3}$$
$$=\mathrm{A}'+\mathrm{B}'+\mathrm{C}'-\mathrm{H}_a$$
$$=\Sigma.m\zeta^2; \qquad (116)$$

which are identical with (73), whereby the principal axes of the body are determined. By similar processes we might find equa-

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tions identical with (74) and (75) in terms of H_b and H_c, which are the moments of inertia about the axes of η and ζ respectively. Thus the geometrical principal axes of the momental ellipsoid lie along the principal axes of the body at the origin; let it be referred to these as axes; it is manifest from (115) that H_a, H_b, H_c are the coefficients of x^2, y^2, z^2 in the reduced equation. Let, however, henceforth A, B, C represent the moments of inertia about the principal axes; then the equation to the momental ellipsoid, referred to the principal axes as coordinate axes of (x, y, z), is

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 = \mu. \tag{117}$$

This result might have been inferred directly from (114). As the position of the coordinate axes is undetermined, let the system be the principal system; then $\text{D} = \text{E} = \text{F} = 0$, and (114) becomes (117).

This is another instance of the simplification which is introduced into the equations of dynamics by the use of principal axes. The form of the equations of motion, and the mathematical values of the moments of inertia which enter into those equations, are much simplified.

In Art. 87 we assumed $\Sigma.mx^2 > \Sigma.my^2 > \Sigma.mz^2$; therefore

$$\Sigma.m(y^2+z^2) < \Sigma.m(z^2+x^2) < \Sigma.m(x^2+y^2);$$

$$\therefore \quad \text{A} < \text{B} < \text{C};$$

so that the moments of inertia are respectively the greatest and the least about the axes of z and x; and the maximum, mean, and minimum axes of the momental ellipsoid lie along the axes of x, y, z respectively, and correspond to the minimum, mean, and maximum axes of the ellipsoid of principal axes.

One word as to the meaning of μ; let us give it a value which will make the equations homogeneous; let M be the mass of the moving body, and let a, b, c be the radii of gyration about the axes of x, y, z respectively; so that

$$\text{A} = \text{M}a^2, \qquad \text{B} = \text{M}b^2, \qquad \text{C} = \text{M}c^2; \tag{118}$$

let therefore

$$\mu = \text{M}g^4; \tag{119}$$

we shall hereafter determine the meaning of g; then (117) becomes

$$a^2x^2 + b^2y^2 + c^2z^2 = g^4; \tag{120}$$

so that the maximum, mean, and minimum axes of the momental ellipsoid are respectively $\frac{g^2}{a}$, $\frac{g^2}{b}$, $\frac{g^2}{c}$, and (120) is homogeneous. Notwithstanding, however, we shall still find it convenient to employ μ.

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99.] Since the moment of inertia of the body about any axis is proportional to the square of the reciprocal of the radius vector of the momental ellipsoid which coincides with that axis, it follows that the moments of inertia and the radii vectores have simultaneously critical values. Thus, as the x-, the y-, the z-axes of the ellipsoid (117) are respectively the greatest, the mean, and the least of all axes of the ellipsoid, so of all rotation-axes passing through the centre of the ellipsoid, those of x, y, z are the rotation-axes relatively to which the moments of inertia are the least, the mean, and the greatest. That is, according to our assumptions, A is the least, B is the mean, and C is the greatest of all moments of inertia relatively to the given origin. And these are the principal moments.

Whereas then we have defined principal axes as those in reference to which $\Sigma . myz = \Sigma . mzx = \Sigma . mxy = 0$, they might have been defined as those axes for which the moments of inertia have critical values. The former conception of them arose first, in the simplification of the equations of the motion, and therefore we pursued it. It is however to be observed that whatever is true of the axes of principal moments of inertia is also true of the principal axes and of the principal planes; and several properties which are true of principal planes as defined in the last section, and which might have been there demonstrated, will be proved in the course of the present section. Henceforth then we shall treat principal axes, and the geometrical principal axes of the momental ellipsoid, as identical.

And all other moments of inertia relatively to the given point are evidently intermediate to C and A; that is, are less than C, and are greater than A. From (109)

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2;$$

which may be expressed in either of the following forms;

$$\text{H} = \text{A} + (\text{B} - \text{A})(\cos\beta)^2 + (\text{C} - \text{A})(\cos\gamma)^2; \qquad (121)$$

$$\text{H} = \text{C} - (\text{C} - \text{A})(\cos\alpha)^2 - (\text{C} - \text{B})(\cos\beta)^2; \qquad (122)$$

and as B − A, C − A, C − B, are positive quantities, C is the greatest and A is the least of all moments of inertia.

If two principal moments are equal, the momental ellipsoid becomes a spheroid; if B = C, the spheroid is prolate and the moments for all axes lying in the plane of (y, z) are equal to one another and to C; and other moments are less than C: if A = B, the momental ellipsoid becomes an oblate spheroid, and the mo-

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ments for all axes lying in the plane of (x, y) are equal to one another and to A, and the moments for all other axes are greater than A. In the former case the ellipsoid of principal axes becomes an oblate spheroid; and in the latter case a prolate spheroid.

If the three principal moments are equal, $A = B = C$, and the momental ellipsoid becomes a sphere, and the moments of inertia for all axes are equal to one another. In this case also the ellipsoid of principal axes becomes a sphere.

100.] All the rotation-axes passing through a given point, for which the moments of inertia are equal to each other, lie on a cone of the second degree, whose vertex is the origin.

Let H be the moment of inertia to which all are to be equal; then, since $\rho^2 = x^2 + y^2 + z^2$, from (112) we have

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = H(x^2 + y^2 + z^2);$$

$$\therefore\ (H - A)x^2 + (H - B)y^2 + (H - C)z^2 + 2Dyz + 2Ezx + 2Fxy = 0; \quad (123)$$

which is the equation to a cone of the second degree whose vertex is at the origin; and the principal axes are evidently coincident with those of the momental ellipsoid. All rotation-axes therefore lying on the surface of this cone are axes of equal moment, and the cone is consequently called equimomental.

If the coordinate axes are principal axes at the origin,

$$D = E = F = 0,$$

and the equation to the equimomental cone is

$$(H - A)x^2 + (H - B)y^2 + (H - C)z^2 = 0, \quad (124)$$

where A, B, C are the principal moments of inertia. As we have proved that C is the greatest and A is the least of the moments of inertia for axes passing through the origin, H must be intermediate to C and A; so that necessarily one of the coefficients in (124) is negative; and not more than two can be negative.

Let H be greater than B; then $H - A$ and $H - B$ are positive, and $H - C$ is negative; in which case the axis of z is the internal principal axis, and the axes of x and y are the external principal axes. See Art. 11. And all plane sections parallel to the plane of (x, y) are ellipses.

Let $H = B$; then

$$(B - A)^{\frac{1}{2}}x = \pm (C - B)^{\frac{1}{2}}z; \quad (125$$

which are the equations to two planes; these are indeed, see

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(105), Art. 13, the cyclic planes of the momental ellipsoid. Thus all the rotation-axes at any point for which the moments of inertia are equal to the mean moment of inertia lie in two planes equally inclined to the axis of greatest moment.

Let H be less than B; then H − B and H − C are negative, and H − A is positive, so that the axis of x is the internal principal axis of the cone, and the axes of y and z are the external principal axes; and all plane sections perpendicular to the axis of x are ellipses.

Thus, according to the configuration which we have chosen, all axes lying within the planes (125) towards the axes of z are rotation-axes of greater moment than the mean; all those lying in the planes (125) are rotation-axes of moment equal to the mean; and all those lying without the planes towards the axis of x are rotation-axes of moment less than the mean. See Fig. 2, in which the cyclic planes are delineated; all axes within the angles UOU′ and VOV′ are of the former kind, and all those within the angles UOV and U′OV′ are of the latter kind.

Also the cyclic planes of the equimomental cone (124) are the cyclic planes of the momental ellipsoid; for, by reason of (96), Art. 12, the equations to the cyclic planes of (124) are

$$\{\text{H}-\text{B}-(\text{H}-\text{A})\}^{\frac{1}{2}}x = \pm\{\text{H}-\text{C}-(\text{H}-\text{B})\}^{\frac{1}{2}}z;$$

$$(\text{B}-\text{A})^{\frac{1}{2}}x = \pm(\text{C}-\text{B})^{\frac{1}{2}}z; \qquad (126)$$

which are the same equations as (125).

If two principal moments are equal, so that the momental ellipsoid becomes a spheroid, the equimomental cones become cones of revolution.

If the three principal moments are equal, the equimomental cone degenerates into a rotation axis.

101.] I propose now to consider the moments of inertia and the momental ellipsoid relatively to any point of a body, with reference to the moments of inertia and the momental ellipsoid relatively to the centre of gravity. We shall hereby be led to general theorems which will clear up many obscurities as to the distribution of principal axes in space, and will indicate remarkable symmetry as to their arrangement.

The following theorem must be demonstrated in the first place.

The moment of inertia of a body, or of a system of particles, about any axis is equal to the sum of the moment of inertia about a parallel axis passing through the centre of gravity, and of

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the product of the mass and the square of the distance between the axes.

Let H be the moment of inertia about the given rotation-axis, and H′ the moment of inertia about a parallel axis passing through the centre of gravity; let h be the perpendicular distance between these two axes. Let r and r' be the distances of m from the axes of H and H′ respectively, and let ϕ be the angle at which r' is inclined to h; so that

$$r^2 = r'^2 - 2r'h\cos\phi + h^2;$$

$$\therefore \quad \Sigma.mr^2 = \Sigma.mr'^2 - 2h\Sigma.mr'\cos\phi + h^2\Sigma.m; \qquad (127)$$

but $r'\cos\phi$ is the perpendicular distance from m on the plane which passes through the centre of gravity and is perpendicular to h; so that $\Sigma.mr'\cos\phi = 0$. Let the mass of the body or system of particles = M; then, since $\Sigma.mr^2 = \text{H}$, $\Sigma.mr'^2 = \text{H}'$, (127) becomes

$$\text{H} = \text{H}' + \text{M}h^2; \qquad (128)$$

which is the mathematical expression of the theorem.

Hence, if k is the radius of gyration relatively to the rotation-axis, and k' is the radius of gyration relatively to the parallel axis through the centre of gravity,

$$k^2 = k'^2 + h^2. \qquad (129)$$

Hence also it follows, that if a line is drawn through the centre of gravity of a body or system, the moments of inertia are equal for all parallel rotation-axes at equal distances from this line; or, in other words, all axes lying on the surface of a right circular cylinder whose axis passes through the centre of gravity of a body or system are rotation-axes of equal moment.

Hence too of all parallel rotation-axes the moment of inertia is the least for that which passes through the centre of gravity of the body.

102.] As we shall often have occasion to refer to the momental ellipsoid at the centre of gravity, it is convenient to give it a distinctive name; I shall call it the Central Ellipsoid; and the principal axes and the principal planes which refer to the centre of gravity will be called the central principal axes and the central principal planes: and the principal moments of inertia at the centre of gravity will be called the principal central moments.

Let the centre of gravity be the origin; and let the equation to the central ellipsoid be

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 = \mu; \qquad (130)$$

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μ being an arbitrary constant which we shall hereafter determine; and where A, B, C are the principal central moments of inertia, arranged in order of magnitude as heretofore; viz. $A < B < C$. Let also A', B', C' refer to the central ellipsoid; where

$$A' = \Sigma.mx^2, \quad B' = \Sigma.my^2, \quad C' = \Sigma.mz^2.$$

Let (ξ, η, ζ) be the point at which the principal moments and the position of the principal axes are to be determined; and let (x', y', z') be the place of m relatively to that origin, the coordinate axes being parallel to the central principal axes; so that

$$x' = x - \xi, \qquad y' = y - \eta, \qquad z' = z - \zeta; \tag{131}$$

and let M be the mass of the body or system of particles. Now the equation to the momental ellipsoid whose centre is at (ξ, η, ζ) is that given in (114), Art. 98, and in this case is

$$\Sigma.m(y'^2+z'^2)x^2+\Sigma.m(z'^2+x'^2)y^2+\Sigma.m(x'^2+y'^2)z^2$$
$$-2\Sigma.my'z'\,yz-2\Sigma.mz'x'\,zx-2\Sigma.mx'y'\,xy-\mu = 0. \tag{132}$$

Let us calculate the coefficients of this equation relatively to the centre of gravity as origin. Here, by (131),

$$\begin{aligned}\Sigma.m(y'^2+z'^2) &= \Sigma.m\{(y-\eta)^2+(z-\zeta)^2\}\\ &= \Sigma.m(y^2+z^2)-2\eta\,\Sigma.my-2\zeta\,\Sigma.mz+(\eta^2+\zeta^2)\Sigma.m\\ &= A+M(\eta^2+\zeta^2),\end{aligned}$$

as is evident also by reason of the last Article.

Let $$\xi^2+\eta^2+\zeta^2 = \rho^2; \tag{133}$$

then
similarly
$$\left.\begin{aligned}\Sigma.m(y'^2+z'^2) &= A+M\rho^2-M\xi^2;\\ \Sigma.m(z'^2+x'^2) &= B+M\rho^2-M\eta^2,\\ \Sigma.m(x'^2+y'^2) &= C+M\rho^2-M\zeta^2.\end{aligned}\right\} \tag{134}$$

Also
$$\begin{aligned}\Sigma.my'z' &= \Sigma.m(y-\eta)(z-\zeta)\\ &= \Sigma.myz-\eta\,\Sigma.mz-\zeta\,\Sigma.my+\eta\zeta\,\Sigma.m;\end{aligned}$$

$$\therefore\quad \Sigma.my'z' = M\eta\zeta;$$

similarly
$$\left.\begin{aligned}\Sigma.mz'x' &= M\zeta\xi,\\ \Sigma.mx'y' &= M\xi\eta.\end{aligned}\right\} \tag{135}$$

Thus the equation to the momental ellipsoid, whose centre is at (ξ, η, ζ), which is the origin of x, y, z, is

$$(A+M\rho^2-M\xi^2)x^2+(B+M\rho^2-M\eta^2)y^2+(C+M\rho^2-M\zeta^2)z^2$$
$$-2M\eta\zeta yz-2M\zeta\xi zx-2M\xi\eta xy-\mu = 0. \tag{136}$$

The equations for determining the position of the principal axes of this ellipsoid are given in Art. 6; and for that principal axis which corresponds to a_1, a_2, a_3 we have

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$$\frac{(\mathrm{A}+\mathrm{M}\rho^2-\mathrm{M}\xi^2)a_1-\mathrm{M}\xi\eta a_2-\mathrm{M}\xi\zeta a_3}{a_1}=\frac{-\mathrm{M}\xi\eta a_1+(\mathrm{B}+\mathrm{M}\rho^2-\mathrm{M}\eta^2)a_2-\mathrm{M}\eta\zeta a_3}{a_2}$$

$$=\frac{-\mathrm{M}\xi\zeta a_1-\mathrm{M}\eta\zeta a_2+(\mathrm{C}+\mathrm{M}\rho^2-\mathrm{M}\zeta^2)a_3}{a_3}; \quad (137)$$

$$=(\mathrm{A}+\mathrm{M}\rho^2-\mathrm{M}\xi^2)a_1^2+(\mathrm{B}+\mathrm{M}\rho^2-\mathrm{M}\eta^2)a_2^2+(\mathrm{C}+\mathrm{M}\rho^2-\mathrm{M}\zeta^2)a_3^2$$
$$-2\mathrm{M}\eta\zeta a_2a_3-2\mathrm{M}\zeta\xi a_3a_1-2\mathrm{M}\xi\eta a_1a_2$$
$$=\mathrm{H}_a, \quad (138)$$

by reason of (108), if H_a is the moment of inertia of the body about the axis (a_1, a_2, a_3) at the given point. We have also analogous equations in terms of $b_1, b_2, b_3, \mathrm{H}_b$, and $c_1, c_2, c_3, \mathrm{H}_c$; and from these equations the direction-cosines of the principal axes may be determined as in Arts. 6 and 7. Let t_1, t_2, t_3 be the direction-cosines of a principal axis, and let H be the type of $\mathrm{H}_a, \mathrm{H}_b, \mathrm{H}_c$; so that from (137) we have

$$\mathrm{A}+\mathrm{M}\rho^2-\frac{\mathrm{M}\xi(t_1\xi+t_2\eta+t_3\zeta)}{t_1}=\mathrm{B}+\mathrm{M}\rho^2-\frac{\mathrm{M}\eta(t_1\xi+t_2\eta+t_2\zeta)}{t_2}$$
$$=\mathrm{C}+\mathrm{M}\rho^2-\frac{\mathrm{M}\zeta(t_1\xi+t_2\eta+t_3\zeta)}{t_3}=\mathrm{H}; \quad (139)$$

$$\mathrm{M}(t_1\xi+t_2\eta+t_3\zeta)=\frac{(\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H})t_1}{\xi}=\frac{(\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H})t_2}{\eta}=\frac{(\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H})t_3}{\zeta}; \quad (140)$$

$$=\frac{t_1\xi+t_2\eta+t_3\zeta}{\dfrac{\xi^2}{\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H}}+\dfrac{\eta^2}{\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H}}+\dfrac{\zeta^2}{\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H}}}; \quad (141)$$

whence we have

$$\frac{\xi^2}{\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H}}+\frac{\eta^2}{\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H}}+\frac{\zeta^2}{\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H}}=\frac{1}{\mathrm{M}}; \quad (142)$$

and
$$\frac{t_1}{\dfrac{\xi}{\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H}}}=\frac{t_2}{\dfrac{\eta}{\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H}}}=\frac{t_3}{\dfrac{\zeta}{\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H}}}. \quad (143)$$

(142) is a cubic equation in H, whose roots are $\mathrm{H}_a, \mathrm{H}_b, \mathrm{H}_c$; that is, are the three principal moments at the point (ξ, η, ζ). And these three roots are respectively less than $\mathrm{A}+\mathrm{M}\rho^2$, greater than $\mathrm{A}+\mathrm{M}\rho^2$ and less than $\mathrm{B}+\mathrm{M}\rho^2$, and greater than $\mathrm{B}+\mathrm{M}\rho^2$ and less than $\mathrm{C}+\mathrm{M}\rho^2$; for if we arrange (142) in descending powers of H,

$$\text{the result is } \left.\begin{array}{l} +^{\text{ve}}, \text{ if } \mathrm{H}=\mathrm{C}+\mathrm{M}\rho^2; \\ -^{\text{ve}}, \text{ if } \mathrm{H}=\mathrm{B}+\mathrm{M}\rho^2; \\ +^{\text{ve}}, \text{ if } \mathrm{H}=\mathrm{A}+\mathrm{M}\rho^2; \\ -^{\text{ve}}, \text{ if } \mathrm{H}=-\infty. \end{array}\right\} \quad (144)$$

thus the roots of (142) are real, and limits of them are assigned;

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and as all the other quantities in (142) are given in terms of the principal central moments, the mass of the moving system, and the coordinates of the given point, we shall henceforth consider H_a, H_b, H_c to be known quantities.

If these several values of H are substituted successively in (143), we have three different sets of direction-cosines, which correspond to the three principal axes at (ξ, η, ζ).

103.] These equations (142) and (143) admit of the following interpretation. The equation

$$\frac{x^2}{A+M\rho^2-H}+\frac{y^2}{B+M\rho^2-H}+\frac{z^2}{C+M\rho^2-H}=\frac{1}{M} \tag{145}$$

represents three confocal surfaces of the second degree; which are an ellipsoid, an hyperboloid of one sheet, and an hyperboloid of two sheets; because, according to (144), the coefficients of x^2, y^2, z^2 (1) are all positive; (2) that of x^2 is negative and the other two are positive; (3) those of x^2 and y^2 are negative, and that of z^2 is positive; and according to our assumption of the order of magnitude of A, B, C, in the ellipsoid the x-, y-, and z- axes, are respectively the greatest, the mean, and the least.

Thus (142) shews that the point (ξ, η, ζ) is at the point of intersection of these three confocal surfaces of the second degree. And since the direction-cosines of the normal of either of these surfaces at the common point are proportional to

$$\frac{\xi}{A+M\rho^2-H}, \quad \frac{\eta}{B+M\rho^2-H}, \quad \frac{\zeta}{C+M\rho^2-H}, \tag{146}$$

(143) shew that the principal axes of the body at the point (ξ, η, ζ) lie along the normals to the three confocal surfaces of the second degree which intersect orthogonally at it. Or if we take one surface only, the three principal axes are respectively normal to it, and touch the lines of curvature of the surface at the point.

As we pass from one point of a body or system of particles to another, ρ^2 and H vary; so that in the denominators of the left-hand member of (145) $M\rho^2-H$ varies according as the point changes at which the principal moments are to be determined; and thus the ellipsoid, with which all the surfaces of the second degree represented by (145) are confocal, is that whose equation is

$$\frac{x^2}{A}+\frac{y^2}{B}+\frac{z^2}{C}=\frac{1}{M}. \tag{147}$$

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Let a, b, c be the principal radii of gyration, so that

$$\text{A} = \text{M}a^2, \qquad \text{B} = \text{M}b^2, \qquad \text{C} = \text{M}c^2; \tag{148}$$

then (147) becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{149}$$

which ellipsoid is called the central ellipsoid of gyration, as its principal axes are the radii of the body relative severally to them. Hence we have the following construction for the position of the principal axes at any point in space. Through the given point let three surfaces of the second degree be drawn which are confocal with the central ellipsoid of gyration, the tangents to the three lines of intersection of these surfaces are the principal axes at the point; and the principal moments at the point are the three roots of the cubic equation (142).

Again, let us multiply the numerators and denominators of the three last terms of (140) severally by

$$(\text{B}-\text{C})t_2t_3, \qquad (\text{C}-\text{A})t_3t_1, \qquad (\text{A}-\text{B})t_1t_2;$$

and let us add the numerators and the denominators respectively; as the sum of the numerators vanishes, so must also the sum of the denominators; and therefore

$$(\text{B}-\text{C})\xi t_2t_3 + (\text{C}-\text{A})\eta t_3t_1 + (\text{A}-\text{B})\zeta t_1t_2 = 0.$$

Now as t_1, t_2, t_3 are the direction-cosines of one of the principal axes at the point (ξ, η, ζ) referred to central principal axes, let us replace them by x, y, z; whence we have

$$(\text{B}-\text{C})\xi yz + (\text{C}-\text{A})\eta zx + (\text{A}-\text{B})\xi xy = 0; \tag{150}$$

which is the equation to a cone of the second degree; on the surface of which therefore are all the principal axes at the point (ξ, η, ζ), this being the vertex of the cone. Since (150) is satisfied when x, y, z are proportional to ξ, η, ζ it follows that the line drawn from (ξ, η, ζ) to the centre of gravity lies on the cone. It is evident also that the three axes of x, y, z lie in the surface of the cone. Hence we have the following geometrical theorem:

The three principal axes at any point of a body, the three lines drawn through that point parallel to the central principal axes, and the line drawn from the point to the centre of gravity all lie in the surface of a cone of the second degree.

104.] The ellipsoid of gyration is the sphero-polar reciprocal of the central ellipsoid with reference to the sphere

$$x^2 + y^2 + z^2 = g^2. \tag{151}$$

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The equation to the central ellipsoid (120) is

$$a^2\xi^2 + b^2\eta^2 + c^2\zeta^2 = g^4; \tag{152}$$

and the polar plane of (151), relatively to (ξ, η, ζ) as the pole, is

$$x\xi + y\eta + z\zeta = g^2;$$

therefore we have

$$\left.\begin{aligned} x\,d\xi + y\,d\eta + z\,d\zeta &= 0, \\ a^2\xi\,d\xi + b^2\eta\,d\eta + c^2\zeta\,d\zeta &= 0; \end{aligned}\right\}$$

$$\therefore \quad \frac{a^2\xi}{x} = \frac{b^2\eta}{y} = \frac{c^2\zeta}{z} = \frac{a^2\xi^2 + b^2\eta^2 + c^2\zeta^2}{x\xi + y\eta + z\zeta} = g^2;$$

and, substituting in (152), we have

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1; \tag{153}$$

which is the central ellipsoid of gyration; and therefore the polar reciprocal of the central ellipsoid with respect to the sphere (151) is an ellipsoid confocal with which are those surfaces of the second degree which intersect orthogonally at any point of a rigid body along the principal axes at that point. Of this ellipsoid of gyration, see Art. 20, the momental ellipsoid is the sphero-reciprocal polar.

If p is the perpendicular from the origin on the tangent plane of (153),

$$\frac{1}{p^2} = \frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}; \tag{154}$$

and the direction-cosines of the perpendicular are severally $\frac{px}{a^2}, \frac{py}{b^2}, \frac{pz}{c^2}$; so that the moment of inertia of the body about p, by (109),

$$\begin{aligned} &= \text{A}\frac{p^2x^2}{a^4} + \text{B}\frac{p^2y^2}{b^4} + \text{C}\frac{p^2z^2}{c^4} \\ &= \text{M}p^2\left\{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right\} \\ &= \text{M}p^2; \end{aligned} \tag{155}$$

so that p is the radius of gyration of the body about the axis which coincides with p. Hence we have the following theorem;

If the central ellipsoid of gyration is described, and on a tangent plane drawn to any point of it a perpendicular is let fall from the centre, the length of that perpendicular is the radius of gyration of the body relatively to it as the rotation-axis.

The locus-surface of the extremity of p is

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2; \tag{156}$$

of this surface any radius vector is the radius of gyration of the body rotating about it.

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This result also follows directly from (109). For if r is the radius of gyration about the rotation-axis (α, β, γ), then its moment of inertia is $\text{M}r^2$, and we have

$$\text{M}r^2 = \text{M}a^2(\cos\alpha)^2 + \text{M}b^2(\cos\beta)^2 + \text{M}c^2(\cos\gamma)^2;$$

$$\therefore \quad (x^2+y^2+z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2;$$

and thus, if we choose to begin with this equation, the theorems already proved may be deduced by an inversion of the processes of the preceding Article. The advantage of the method would be that we should be rid of the undetermined constant g until it was introduced as the radius of the sphere, relatively to which the sphero-polar reciprocal of the central ellipsoid of gyration would be the central momental ellipsoid.

105.] Now the three surfaces of the second degree, which equation (145) represents, and which are confocal with the central ellipsoid of gyration, intersect orthogonally at not only (ξ, η, ζ), but at seven other points which are situated symmetrically in the other octants, and which correspond to the several combinations of the double signs of ξ, η, and ζ. The equation (142) is the same whatever are the signs of ξ, η, and ζ; so that the principal moments are the same at each of the eight points, which are the angles of a rectangular parallelepipedon whose centre is at the centre of gravity, and whose sides are parallel to the central principal axes: and as the equations for determining the position of the principal axes (143) are the same when the signs of ξ, η, ζ are all changed, so the principal axes at (ξ, η, ζ) are parallel to those at $(-\xi, -\eta, -\zeta)$; and similarly the principal axes at the other six points of symmetry are arranged in pairs corresponding to the ends of a diameter.

Thus the body or system of particles is symmetrically arranged as to principal axes, principal moments, and all moments of inertia, relative to the centre of gravity, the central axes, and the central principal planes. And as space is divided by the principal central planes into eight portions, so to a point in any octant a point of symmetry corresponds in each of the other seven octants, at which the principal moments are equal, and the momental ellipsoid is similarly situated with respect to the centre and the principal central axes. Therefore whatever is the form of the moving system, be it a continuous body or a system of disconnected particles, however various the distribution of its parts, however unsymmetrical its bounding surface, yet it has a centre of gravity,

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central axes, and a central momental ellipsoid; and the arrangement of all other moments and axes is symmetrical relatively to that point. In discussing therefore the rotation of an irregular mass about an axis passing through a fixed point, we may dismiss from our minds all the irregularities of the mass, and consider in its stead either the regular and symmetrical central ellipsoid, or, as the late Professor Maccullagh taught in his lectures, the central ellipsoid of gyration; for the properties of either of these surfaces will express all the possible circumstances of motion of the system.

106.] I must also mention another construction whereby the position of the principal axes at a given point may be determined.

From the point as a vertex let a cone be described enveloping the central ellipsoid of gyration (149); then, as we have shewn in Art. 19, the principal axes of the cone at its vertex are the normals to the three surfaces of the second degree which intersect at it and are confocal with the enveloped ellipsoid; and as the principal axes lie along these three normals, so they also coincide with the principal axes of the cone at the point.

The cone which is reciprocal to this enveloping cone is an equimomental cone: this might be demonstrated directly from the equation to the enveloping cone which is given in (134), Art. 19; for if we determined the equation to its reciprocal cone, it would be identical with (123), Art. 100. The following proof however is more concise. Through the given point let a tangent plane be drawn to the ellipsoid of gyration; this plane being evidently a tangent plane to the enveloping cone. To it let perpendiculars be drawn from the centre of the ellipsoid and from the given point: let the distance between these perpendiculars be q, say; then, since by (155) the moment of inertia of the body about the former line $= \text{M}p^2$, therefore the moment of inertia about the latter line, by reason of Art. 101, is

$$\text{M}p^2 + \text{M}q^2 = \text{M}r^2,$$

if r is the distance of the given point from the centre of gravity: but this latter line is a generating line of the reciprocal cone, and r^2 is the same for all generating lines of the cone; and therefore the moment of inertia is the same for all generating lines of the reciprocal cone; and consequently the reciprocal cone is equimomental.

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Thus also may a series of equimomental cones be described; for the number of surfaces of the second degree confocal with the central ellipsoid of gyration is unlimited: and as to each of these an enveloping cone may be drawn, so may the reciprocal cone of each be described, and thus will there be a series of equimomental cones.

The focal conics are particular and degenerate forms of surfaces of the second degree which are confocal with the central ellipsoid of gyration. Their equations have been found in Art. 17; and are

$$\left.\begin{aligned} \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2} &= 1, \\ \frac{z^2}{c^2-b^2}+\frac{x^2}{a^2-b^2} &= 1, \\ \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2} &= 1; \end{aligned}\right\} \qquad (157)$$

and as by our hypothesis $c > b > a$, so are these curves respectively an ellipse in the plane of yz, an hyperbola in the plane of zx, and an imaginary conic in the plane of xy. These curves then in their several planes may be taken to be directors of the cones whose geometrical principal axes are the principal axes of the body at the vertex of the cone.

If the surface of the second order, confocal with the ellipse of gyration, passes through the point at which the principal axes are to be constructed, the enveloping cones degenerate into tangent planes; and the reciprocal cones become normals to these planes: thus the two constructions, by means of the three confocal conics, and by the axes of the enveloping cone, become identical.

107.] I propose in the next place to inquire into the locus-surface of those points (ξ, η, ζ) at which *one* of the principal moments has a constant value. Let k be the radius of gyration corresponding to this given value of the principal moment of inertia; so that if H is the principal moment, $\text{H} = \text{M}k^2$: also let a, b, c be the principal central radii of gyration; so that

$$\text{A} = \text{M}a^2, \qquad \text{B} = \text{M}b^2, \qquad \text{C} = \text{M}c^2;$$

and thus (145) becomes

$$\frac{x^2}{r^2+a^2-k^2}+\frac{y^2}{r^2+b^2-k^2}+\frac{z^2}{r^2+c^2-k^2} = 1; \qquad (158)$$

where $$r^2 = x^2+y^2+z^2;$$

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and if we replace 1 in the right-hand member by $\frac{x^2+y^2+z^2}{r^2}$, we have

$$\frac{x^2(a^2-k^2)}{r^2+a^2-k^2}+\frac{y^2(b^2-k^2)}{r^2+b^2-k^2}+\frac{z^2(c^2-k^2)}{r^2+c^2-k^2}=0. \qquad (159)$$

The surfaces which these equations represent have been named by Professor William Thomson Equimomental Surfaces. As generally at every point k has three different values, so will three equimomental surfaces pass through every point.

By giving different values to k we have different equimomental surfaces. According to our hypothesis a is the least radius of gyration for all axes passing through the centre of gravity; it is therefore absolutely the least of all radii of gyration, but they have no superior limit; so that k may have all values from a to $+\infty$. Now (159) will express surfaces different in form according as k is greater than c; lies between c and b; is equal to b; and lies between b and a. If k is greater than c, the equimomental surface is the same as the wave surface in biaxal crystals*.

The principal axis at a given point (x, y, z) of the equimomental surface lies in the tangent plane at that point; and passes through the point where a perpendicular from the origin on the tangent plane meets it.

Let us replace A, B, C, H in (143), severally by Ma^2, Mb^2, Mc^2, Mh^2; then the direction-cosines at the point (x, y, z) are respectively proportional to

$$\frac{x}{r^2+a^2-k^2}, \quad \frac{y}{r^2+b^2-k^2}, \quad \frac{z}{r^2+c^2-k^2}. \qquad (160)$$

Now let l, m, n be the direction-cosines of the line drawn through (x, y, z) to the point of intersection of the tangent plane of (158) with the perpendicular on it from the centre. Then, if $F(x,y,z)=0$ is the equation to the surface, the direction-cosines of this line are easily shewn to be proportional to

$$x\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}-\left(\frac{dF}{dx}\right)\left\{x\left(\frac{dF}{dx}\right)+y\left(\frac{dF}{dy}\right)+z\left(\frac{dF}{dz}\right)\right\},$$

$$y\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}-\left(\frac{dF}{dy}\right)\left\{x\left(\frac{dF}{dx}\right)+y\left(\frac{dF}{dy}\right)+z\left(\frac{dF}{dz}\right)\right\},$$

$$z\left\{\left(\frac{dF}{dx}\right)^2+\left(\frac{dF}{dy}\right)^2+\left(\frac{dF}{dz}\right)^2\right\}-\left(\frac{dF}{dz}\right)\left\{x\left(\frac{dF}{dx}\right)+y\left(\frac{dF}{dy}\right)+z\left(\frac{dF}{dz}\right)\right\};$$

* A full discussion of this surface will be found in a Thèse de Mécanique, by M. Peslin. Mallet-Bachelier, Paris, 1858.

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Let $$\frac{x^2}{(r^2+a^2-k^2)^2}+\frac{y^2}{(r^2+b^2-k^2)^2}+\frac{z^2}{(r^2+c^2-k^2)^2}=\mathrm{s}\,; \quad (161)$$
then from (158)
$$\left(\frac{d\mathrm{F}}{dx}\right)=\frac{2x}{r^2+a^2-k^2}-2x\mathrm{s},$$
$$\left(\frac{d\mathrm{F}}{dy}\right)=\frac{2y}{r^2+b^2-k^2}-2y\mathrm{s},$$
$$\left(\frac{d\mathrm{F}}{dz}\right)=\frac{2z}{r^2+c^2-k^2}-2z\mathrm{s}\,;$$
so that l, m, n are proportional to
$$\frac{x}{r^2+a^2-k^2},\quad \frac{y}{r^2+b^2-k^2},\quad \frac{z}{r^2+c^2-k^2}; \quad (162)$$
which are the same as (160): and thus the theorem as enunciated is proved.

108.] I must also explain another process by which the equimomental surface is found. It occurred to Professor Maccullagh to draw through a fixed point a series of planes intersecting a given surface, and at the point to draw perpendiculars to the several planes of lengths equal to the apsidal radii vectores of the section of the surface by the plane; one or more surfaces would be the locus of the extremities of these perpendiculars, and these he called the apsidals or apsidal surfaces.

Let this process be performed on a central surface of the second degree at its centre; and let the equation to the surface be
$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1\,; \quad (163)$$
let any plane of section be
$$lx+my+nz=0\,; \quad (164)$$
and let the apsidal radius vector in this plane be r; so that
$$r^2=x^2+y^2+z^2\,; \quad (165)$$
let (ξ, η, ζ) be the end of the perpendicular; then, as the perpendicular is to be equal to r, and to be perpendicular to the plane (164), we have
$$\frac{\xi}{l}=\frac{\eta}{m}=\frac{\zeta}{n}=r. \quad (166)$$
Differentiating (163), (164), and (165), and putting $\mathrm{D}r=0$, as r is an apsidal radius vector of the section of (163), we have
$$\frac{x}{\alpha^2}dx+\frac{y}{\beta^2}dy+\frac{z}{\gamma^2}dz=0\,;$$
$$l\,dx+m\,dy+n\,dz=0\,;$$
$$x\,dx+y\,dy+z\,dz=0:$$

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multiplying the first of these by λ, and the second by μ, adding, and equating to zero the coefficients of dx, dy, dz, we have

$$\lambda \frac{x}{a^2} + \mu l + x = 0,$$

$$\lambda \frac{y}{\beta^2} + \mu m + y = 0,$$

$$\lambda \frac{z}{\gamma^2} + \mu n + z = 0;$$

multiplying these severally by x, y, z, and adding, we have

$$\lambda + r^2 = 0;$$

$$\left.\begin{aligned} \therefore\quad x\left(1-\frac{r^2}{a^2}\right) + \mu l &= 0, \\ y\left(1-\frac{r^2}{\beta^2}\right) + \mu m &= 0, \\ z\left(1-\frac{r^2}{\gamma^2}\right) + \mu n &= 0; \end{aligned}\right\} \qquad \left.\begin{aligned} \therefore\quad x + \frac{a^2\mu l}{a^2-r^2} &= 0, \\ y + \frac{\beta^2\mu m}{\beta^2-r^2} &= 0, \\ z + \frac{\gamma^2\mu n}{\gamma^2-r^2} &= 0; \end{aligned}\right\} \quad (167)$$

and multiplying these last severally by l, m, n, and adding,

$$\frac{a^2 l^2}{a^2-r^2} + \frac{\beta^2 m^2}{\beta^2-r^2} + \frac{\gamma^2 n^2}{\gamma^2-r^2} = 0;$$

and from (166)

$$\frac{a^2\xi^2}{r^2-a^2} + \frac{\beta^2\eta^2}{r^2-\beta^2} + \frac{\gamma^2\zeta^2}{r^2-\gamma^2} = 0; \qquad (168)$$

and adding to both sides r^2, which is equal to $\xi^2+\eta^2+\zeta^2$,

$$\frac{\xi^2}{r^2-a^2} + \frac{\eta^2}{r^2-\beta^2} + \frac{\zeta^2}{r^2-\gamma^2} = 1; \qquad (169)$$

which is the same equation as (158) if

$$a^2 = k^2-a^2, \qquad \beta^2 = k^2-b^2, \qquad \gamma^2 = k^2-c^2, \qquad (170)$$

and therefore the equimomental surface is the apsidal of the surface of the second order (163); which is an ellipsoid, an hyperboloid of one sheet, or an hyperboloid of two sheets, according as

$$k^2 \text{ is } > c^2, \text{ is } > b^2 < c^2, \text{ is } > a^2 < b^2.$$

109.] We must not continue the discussion of the properties of the equimomental surface; let us however investigate the positions of those points at which (1) two of the three principal moments, (2) all three principal moments, are equal.

If (ξ, η, ζ) is a point at which two principal moments of inertia are equal, two of the three values of H in (142) are equal to one

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another; and consequently (142) and its H-differential are simultaneously true. Now the H-differential may be put into the form

$$\frac{\xi^2}{(A+M\rho^2-H)^2}+\frac{\eta^2}{(B+M\rho^2-H)^2}+\frac{\zeta^2}{(C+M\rho^2-H)^2}=0; \quad (171)$$

or $\xi^2(B+M\rho^2-H)^2(C+M\rho^2-H)^2+\eta^2(C+M\rho^2-H)^2(A+M\rho^2-H)^2$

$$+\zeta^2(A+M\rho^2-H)^2(B+M\rho^2-H)^2=0. \quad (172)$$

All the roots of this equation as it stands are imaginary; and as the reality of the roots of (142) has been demonstrated, (172) must be satisfied identically: this may be done as follows:

(1) Let $\xi=0$, and $H=A+M\rho^2$; in which case (142) becomes

$$\frac{\eta^2}{B-A}+\frac{\zeta^2}{C-A}=\frac{1}{M}; \quad (173)$$

and since $A=Ma^2$, $B=Mb^2$, $C=Mc^2$, this becomes

$$\frac{\eta^2}{b^2-a^2}+\frac{\zeta^2}{c^2-a^2}=1; \quad (174)$$

and, according to our assumption, $c>b>a$; thus (174) represents an ellipse in the plane of (η,ζ), and is in that plane the focal conic of the ellipsoid of gyration. Two of the principal moments of inertia $=A+M\rho^2$; and the third $=B+C-A$.

(2) Let $\eta=0$, and $H=B+M\rho^2$; in which case (142) becomes

$$\frac{\xi^2}{a^2-b^2}+\frac{\zeta^2}{c^2-b^2}=1; \quad (175)$$

which represents an hyperbola in the plane of (ζ,ξ), whose real and imaginary axes lie respectively along the axes of ζ and ξ, and is another focal conic of the ellipsoid of gyration. Two of the principal moments of inertia $=B+M\rho^2$; and the third principal moment $=C+A-B$.

(3) Let $\zeta=0$; and $H=C+M\rho^2$; then (142) becomes

$$\frac{\xi^2}{a^2-c^2}+\frac{\eta^2}{b^2-c^2}=1; \quad (176)$$

which is the other focal conic of the ellipsoid of gyration, and is imaginary. At all points therefore of the real focal conics of the ellipsoid of gyration, two roots of (142) are equal; and two principal moments are equal: the tangent line to the focal conic is the axis of the unequal principal moment, and the normal plane to the focal conic is the plane which contains the axes of equal moments. All axes therefore in this plane which pass through the point of contact are axes of equal moment: so that the number of axes of equal moment is infinite. Indeed the other two

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surfaces of the second degree, which with the focal conic are confocal with the ellipsoid of gyration, become flat, and infinitesimally thin; so that any plane which passes through the tangent of the focal conic is a tangent plane to one of these surfaces, and the perpendiculars to these planes at the point of contact are principal axes.

This result is also evident from the construction of principal axes which is given by the enveloping cone of the ellipsoid of gyration: the enveloping conic is a cone of revolution if its vertex is on a focal conic; the tangent of the conic is the internal axis of the cone; and any two lines in the plane through the vertex of the cone, which is perpendicular to the internal axis, are external axes. In this construction however it is to be observed that the enveloping cones may be imaginary.

Hence we have two distinct curves of the second degree in the planes of (η, ζ) and of (ζ, ξ), which are respectively an ellipse and hyperbola, at every point of which the position of one principal axis of the body is determinate; but as the moments corresponding to the other two principal axes are equal, the position of such axes is indeterminate.

At every point on a focal conic, the momental ellipsoid becomes a spheroid, whose axis of revolution is the tangent to the focal conic.

110.] If (ξ, η, ζ) is a point at which all the principal moments are equal, the three roots of (142) are equal; and (136) represents a sphere; so that

$$\eta\zeta = \zeta\xi = \xi\eta = 0; \tag{177}$$

$$\text{A} + \text{M}(\eta^2 + \zeta^2) = \text{B} + \text{M}(\zeta^2 + \xi^2) = \text{C} + \text{M}(\xi^2 + \eta^2). \tag{178}$$

From (177) it follows that of the three quantities ξ, η, ζ two must be equal to zero; and as C is $>$ B $>$ A, the only possible supposition is

$$\xi = \eta = 0;$$

therefore

$$\text{A} + \text{M}\zeta^2 = \text{B} + \text{M}\zeta^2 = \text{C};$$

$$\therefore \quad \text{A} = \text{B}; \tag{179}$$

$$\zeta = \pm\left(\frac{\text{C} - \text{A}}{\text{M}}\right)^{\frac{1}{2}}; \tag{180}$$

therefore two of the central principal moments, viz. the mean and the least, must be equal to each other; and thus the central ellipsoid must be a prolate spheroid; in which case, on the axis of greatest central moment there are two points, viz. the foci of the spheroid of gyration, equally distant from the centre of

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gravity, at which all the principal moments are equal, and therefore all axes are principal axes. At each of these points the momental ellipsoid becomes a sphere.

If $C = A$, that is, if all the central principal moments are equal, from (180), $\zeta = 0$; and at no other point in the body, but the centre of gravity, are all the principal moments equal.

Now these results might have been arrived at from considerations founded on the properties of the focal conics: the three principal moments can be equal only when the focal conics of the ellipsoid of gyration have a common point; and as (174) and (175) can have a common point only on the central axis of ζ, $\xi = \eta = 0$; in which case

$$\zeta = \pm (c^2 - a^2)^{\frac{1}{2}} = \pm (c^2 - b^2)^{\frac{1}{2}};$$

$$\therefore \quad a = b, \quad \text{or} \quad A = B;$$

and
$$\zeta = \pm \left(\frac{C - A}{M}\right)^{\frac{1}{2}}.$$

The results of this and the preceding Articles are important in reference to principal axes, and to their properties as permanent axes: the couple of the centrifugal forces vanishes whenever the rotation-axis is a principal axis; and if it is a central principal axis, having its origin at the centre of gravity, the pressure at the origin, which is due to the centrifugal forces, also vanishes: at the centre of gravity therefore there are three, and generally only three, permanent axes, which are the principal central axes; unless two of the three quantities A, B, C are equal, in which case all axes perpendicular to the unequal principal axis are permanent axes. And if $A = B = C$, every axis through the centre of gravity is a permanent axis.

For all points on the focal conics the position of one principal axis is determinate, and every axis which is perpendicular to that axis is a permanent axis; so that the number of permanent axes is infinite; in this case however the resultant pressure of the centrifugal forces does not vanish; and the axes do not generally pass through the centre of gravity; in fact a permanent axis passes through the centre of gravity only when the origin is at a vertex of a focal conic.

At the two points on the central axis of greatest moment, given by (180), all axes are principal, and therefore permanent axes; and of course only one axis at each point passes through the centre of gravity.

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111.] The investigations of the preceding Articles, and the methods given for the construction of principal axes, shew that an axis taken arbitrarily on a body may not be a principal axis at any point on it; because those axes alone are principal which are normal to some surface of the second degree which is confocal with the ellipsoid of gyration.

If an axis of a body is a principal axis, let us call that point at which it is principal, its principal point; and let us call the plane which is perpendicular to it at its principal point, and which contains the other two principal axes, its principal plane; so that a principal plane of an axis is a plane tangent to a surface of the second degree, confocal with the central ellipsoid of gyration at the point where its axis cuts that surface.

In considering therefore the axes of a body, we may distinguish (1) those which are principal at every point along them; (2) those which are principal at one point; (3) those which are not principal at all. Let us consider them in order.

The three principal central axes cut at right angles all the surfaces of the second degree which are confocal with the central ellipsoid of gyration; and as the number of such surfaces is infinite, so every point on a central principal axis is principal; and as the other two confocal surfaces at these points degenerate into the coordinate planes, the other two principal axes are always parallel to the central principal axes.

The three central principal axes are the only lines which have the property of being principal at every point on them: in this respect then, as in others, they form an unique system.

Some special cases of axes which are principal at a particular point deserve consideration.

The sphere whose centre is at the centre of gravity, and whose radius is infinitely great, is a surface confocal with the central ellipsoid of gyration; and as all lines drawn through the centre of gravity are normal to this sphere at the infinity point, so is every line drawn through the centre of gravity a principal axis at a point which is at an infinite distance along it.

If three confocal surfaces of the second order pass through a point in one of the central principal planes, one of the confocal surfaces becomes flat, and the normal to this surface is a perpendicular to the central plane; so that one of the principal axes at that point is always normal to the principal central plane; and thus all axes parallel to a central principal axis are

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principal at the points where they intersect a central principal plane: the other two principal axes are the tangent and the normal respectively to a line passing through the point, and which is confocal with the trace of the central ellipsoid of gyration in that plane.

112.] Let us however determine the general conditions which are to be satisfied when a line is a principal axis at one of its points; and let us find its principal point, and the equation to its principal plane.

Let the centre of gravity be the origin, and let the central principal axes be the coordinate axes; let the equations to a certain line be

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}=s, \text{ (say)}; \tag{181}$$

and let the equation to a plane perpendicular to it be

$$lx+my+nz-p=0. \tag{182}$$

Now if (181) is a principal axis, and (182) is its principal plane, (181) is the normal to, and (182) is the tangent plane at the same point to, a surface of the second degree which is confocal with the central ellipsoid of gyration.

Let the equation to this confocal surface be

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1; \tag{183}$$

as (181) is to be normal to this surface we have

$$\frac{x}{l(a^2+\theta)}=\frac{y}{m(b^2+\theta)}=\frac{z}{n(c^2+\theta)}; \tag{184}$$

$$\therefore\quad \frac{b^2-c^2}{l}x+\frac{c^2-a^2}{m}y+\frac{a^2-b^2}{n}z=0; \tag{185}$$

which is a condition to be satisfied by the coordinates of the point on (183), at which the given line pierces the surface.

Also, since from (181),

$$\frac{x}{l}=\frac{x_0}{l}+s,\quad \frac{y}{m}=\frac{y_0}{m}+s,\quad \frac{z}{n}=\frac{z_0}{n}+s, \tag{186}$$

(185) becomes

$$\frac{b^2-c^2}{l}x_0+\frac{c^2-a^2}{m}y_0+\frac{a^2-b^2}{n}z_0=0,$$

or $$mn(b^2-c^2)x_0+nl(c^2-a^2)y_0+lm(a^2-b^2)z_0=0; \tag{187}$$

which is a relation between the elements of the line (181) and the central principal radii of gyration, which must be satisfied, when (181) is a principal axis at one of its points.

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Now (187) deserves consideration, as it expresses geometrical theorems of great importance. It is the equation to a plane in terms of x_0, y_0, z_0, if l, m, n are constant; therefore all parallel straight lines, which are principal axes at some one point on them, are in the same plane.

Also this plane contains the central radius vector of the central ellipsoid which is parallel to the system of lines, and the normal to the ellipsoid at the point where the radius vector meets the vector; because (187) is the condition of the coexistence of the three equations

$$\left.\begin{aligned} \text{L}\, x_0 + \text{M}\, y_0 + \text{N}\, z_0 &= 0, \\ \text{L}\, a^2 l + \text{M}\, b^2 m + \text{N}\, c^2 n &= 0, \\ \text{L}\, l + \text{M}\, m + \text{N}\, n &= 0; \end{aligned}\right\} \tag{188}$$

which express a plane fulfilling the above stated conditions.

Hence also if a straight line is a principal axis, it is parallel to that central radius vector of the central ellipsoid which is drawn to the point at which the normal to the ellipsoid meets the given straight line.

Again, let us consider (187) when x_0, y_0, z_0 are constant, and l, m, n vary; let us take the point (x_0, y_0, z_0) to be an origin, and on the line (l, m, n) passing through it let us take a point (x, y, z); so that we may replace l, m, n in (187) generally by x, y, z; then (187) becomes

$$(b^2 - c^2) x_0 yz + (c^2 - a^2) y_0 zx + (a^2 - b^2) z_0 xy = 0; \tag{189}$$

which is the equation of a cone of the second degree. This shews that all principal axes which pass through the point (x_0, y_0, z_0) lie on a cone of the second degree. It is also evident from the equation (189) that the three lines parallel to the central principal axes, and the line drawn from the centre of gravity to the given point, lie on the surface of this cone. This is an extension of the theorem given by (150).

It is hence evident that all straight lines passing through the centre of gravity are principal axes at some point on them.

Now the principal point of (181) is its point of section with the surface (183), at which point also (182) touches the surface; but, if (182) is the equation to a tangent plane of (183), we have

$$\frac{x}{l(a^2+\theta)} = \frac{y}{m(b^2+\theta)} = \frac{z}{n(c^2+\theta)} = \frac{1}{p}$$

$$= \frac{lx + my + nz}{a^2 l^2 + b^2 m^2 + c^2 n^2 + \theta} = \frac{p}{a^2 l^2 + b^2 m^2 + c^2 n^2 + \theta}; \tag{190}$$

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therefore $$a^2l^2+b^2m^2+c^2n^2+\theta=p^2;$$
and $$\theta=p^2-(a^2l^2+b^2m^2+c^2n^2).$$
Also substituting in (190) the values of x, y, z, given in (186), and eliminating s and θ, we have

$$p=\frac{mn(b^2-c^2)}{ny_0-mz_0}=\frac{nl(c^2-a^2)}{lz_0-nx_0}=\frac{lm(a^2-b^2)}{mx_0-ly_0}; \qquad (191)$$

whereby p is given; and thus θ is determined; and therefore the confocal surface of the second degree is determined.

And from (190) we have

$$\left.\begin{aligned} x&=\frac{l(a^2+\theta)}{p}=\frac{l}{p}\{p^2+a^2-a^2l^2-b^2m^2-c^2n^2\},\\ y&=\frac{m(b^2+\theta)}{p}=\frac{m}{p}\{p^2+b^2-a^2l^2-b^2m^2-c^2n^2\},\\ z&=\frac{n(c^2+\theta)}{p}=\frac{n}{p}\{p^2+c^2-a^2l^2-b^2m^2-c^2n^2\};\end{aligned}\right\} \qquad (192)$$

which are the coordinates of the principal point of the line (181).

As p is given in (191), the equation to the plane which is principal to (181) is

$$lx+my+nz=\frac{mn(b^2-c^2)}{my_0-nz_0}=\frac{nl(c^2-a^2)}{lz_0-nx_0}=\frac{lm(a^2-b^2)}{mx_0-ly_0}. \qquad (193)$$

If $a=b=c$, this principal plane passes through the centre of gravity of the system; and the principal point on the axis is that where the perpendicular from the origin falls on it.

Now of all principal axes passing through the point (x_0, y_0, z_0) and lying on the cone (189), the principal points are the points of intersection of (193) with the axis (181), so that if we replace l, m, n, in (193) by their proportionals $x-x_0$, $y-y_0$, $z-z_0$, which are given in (181), we have the equation of a surface on which these principal points are: this process gives

$$\begin{aligned} x(x-x_0)+y(y-y_0)+z(z-z_0)&=\frac{(y-y_0)(z-z_0)(b^2-c^2)}{zy_0-yz_0}\\ &=\frac{(z-z_0)(x-x_0)(c^2-a^2)}{xz_0-zx_0}\\ &=\frac{(x-x_0)(y-y_0)(a^2-b^2)}{yx_0-xy_0};\end{aligned}$$

which are three surfaces of the third degree; any two of which along their line of intersection give the principal points of all axes passing through (x_0, y_0, z_0); or any one of which, together with the cone (189), determines these points.

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The lines of the principal points of the system of parallel principal axes, which lie in the plane (187), is an equilateral hyperbola, which may thus be found: Let the axis of x lie along the radius vector of the central ellipsoid so that $p = x$; and let the centre of gravity be the origin; then

$$y^2 = (my_0 - nz_0)^2 + (lz_0 - nx_0)^2 + (mx_0 - ly_0)^2);$$

then from (191) we have

$$xy = \{(b^2-c^2)^2m^2n^2 + (c^2-a^2)^2n^2l^2 + (a^2-b^2)^2l^2m^2\}^{\frac{1}{2}};$$

which is the equation to an equilateral hyperbola, of which axes of x and y are the asymptotes. Hence it appears that the principal points of lines passing through the centre of gravity of a body are at an infinite distance.

As the relation (187) must be satisfied by the elements of a line capable of being a principal axis at some one of its points, it is evident that a line taken arbitrarily may not be a principal axis at any point.

113.] All lines which lie in a central principal plane are principal axes at some point. This is evident, because the conditions (187) will be satisfied in these cases. Thus, let the line be in the plane of (x, y), $n = 0$, and $z_0 = 0$; and similarly for the other central principal planes.

It may however be proved, independently of the preceding Article, by the following process. Whatever is the position of the line in the plane (say) of (x, y), a conic, confocal with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

can always be drawn to which this line shall be a normal.

Thus, let the equation to the line be

$$lx + my = p, \tag{194}$$

where $l^2 + m^2 = 1$. And let the equation to the conic, confocal with the conic of gyration, be

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} = 1; \tag{195}$$

where θ is to be determined so that (194) may be a normal to (195); whence we have

$$\frac{lx}{a^2+\theta} = -\frac{my}{b^2+\theta} = \frac{p}{a^2-b^2}; \tag{196}$$

$$= \left\{\frac{a^2+\theta}{l^2} + \frac{b^2+\theta}{m^2}\right\}^{-\frac{1}{2}};$$

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therefore $$\theta = \frac{l^2 m^2}{p^2}(a^2-b^2)^2 - a^2m^2 - b^2l^2\,; \tag{197}$$

and therefore from (196)

$$x = \frac{l}{p}\{a^2m^2 - b^2m^2 + p^2\}\,; \tag{198}$$

$$y = -\frac{m}{p}\{a^2l^2 - b^2l^2 - p^2\}\,; \tag{199}$$

which determine the principal point in the line (194); and shew that whatever that line is, it is always a principal axis, and has consequently a principal point.

If $p = 0$, $x = y = \infty$; that is, if a line in a principal central plane passes through the centre of gravity, the principal point of that line is at an infinite distance; a theorem which has been stated before. If $p = 0$, and $l = 0$, $x = \frac{0}{0}$, $y = 0$; thus on the central axis of x every point is a principal point. A similar theorem is true of the axis of y.

Also, since through the point (x, y), as defined by (198) and (199), two confocal conics can be drawn which intersect it at right angles, the given line (194) will be a normal to one conic, and a tangent to the other confocal conic. The other principal axis is perpendicular to the central plane. It is easy therefore to construct the principal axes at a point in a central principal plane; through it describe the two conics which are confocal with the focal conic in that plane, the tangents and the normals to these two confocal cones are the principal axes.

It is unnecessary to say more as to lines in space which may not be principal axes at all; the criterion of such lines is, that their equations do not satisfy the condition (187). I will however again observe that this fact is evident from this consideration. Let the given straight line be produced to meet one of the principal central planes, and let the polar of that point be drawn relatively to the focal conic in that plane; it is evident that the trace of a plane perpendicular to the given line need not be parallel with that line. If it is parallel, the original line has a principal point, and is a principal axis. The condition of parallelism is expressed by (187).

114.] Although a line in space may not be a principal axis at all, yet every plane is a principal plane for some point in it, because, whatever is the position of the plane, it is a tangent plane to some surface of the second degree which is confocal with the

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central ellipsoid of gyration. And the principal axis may be found in the following way. Let us consider the trace of the plane on one of these central principal planes in which the focal conic is real; and let this trace be considered a polar relatively to that conic; let the corresponding pole be then determined, and from it let a perpendicular be drawn to the given plane; that perpendicular is evidently the principal axis of the plane, and the point of intersection of it with the plane is the principal point of the plane. But we may investigate these results mathematically, by the process which we have followed in Art. 112.

Let the equation to the plane be

$$lx + my + nz = p, \tag{200}$$

where $l^2 + m^2 + n^2 = 1$; and let (ξ, η, ζ) be the point where this plane touches the surface

$$\frac{\xi^2}{a^2+\theta} + \frac{\eta^2}{b^2+\theta} + \frac{\zeta^2}{c^2+\theta} = 1; \tag{201}$$

of which the tangent plane at (ξ, η, ζ) is

$$\frac{\xi x}{a^2+\theta} + \frac{\eta y}{b^2+\theta} + \frac{\zeta z}{c^2+\theta} = 1; \tag{202}$$

and as (200) and (202) are identical, we have

$$\frac{l(a^2+\theta)}{\xi} = \frac{m(b^2+\theta)}{\eta} = \frac{n(c^2+\theta)}{\zeta} = p \tag{203}$$

$$= \{l^2(a^2+\theta) + m^2(b^2+\theta) + n^2(c^2+\theta)\}^{\frac{1}{2}}; \tag{204}$$

$$\therefore \quad \theta = p^2 - (a^2l^2 + b^2m^2 + c^2n^2); \tag{205}$$

$$\left.\begin{aligned} \therefore \quad \xi &= \frac{l}{p}\{p^2 + a^2 - (a^2l^2 + b^2m^2 + c^2n^2)\}, \\ \eta &= \frac{m}{p}\{p^2 + b^2 - (a^2l^2 + b^2m^2 + c^2n^2)\}, \\ \zeta &= \frac{n}{p}\{p^2 + c^2 - (a^2l^2 + b^2m^2 + c^2n^2)\}; \end{aligned}\right\} \tag{206}$$

so that (205) gives the particular ellipsoid surface which is confocal with the ellipsoid of gyration; and (206) assigns the principal point in the plane, and is that point at which the plane touches the surface (201).

Hence it appears that if a plane is given, a confocal surface can be assigned which shall be touched by that plane; and also the point of contact can be determined, and this is the principal point of the plane.

Let the trace of (200) be taken on the plane of (x, y); then, if

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it is considered a polar relatively to the focal conic in that plane, the pole is $\left(\frac{a^2-c^2}{p}l, \frac{b^2-c^2}{p}m\right)$; and therefore the equations to the axis, which is principal to the plane (200), are

$$\frac{\xi - \frac{a^2-c^2}{p}l}{l} = \frac{\eta - \frac{b^2-c^2}{p}m}{m} = \frac{\zeta}{n};$$

or, as they may be expressed,

$$\frac{\xi - \frac{a^2 l}{p}}{l} = \frac{\eta - \frac{b^2 m}{p}}{m} = \frac{\zeta - \frac{c^2 n}{p}}{n}. \tag{207}$$

Hence every plane is a principal plane at some one point of it. A central principal plane is a principal plane for every point of it, because every axis which is perpendicular to a central principal plane is a principal axis with its principal point in the central principal plane.

Also the plane at an infinite distance is a principal plane at every point of it; and all the corresponding principal axes pass through the centre of gravity. The three central principal planes and the plane at infinity alone have this property, that every point in them is a principal point.

SECTION 4.—*Examples of moments of inertia.*

115.] In this section I propose to apply the general formulæ of the preceding section to the calculation of moments of inertia and radii of gyration, relatively to certain given axes, of material lines or wires, of thin plates and curved shells, and of solid bodies. It will be found most convenient to make the calculations with reference to certain axes to which the bodies are geometrically related, and which yield the most simple forms of integration. And by means of them, and the theorems of the preceding section, to investigate the moments of inertia about the given axes. The following theorems are most useful for the purpose.

(1) If at any point of a body A, B, C are the principal moments of inertia, and H is the moment of inertia about the axis (α, β, γ) passing through that point, then

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2; \tag{208}$$

and if $\quad \text{A}' = \Sigma.mx^2, \quad \text{B}' = \Sigma.my^2, \quad \text{C}' = \Sigma.mz^2,$

$$\text{H} = \text{A}'(\sin\alpha)^2 + \text{B}'(\sin\beta)^2 + \text{C}'(\sin\gamma)^2. \tag{209}$$

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(2) If H and H′ are the moments of inertia of the mass M about two parallel axes, one of which passes through a given point, and the other passes through the centre of gravity; and if h is the distance between these axes, then

$$\text{H} = \text{H}' + \text{M}h^2; \tag{210}$$

and therefore if k and k' are the radii of gyration about the axis through the given point, and the parallel axis through the centre of gravity respectively, then

$$\text{H} = \text{M}k^2, \qquad \text{H}' = \text{M}k'^2;$$

$$\therefore \quad k^2 = k'^2 + h^2. \tag{211}$$

116.] The moments of inertia of material lines or wires.

Ex. 1. The moment of inertia of a straight wire of uniform thickness and density.

Let the length of the wire be $2a$, ρ = its density, ω = the area of a transverse section; and let it lie along the axis of x.

(1) Let the rotation-axis be perpendicular to its length, and pass through its middle point; then

$$\text{the moment of inertia} = \int_{-a}^{a} \rho\omega x^2 dx$$

$$= \frac{2}{3}\rho\omega a^3.$$

(2) Let the rotation-axis be perpendicular to the wire, and at a distance c from the middle point which is its centre of gravity; then, by (210), since the mass of the wire $= 2\rho\omega a$,

$$\text{the moment of inertia} = \frac{2}{3}\rho\omega a^3 + 2\rho\omega a c^2.$$

Hence, if an equilateral triangle is formed of a wire whose length is $6a$, the moment of inertia relatively to an axis passing through the centre of gravity of the triangle and perpendicular to its plane is $4\rho\omega a^3$.

(3) Let the rotation-axis be perpendicular to the wire and pass through one of its ends; then

$$\text{the moment of inertia} = \int_{0}^{2a} \rho\omega x^2 dx$$

$$= \frac{8\rho\omega a^3}{3}.$$

(4) Let the rotation-axis intersect the wire in its middle point at an angle α; then

$$\text{the moment of inertia} = \frac{2}{3}\rho\omega a^3 (\sin\alpha)^2.$$

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Ex. 2. The moment of inertia of a wire of uniform thickness and density whose form is a circular arc.

Let ρ = the density, ω = the area of a transverse section, a = the radius of the circle, 2α = the angle subtended by the arc at the centre of the circle.

(1) Let the rotation-axis pass through the centre and be perpendicular to the plane of the arc; then

$$\text{the moment of inertia} = \int_0^{2\alpha} \rho\omega a^3 d\theta$$
$$= 2\rho\omega a^3 \alpha;$$

and therefore the moment of inertia of a complete circular wire about an axis which passes through its centre and is perpendicular to its plane is $2\pi\rho\omega a^3$.

(2) Let the rotation-axis be perpendicular to the plane of the wire and pass through its middle point; then

$$y^2 = 2ax - x^2;$$
$$\therefore \quad \frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a}:$$
$$\therefore \quad \text{the moment of inertia} = \int \rho\omega(x^2+y^2)\,ds$$
$$= 4\rho\omega a^3\{\alpha - \sin\alpha\};$$

and the moment of inertia of a complete circle $= 4\pi\rho\omega a^3$.

(3) Let the rotation-axis be in the plane of the wire and pass through the centre and its middle point; then

$$\text{the moment of inertia} = \int_{-\alpha}^{\alpha} \rho\omega a^3 (\sin\theta)^2 d\theta$$
$$= \rho\omega a^3\{\alpha - \sin\alpha\cos\alpha\};$$

and therefore the moment of inertia of a complete circular wire about its diameter is $\pi\rho\omega a^3$.

(4) Let the rotation-axis pass through the centre of a complete circular ring, and be inclined at an angle γ to the plane of the circle; then, by (208),

$$\text{the moment of inertia} = \pi\rho\omega a^3(\cos\gamma)^2 + 2\pi\rho\omega a^3(\sin\gamma)^2$$
$$= \pi\rho\omega a^3\{1+(\sin\gamma)^2\}.$$

Ex. 3. A wire of uniform thickness and density, whose length is a, is bent into the form of a complete cycloidal arc: the moment of inertia of it about a rotation-axis which joins its two ends is $\dfrac{\rho\omega a^3}{30}$.

In each of the preceding examples the mass of the wire can be easily found: and as the square of the radius of gyration is

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the moment of inertia divided by the mass, so the radius of gyration can be found without difficulty.

If the wire lies wholly in one plane, say in the plane of (x, y), that plane is a principal plane of it; because in this case $z = 0$ for all elements of it; and therefore $\Sigma . mxz = \Sigma . myz = 0$, and the axis of z is a principal axis. The other two principal axes must be found by the process of Art. 90.

117.] The moment of inertia of thin plates and of curved shells.

In all cases we shall assume the thickness of the plates and shells to be infinitesimal, and to be represented by the symbol τ; and thus, if it is convenient, we shall take the plate-plane to be the plane of (x, y); in this case, as $z = 0$ for all elements of the plate, $\Sigma . mxz = \Sigma . myz = 0$, and the plane of (x, y) is a principal plane and the axis of z is a principal axis. The other principal axes will be found by the method of Art. 90; and the principal moments of inertia having been determined, the moment of inertia about any other axis may be determined by means of the theorems given in (208) and (210).

Also, since $z = 0$, the moments of inertia about the axes of x and y are respectively $\Sigma . my^2$ and $\Sigma . mx^2$; and as $\Sigma . m(x^2 + y^2)$ is the moment of inertia about the axis of z, it follows that the moment of inertia about an axis perpendicular to the plate is equal to the sum of the moments of inertia about any two axes at right angles to each other in the plate.

If the axes of coordinates are principal axes, from (208) we have

$$\begin{aligned} \text{H} &= \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + (\text{A} + \text{B})(\cos\gamma)^2 \\ &= \text{A}\{(\cos\alpha)^2 + (\cos\gamma)^2\} + \text{B}\{(\cos\beta)^2 + (\cos\gamma)^2\} \\ &= \text{A}(\sin\beta)^2 + \text{B}(\sin\alpha)^2; \end{aligned} \tag{212}$$

and if the rotation-axis is in the plane of (x, y), $\sin\beta = \cos\alpha$; and

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\sin\alpha)^2. \tag{213}$$

Ex. 1. The moment of inertia of a square plate.

Let a = the side of the plate, ρ = the density at the point (x, y).

(1) Let the rotation-axis pass through the centre of the plate and be perpendicular to its plane; then

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho\tau(x^2 + y^2)\, dy\, dx$$
$$= \frac{\rho\tau a^4}{6}.$$

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(2) Let the rotation-axis be the line joining the middle points of two opposite sides; then

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho\tau y^2\, dy\, dx$$

$$= \frac{\rho\tau a^4}{12}.$$

(3) Let the rotation-axis pass through an angular point of the plate, and be perpendicular to its plane; then

$$\text{the moment of inertia} = \frac{2\rho\tau a^4}{3}.$$

(4) Let the rotation-axis pass through the centre of the plate; and let its direction-angles, with reference to two lines bisecting the opposite sides of the plate and the perpendicular through its centre, be α, β, γ; then, as these lines are the principal axes of the plate,

$$\text{the mom. of in.} = \frac{\rho\tau a^4}{12}\{(\cos\alpha)^2 + (\cos\beta)^2\} + \frac{\rho\tau a^4}{6}(\cos\gamma)^2$$

$$= \frac{\rho\tau a^4}{12}(\sin\gamma)^2 + \frac{\rho\tau a^4}{6}(\cos\gamma)^2.$$

Also, if the rotation-axis is the diagonal of the plate, $\gamma = 90°$, and

$$\text{the moment of inertia} = \frac{\rho\tau a^4}{12}.$$

As this is a case in which the two principal moments of inertia in the plane of (x, y) are equal, and the third is greater than each of them, two points on the axis of z, which are at distances from the origin, which is the centre of gravity of the plate, (see Art. 110), equal to

$$\pm\left(\frac{C-A}{M}\right)^{\frac{1}{2}} = \pm\frac{a}{6}3^{\frac{1}{2}},$$

are such that at them the principal moments of inertia, and therefore all the moments of inertia, are equal. At these points the momental ellipsoid becomes a sphere.

Ex. 2. The moment of inertia of a triangular plate.

(1) Let the triangular plate be isosceles; and let the rotation-axis pass through its vertex and be perpendicular to its plane; let a = the altitude, $2b$ = the base; then

$$\text{the moment of inertia} = 2\int_0^a \int_0^{\frac{bx}{a}} \rho\tau(x^2+y^2)\, dy\, dx$$

$$= \frac{\rho\tau ab}{6}(3a^2+b^2).$$

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(2) Let the triangular plate be isosceles; and let the rotation-axis be the line which passes through the vertex and bisects the base; then

$$\text{the moment of inertia} = 2\int_0^a \int_0^{\frac{bx}{a}} \rho\tau y^2 dy\, dx$$

$$= \frac{\rho\tau a b^3}{6}.$$

(3) Let the triangular plate be that whose sides and angles are a, b, c, A, B, C; and let the rotation-axis pass through C and be perpendicular to the plane of the plate; let C be the origin, and let the lines lying along the sides a and b respectively be the axes of x and y; so that the equation to the side c is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

and let $$Y = \frac{b}{a}(a-x);$$

$$\text{the mom. of in.} = \int_0^a \int_0^Y \rho\tau(x^2 + 2xy\cos C + y^2)\, dy\, dx \sin C$$

$$= \frac{\rho\tau a b}{24}(3a^2 + 3b^2 - c^2)\sin C.$$

(4) Let the triangular plate be that of the preceding case; and let the rotation-axis pass through the centre of gravity of the plate and be perpendicular to its plane; then, if G is the centre of gravity, $(CG)^2 = \frac{2a^2 + 2b^2 - c^2}{9}$; and the mass of the plate $= \frac{\rho\tau a b \sin C}{2}$; therefore, by reason of (210),

$$\text{mom. of in.} = \frac{\rho\tau a b}{24}(3a^2 + 3b^2 - c^2)\sin C - \frac{\rho\tau a b}{18}(2a^2 + 2b^2 - c^2)\sin C$$

$$= \frac{\rho\tau a b}{72}(a^2 + b^2 + c^2)\sin C;$$

and therefore, if k is the radius of gyration relative to a rotation-axis passing through the centre of gravity of a triangular plate and perpendicular to its plane,

$$k^2 = \frac{a^2 + b^2 + c^2}{36}.$$

Ex. 3. The moment of inertia of a circular plate, and of a circular annulus.

Let the radius of the plate $= a$; and let ρ and τ express the same quantities as heretofore.

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(1) Let the rotation-axis pass through the centre and be perpendicular to the plane of the plate; then

$$\text{the moment of inertia} = \int_0^{2\pi}\int_0^a \rho\tau r^3 dr d\theta$$
$$= \frac{\pi\rho\tau a^4}{2}.$$

(2) Let the rotation-axis pass through the circumference and be perpendicular to the plate; then, by (210),

$$\text{the moment of inertia} = \frac{\pi\rho\tau a^4}{2} + \pi\rho\tau a^4$$
$$= \frac{3\pi\rho\tau a^4}{2}.$$

(3) Let the rotation-axis be the diameter of the plate; then

$$\text{the moment of inertia} = \int_0^{2\pi}\int_0^a \rho\tau r^3 (\sin\theta)^2 dr d\theta$$
$$= \frac{\pi\rho\tau a^4}{4}.$$

(4) Let the rotation-axis be a tangent to the plate; then, by (210),

$$\text{the moment of inertia} = \frac{5\pi\rho\tau a^4}{4}.$$

(5) Let the interior of the circular plate be removed, so that the remainder is a circular annulus, the radii of the exterior and interior bounding circles of which are a and b: then the moment of inertia relative to a rotation-axis passing through the centre of the annulus and perpendicular to its plane is

$$\frac{\pi\rho\tau(a^4 - b^4)}{2}.$$

Also the moment of inertia of the annulus relative to its diameter is

$$\frac{\pi\rho\tau(a^4 - b^4)}{4}.$$

Ex. 4. The moment of inertia of an elliptical plate.

Let the equation to the bounding ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1;$$

and let

$$Y = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}.$$

(1) Let the rotation-axis be the major axis of the ellipse; then

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$$\text{the moment of inertia} = 4\rho\tau\int_0^a\int_0^Y y^2\,dy\,dx$$

$$= \frac{4\rho\tau b^3}{3a^3}\int_0^a (a^2-x^2)^{\frac{3}{2}}\,dx$$

$$= \frac{\pi\rho\tau a b^3}{4}.$$

(2) Let the rotation-axis be the minor axis of the ellipse; then

$$\text{the moment of inertia} = 4\rho\tau\int_0^a\int_0^Y x^2\,dy\,dx$$

$$= \frac{\pi\rho\tau a^3 b}{4}.$$

(3) Let the rotation-axis be a line perpendicular to the plane of the plate and passing through its centre; then

$$\text{the moment of inertia} = 4\rho\tau\int_0^a\int_0^Y (x^2+y^2)\,dy\,dx$$

$$= \frac{\pi\rho\tau ab}{4}(a^2+b^2).$$

(4) Let the rotation-axis pass through the centre of the plate and make angles α, β, γ severally with the major axis, the minor axis, and the perpendicular to the plate through its centre; then, as these are the principal axes of the plate, we have, by (208),

$$\text{the mom. of in.} = \frac{\pi\rho\tau ab}{4}\{b^2(\cos\alpha)^2 + a^2(\cos\beta)^2 + (a^2+b^2)(\cos\gamma)^2\}$$

$$= \frac{\pi\rho\tau ab}{4}\{a^2(\sin\alpha)^2 + b^2(\sin\beta)^2\}.$$

(5) Let the rotation-axis be a central radius vector r of the plate, making an angle α with the major-axis; then, from the last result, as $\alpha+\beta = 90°$, we have

$$\text{the moment of inertia} = \frac{\pi\rho\tau ab}{4}\{a^2(\sin\alpha)^2 + b^2(\cos\alpha)^2\};$$

but by the equation to the ellipse

$$a^2(\sin\alpha)^2 + b^2(\cos\alpha)^2 = \frac{a^2b^2}{r^2};$$

$$\therefore \quad \text{the moment of inertia} = \frac{\pi\rho\tau a^3 b^3}{4r^2}.$$

Ex. 5. The moment of inertia of a spherical shell of radius a and thickness τ about its diameter $= \frac{8}{3}\pi\rho\tau a^4$.

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118.] The moment of inertia of a solid body bounded by a surface of revolution relative to its geometrical axis as its rotation-axis.

Let the axis be that of x; and let the equation to the curve, by the revolution of which about the axis of x the bounding surface is formed, be $y = f(x)$.

Let the solid be divided into a series of circular plates by planes at an infinitesimal distance apart and perpendicular to the axis of revolution; let the density be uniform and be ρ; then, at the distance x from the origin, y is the radius of a circular plate whose thickness is dx; and therefore, by Ex. 3, Art. 117, the moment of inertia of this circular slice, relative to an axis passing through its centre and perpendicular to its plane, is

$$= \frac{\pi\rho y^4 dx}{2}$$

$$= \frac{\pi\rho}{2}\{f(x)\}^4 dx;$$

and therefore, if x_n and x_0 are the limits of x,

$$\text{the moment of inertia} = \frac{\pi\rho}{2}\int_{x_0}^{x_n}\{f(x)\}^4 dx.$$

Ex. 1. The moment of inertia of a cylinder.

Let the altitude of the cylinder $= a$, and the radius of the base $= b$; therefore

$$\text{the moment of inertia} = \frac{\pi\rho a b^4}{2}.$$

Ex. 2. The moment of inertia of a cone; let the altitude $= a$, and the radius of the base $= b$; then

$$\text{the moment of inertia} = \frac{\pi\rho b^4}{2a^4}\int_0^a x^4 dx$$

$$= \frac{\pi\rho a b^4}{10}.$$

Ex. 3. If $a =$ the altitude, and $b =$ the radius of the base of a paraboloid, then

$$\text{the moment of inertia} = \frac{\pi\rho a b^4}{6}.$$

Ex. 4. If $a =$ the radius of a sphere, then relatively to a diameter as the rotation-axis,

$$\text{the moment of inertia} = \frac{8\pi\rho a^5}{15}.$$

Hence the moment of inertia of a spherical shell contained

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between two concentric spheres whose radii are a and b respectively, relatively to the diameter as the rotation-axis,

$$= \frac{8\pi\rho(a^5-b^5)}{15}.$$

Ex. 5. The moment of inertia of a prolate spheroid relatively to its axis as the rotation-axis $= \dfrac{8\pi\rho ab^4}{15}$.

Ex. 6. The moment of inertia of an oblate spheroid, whose axis is the rotation-axis, $= \dfrac{8\pi\rho a^4 b}{15}$.

Ex. 7. If the radius of each surface of an equiconvex lens is a, and the thickness of the lens is $2t$, then the moment of inertia of the lens relative to its axis as the rotation-axis

$$= \pi\rho\int_0^t (2ax-x^2)^2\,dx$$

$$= \frac{\pi\rho t^3}{15}(20a^2-15at+3t^2).$$

119.] The moment of inertia of a solid body bounded by a surface of revolution relative to an axis perpendicular to its geometrical axis.

Let the point in which the rotation-axis intersects the axis of revolution be the origin; and let $y=f(x)$ be the equation of the generating curve of the bounding surface; then, using the notation of the preceding Article, and applying the result of Ex. 3, Art. 117, the moment of inertia of the type-slice relative to its own diameter

$$= \frac{\pi\rho y^4\,dx}{4};$$

and therefore by (210) the moment of inertia of this slice about the actual rotation-axis is

$$= \frac{\pi\rho y^4\,dx}{4} + \pi\rho y^2 x^2\,dx;$$

and if x_n and x_0 are the limits of the x-integration,

$$\text{the moment of inertia} = \pi\rho\int_{x_0}^{x_n}\left(\frac{y^4}{4}+y^2x^2\right)dx.$$

Ex. 1. The moment of inertia of a cone relative to a rotation-axis passing through its vertex and perpendicular to its own axis.

Let the altitude of the cone $= a$; let the radius of the base $= b$; then

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$$\text{the moment of inertia} = \pi\rho\int_0^a \left(\frac{b^4}{4a^4} + \frac{b^2}{a^2}\right) x^4 dx$$
$$= \frac{\pi\rho a b^3}{20}(4a^2 + b^2).$$

It is evident that relative to the vertex of a cone the principal axes are the axis of the cone and any two lines perpendicular to each other and to the axis of the cone. So that the moment of inertia relative to a rotation-axis passing through the vertex of the cone and inclined at an angle α to the axis

$$= \frac{\pi\rho a b^3}{20}(4a^2 + b^2)(\sin\alpha)^2 + \frac{\pi\rho a b^4}{10}(\cos\alpha)^2.$$

Ex. 2. The moment of inertia of a cone of which the altitude $= a$, and the radius of whose base $= b$, relative to a rotation-axis passing through its centre of gravity and perpendicular to its own axis,

$$= \frac{\pi\rho a b^3}{80}(a^2 + 4b^2).$$

Ex. 3. If the altitude of a paraboloid of revolution is a, and the radius of the base $= b$, the moment of inertia relative to a rotation-axis passing through its vertex and perpendicular to its own axis

$$= \frac{\pi\rho a b^3}{8}(b^2 + 2a^2).$$

Ex. 4. If the altitude of a cylinder is a, and the radius of its base $= b$; and if the rotation-axis is perpendicular to the axis, and at a distance c from its end, then

$$\text{the moment of inertia} = \int_c^{a+c} \left(\frac{\pi\rho b^4}{4} + \pi\rho b^2 x^2\right) dx$$
$$= \frac{\pi\rho a b^4}{4} + \frac{\pi\rho a b^3}{3}(a^2 + 3ac + 3c^2).$$

Hence, if the rotation-axis passes through the end of the axis,

$$\text{the moment of inertia} = \frac{\pi\rho a b^3}{12}(3b^2 + 4a^2);$$

and if the rotation-axis passes through the middle point of the axis of the cylinder,

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \left(\frac{\pi\rho b^4}{4} + \pi\rho b^2 x^2\right) dx$$
$$= \frac{\pi\rho a b^3}{12}(a^2 + 3b^2).$$

120.] The moment of inertia of various solid bodies.

Ex. 1. The moment of inertia of a rectangular parallelepipedon about an edge.

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Let the edges be a, b, c; and let the lines which coincide with the edges be the axes of x, y, z respectively; let the density $= \rho$; then the moment of inertia relative to the edge a

$$= \int_0^a \int_0^b \int_0^c \rho (y^2 + z^2)\, dz\, dy\, dx$$

$$= \frac{\rho abc}{3}(b^2 + c^2);$$

and symmetrical values are of course true for the moments of inertia relative to the edges b and c.

Thus the moment of inertia of a cube whose side is a, relative to one of its edges as a rotation-axis, $= \dfrac{2\rho a^5}{3}$.

Ex. 2. The moment of inertia of a cube relative to a diagonal.

Let the side of the cube be a; and let the centre of the cube be the origin, and let the three lines which pass through the centres of the opposite sides be the coordinate axes; these lines are evidently principal axes; and relatively to either of them

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho (y^2 + z^2)\, dz\, dy\, dx$$

$$= \frac{\rho a^5}{6};$$

and as the moment of inertia is the same for each of these principal axes, it is the same for every axis passing through this point; thus, the central ellipsoid is a sphere, and all its radii vectores are equal; and therefore relative to the diagonal of the cube,

$$\text{the moment of inertia} = \frac{\rho a^5}{6}.$$

Ex. 3. The moment of inertia of an ellipsoid.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

The axes of the ellipsoid are evidently the principal axes of the body; so that when the moments of inertia relative to these axes are determined, that about any other axis may be found from (208).

Let $\quad c\left\{1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}\right\}^{\frac{1}{2}} = \text{Z}, \qquad b\left\{1 - \dfrac{x^2}{a^2}\right\}^{\frac{1}{2}} = \text{Y}.$

Now $\quad \text{Z}.mx^2 = 8\displaystyle\int_0^a \int_0^{\text{Y}} \int_0^{\text{Z}} \rho x^2\, dz\, dy\, dx$

$$= \frac{4\pi\rho a^3 bc}{15};$$

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similarly $\quad \Sigma.my^2 = \dfrac{4\pi\rho ab^3c}{15}, \quad \Sigma.mz^2 = \dfrac{4\pi\rho abc^3}{15};$

$$\therefore \quad \text{A} = \Sigma.m(y^2+z^2) = \frac{4\pi\rho abc}{15}(b^2+c^2),$$

$$\text{B} = \Sigma.m(z^2+x^2) = \frac{4\pi\rho abc}{15}(c^2+a^2),$$

$$\text{C} = \Sigma.m(x^2+y^2) = \frac{4\pi\rho abc}{15}(a^2+b^2);$$

and therefore the moment of inertia relative to the axis (α, β, γ)

$$= \frac{4\pi\rho abc}{15}\{(b^2+c^2)(\cos\alpha)^2+(c^2+a^2)(\cos\beta)^2+(a^2+b^2)(\cos\gamma)^2\}$$

$$= \frac{4\pi\rho abc}{15}\{a^2(\sin\alpha)^2+b^2(\sin\beta)^2+c^2(\sin\gamma)^2\}.$$

Ex. 4. If in the preceding example $a = b$, and a is $> c$, the ellipsoid becomes an oblate spheroid, and

$$\text{A} = \text{B} = \frac{4\pi\rho a^2c}{15}(a^2+c^2),$$

$$\text{C} = \frac{8\pi\rho a^4c}{15}.$$

Therefore, by Art. 110, at two points on the axis of z all the moments of inertia are equal, and at them the momental ellipsoid becomes a sphere: the distances of them from the centre

$$= \pm\left\{\frac{\text{C}-\text{A}}{\text{M}}\right\}^{\frac{1}{2}}$$

$$= \pm\left\{\frac{a^2-c^2}{5}\right\}^{\frac{1}{2}};$$

and if these points are at the poles of the spheroid, $a^2 = 6c^2$.

121.] From the preceding results the moments of inertia of many curved shells and of systems of thin plates may be deduced.

For if the equation of the bounding surface of the solid contains a single parameter, by the infinitesimal variation of that parameter, the content of the solid will receive an infinitesimal variation in the form of a thin shell, the thickness of which will be the variation of the parameter. Thus, if the radius of a sphere is increased by an infinitesimal variation, say dr, the content will be increased by a spherical shell of thickness dr. Similarly, if a solid is increased by the variation of the parameter on which the bounding surface depends, the moment of inertia of

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that increase is the increase of the moment of inertia of the solid; and the former is generally a thin shell or a system of thin plates, so that the moment of inertia of these may be determined by the variation of the moment of inertia of the solid.

Thus by the preceding Article the moment of inertia of a cube about a diagonal is $\frac{\rho a^5}{6}$; let the edge of the cube be increased by da; then all the sides of the cube receive increments in the form of thin plates, the thickness of which $= da = \tau$, say; and therefore the moment of inertia of the hollow box, formed by these six plates relative to a diagonal $= \frac{5\rho a^4 \tau}{6}$.

Similarly, by reason of Ex. 1 in the preceding Article, the moment of inertia of the box relative to an edge $= \frac{10\rho a^4 \tau}{3}$.

As the moment of inertia of a sphere relative to a diameter is $\frac{8\pi\rho a^5}{15}$, so that if a spherical shell of thickness τ, relatively to the same rotation-axis, $= \frac{8\pi\rho\tau a^4}{3}$.

As the moment of inertia of a cylinder, relative to its own axis as rotation-axis, is $\frac{\pi\rho a b^4}{2}$, so the moment of inertia of a cylindrical shell whose thickness is $db = \tau$, is, relatively to its own axis, $2\pi\rho\tau a b^3$.

In all the preceding examples we have calculated moments of inertia; and as the masses of the rotating bodies may be found in all the cases, the corresponding radii of gyration can be determined without difficulty.

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CHAPTER V.

THE ROTATION OF A BODY ABOUT A FIXED AXIS.

SECTION 1.—*The rotation of a rigid body about a fixed axis under the action of instantaneous forces.*

122.] In the last two sections of the preceding Chapter we have considered that part of our subject which has been called the Geometry of Masses: it has indeed nothing directly mechanical in it, but the theorems which have been proved are useful and necessary on account of the form which the process of transformation into angular velocities has given to the equations of rotatory motion. We come now to the consideration of the most simple case of dynamics proper; that, namely, in which a rigid body under the action of given forces revolves about an axis fixed in it and in space. Every particle of the body thus moves in a circle, the plane of which is perpendicular to the rotation-axis, and the centre of which is in that axis.

We shall suppose the form, matter, and density of every part of the moving body or system to be given; and we shall suppose the body to be capable of an unfettered rotation about the axis. This axis may be fixed at many points, or, in the language of machinery, may have many bearings; we shall however suppose that it has only two fixed points; because these are sufficient to fix the axis; and if there are more, the pressures become indeterminate at them both in intensity and in line of action. We shall indeed find that even in the case of two points, the components of the pressures on the fixed points along the rotation-axis are indeterminate. We have already had a similar instance in Art. 62, Vol. III.

Let us in the first place consider the circumstances of rotation of the body, when it is acted on by instantaneous or impulsive forces; that is, we shall investigate the resulting angular velocity of the body, the pressures on the fixed points, and their incidents, which are due to one or more blows impressed at given points of the body. To simplify the formulæ, we shall generally assume the body to be at rest when the impulsive force acts, although the results will be equally applicable if the body is moving with a given angular velocity.

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123.] Let the rotation-axis, on which are the two fixed points, be the axis of z; and let the two fixed points be at distances z_1, z_2 from the origin; let the pressure at these two points be P_1, P_2, and let the direction-angles of the lines along which they respectively act be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$. Let m be the type-particle, and let (x, y, z) be its place at the time t, when the instantaneous force acts on it; let $m\mathrm{v}$ be the momentum impressed by this force, of which let the axial components be $m\mathrm{v}_x$, $m\mathrm{v}_y$, $m\mathrm{v}_z$; let $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$ be the components of the actual velocity (or increase of velocity) with which m moves in consequence of this instantaneous force; all these being type-expressions, and therefore applicable to each particle on which forces act. Thus the equations of motion, (34) and (35), Art. 48, become

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{v}_x - \frac{dx}{dt}\right) - \mathrm{P}_1\cos\alpha_1 - \mathrm{P}_2\cos\alpha_2 &= 0, \\ \Sigma.m\left(\mathrm{v}_y - \frac{dy}{dt}\right) - \mathrm{P}_1\cos\beta_1 - \mathrm{P}_2\cos\beta_2 &= 0, \\ \Sigma.m\left(\mathrm{v}_z - \frac{dz}{dt}\right) - \mathrm{P}_1\cos\gamma_1 - \mathrm{P}_2\cos\gamma_2 &= 0; \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{v}_z - \frac{dz}{dt}\right) - z\left(\mathrm{v}_y - \frac{dy}{dt}\right)\right\} + z_1\mathrm{P}_1\cos\beta_1 + z_2\mathrm{P}_2\cos\beta_2 &= 0, \\ \Sigma.m\left\{z\left(\mathrm{v}_x - \frac{dx}{dt}\right) - x\left(\mathrm{v}_z - \frac{dz}{dt}\right)\right\} - z_1\mathrm{P}_1\cos\alpha_1 - z_2\mathrm{P}_2\cos\alpha_2 &= 0, \\ \Sigma.m\left\{x\left(\mathrm{v}_y - \frac{dy}{dt}\right) - y\left(\mathrm{v}_x - \frac{dx}{dt}\right)\right\} &= 0; \end{aligned}\right\} \qquad (2)$$

Let us express these equations, as in Art. 73, in terms of angular velocities. Let Ω be the angular velocity which results from the instantaneous forces; then, as its rotation-axis is the axis of z, and as there is no motion parallel to the axis of z,

$$\frac{dx}{dt} = -\Omega y, \qquad \frac{dy}{dt} = \Omega x, \qquad \frac{dz}{dt} = 0. \qquad (3)$$

Let the moments of the axial components of the couple of the impressed momenta be L, M, N; then (1) and (2) become

$$\left.\begin{aligned} \Sigma.m\mathrm{v}_x + \Omega\Sigma.my - \mathrm{P}_1\cos\alpha_1 - \mathrm{P}_2\cos\alpha_2 &= 0, \\ \Sigma.m\mathrm{v}_y - \Omega\Sigma.mx - \mathrm{P}_1\cos\beta_1 - \mathrm{P}_2\cos\beta_2 &= 0, \\ \Sigma.m\mathrm{v}_z \qquad\qquad - \mathrm{P}_1\cos\gamma_1 - \mathrm{P}_2\cos\gamma_2 &= 0; \end{aligned}\right\} \qquad (4)$$

$$\left.\begin{aligned} \mathrm{L} + \Omega\Sigma.mzx + \mathrm{P}_1\cos\beta_1 z_1 + \mathrm{P}_2\cos\beta_2 z_2 &= 0, \\ \mathrm{M} + \Omega\Sigma.myz - \mathrm{P}_1\cos\alpha_1 z_1 - \mathrm{P}_2\cos\alpha_2 z_2 &= 0, \\ \mathrm{N} - \Omega\Sigma.m(x^2 + y^2) &= 0; \end{aligned}\right\} \qquad (5)$$

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which six equations assign the incidents of motion, and the pressures on the two fixed points.

These equations admit of dissection by means of first principles, in a manner similar to that which has been employed in Art. 81 and 82. As Ω is the expressed angular velocity about the axis of z, Ωr is the expressed velocity of m at a distance r from that axis; and $m\Omega r$ is the expressed momentum; the x- and y- axial components of which are $-m\Omega y$ and $m\Omega x$. Let us introduce pairs of momenta equal and opposite to these at the origin and in the plane of (x,y) at the foot of the z-ordinate of m; then the momentum $m\Omega r$ of m at the point (x,y,z) is equivalent to (1) a momentum $-m\Omega y$ acting at the origin and along the axis of x; (2) a momentum $m\Omega x$ also acting at the origin along the axis of y; (3) three couples $-m\Omega zx$, $-m\Omega yz$, $m\Omega(x^2+y^2)$ whose axes are respectively the coordinate axes of x, y, and z; and a similar result is true for every element of the body. Now, by D'Alembert's principle, the sum of all these expressed momenta, together with the pressures at the fixed points, are in equilibrium with the impressed momenta; and the conditions requisite for the equilibrium are evidently the six equations (4) and (5). We have hereby an intelligible meaning of their several terms. We proceed to deduce from them the value of the angular velocity which results from the impressed forces, and the pressures on the fixed points.

124.] The angular velocity is given by the last equation of (5), and we have

$$\Omega = \frac{N}{\Sigma . m(x^2+y^2)}$$

$$= \frac{N}{\Sigma . mr^2}$$

$$= \frac{\text{The moment of the impressed momenta}}{\text{The moment of inertia}}; \qquad (6)$$

which is the same result as (16), Art. 73. It appears therefore that the resulting angular velocity does not depend on the pressures at the fixed points, or on the distance between them, but only on the moment of the impressed momenta, and on the moment of inertia of the body or system. It is also the same whatever is the number of the bearings. And if no force external to the system acts, the system continues to rotate uniformly with this angular velocity.

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Now let us suppose a body capable of rotating about a fixed axis to be at rest, and let us suppose it to be struck by a blow of given momentum at a given point and in a determinate line: we must first resolve the blow into two parts, of one of which the line of action shall be parallel to the rotation-axis, so that the angular velocity will not be affected thereby, for it will only produce pressures at the fixed points along the rotation-axis; of the other, let the line of action be in the plane of (x, y) which is perpendicular to the rotation-axis; let the momentum of this latter be Q, and let a be the perpendicular distance from the axis on its line of action; then (6) becomes

$$\Omega = \frac{\text{Q}a}{\text{The moment of inertia}}. \tag{7}$$

The following are examples of this equation.

Ex. 1. A body M at rest, and capable of moving about a fixed rotation-axis, is simultaneously struck by several masses m_1, m_2, ... m_n, moving with velocities v_1, v_2, ... v_n in planes perpendicular to the fixed axis; the masses adhere to the body: it is required to find the angular velocity of the body.

Let the distances of the points of impact of the masses severally from the rotation-axis be $l_1, l_2, \ldots\ldots l_n$; and let $p_1, p_2, \ldots\ldots p_n$ be the perpendiculars from the rotation-axis on the lines of the velocities $v_1, v_2, \ldots\ldots v_n$; then, if k is the radius of gyration of the body relative to the rotation-axis,

$$\Omega = \frac{m_1 v_1 p_1 + m_2 v_2 p_2 + \ldots\ldots + m_n v_n p_n}{\text{M}k^2 + m_1 l_1^2 + m_2 l_2^2 + \ldots\ldots + m_n l_n^2}$$

$$= \frac{\Sigma . m v p}{\text{M}k^2 + \Sigma . m l^2}.$$

Ex. 2. A body M revolving about a fixed axis with an angular velocity Ω, is struck by a particle m, moving with a velocity v in a line perpendicular to the plane containing the rotation-axis and the point of impact; it is required to determine the resulting angular velocity of the rotating body, the velocity of rebound of the striking particle, and the place of percussion when the velocity of rebound is a maximum, the elasticity of the body and particle being e.

Let $\text{M}k^2$ be the moment of inertia of the body relative to the rotation-axis; $p =$ the distance of the point of impact from the axis; $\Omega' =$ the angular velocity of the body after collision; $\text{v} =$ the velocity of m after rebound; and let us suppose Ω and v to

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be such that the motion of m and of the point of impact may be in the same direction at the instant of collision.

Let v' be the velocity of that point of M at which the impact takes place; so that $v' = \Omega p$;

and let m' be the mass of a particle which, moving with the velocity v', would produce the same circumstances of velocity &c. in m after impact on m', as the rotating body M; so that $m'v'$ is the momentum with which M would strike a body at the point of impact of m, and in the line of m's motion: therefore by (6)

$$m'v' = \frac{\mathrm{M}k^2\Omega}{p}.$$

$$\therefore \quad m' = \frac{\mathrm{M}k^2}{p^2}.$$

Let $\mathrm{v}' = p\Omega'$ be the velocity of the point of impact after collision has ceased; then, by (8) and (9), Art. 215, Vol. III,

$$\mathrm{v} = \frac{mp^2v + \Omega p\mathrm{M}k^2 - e\mathrm{M}k^2(v - p\Omega)}{mp^2 + \mathrm{M}k^2},$$

$$\mathrm{v}' = \frac{mp^2v + \Omega p\mathrm{M}k^2 + mep^2(v - p\Omega)}{mp^2 + \mathrm{M}k^2};$$

$$\Omega' = \frac{mpv + \mathrm{M}k^2\Omega + mep(v - p\Omega)}{mp^2 + \mathrm{M}k^2};$$

$$\Omega' - \Omega = \frac{mp(1 + e)(v - p\Omega)}{mp^2 + \mathrm{M}k^2},$$

$$v - \mathrm{v} = \frac{\mathrm{M}k^2(1 + e)(v - p\Omega)}{mp^2 + \mathrm{M}k^2};$$

whereby we know the velocity of m after collision and the angular velocity of M.

Thus, let M be a cricket bat, and m a ball; let us suppose the ball to meet the bat; then the sign of v must be changed; and if v = the velocity of rebound of the ball,

$$\mathrm{v} = \frac{\mathrm{M}k^2\Omega p - mp^2v + e\mathrm{M}k^2(v + \Omega p)}{mp^2 + \mathrm{M}k^2};$$

and to determine the point of impact so that v may be a maximum, the p-differential of v must be equated to zero; whereby we have

$$p = -\frac{v}{\Omega} \pm \left\{\frac{v^2}{\Omega} + \frac{\mathrm{M}k^2}{m}\right\}^{\frac{1}{2}}.$$

If m is at rest when it is struck by M, $v = 0$, and

$$p = k\left(\frac{\mathrm{M}}{m}\right)^{\frac{1}{2}}.$$

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Again, let M be a rectangular plate whose sides are a and b, and let the rotation-axis lie along the side a: let us suppose it to be at rest and to be struck by m at a point on the side opposite to the rotation-axis; then $\mathrm{M}k^2 = \frac{\mathrm{M}b^2}{3}$; and

$$\Omega' = \frac{3mv(1+e)}{(3m+\mathrm{M})b}.$$

125.] In the next place let us consider the pressures on the two fixed points of the axis; the x- and y- components of P_1 and P_2 can be determined from the first two of (4), and from the first two of (5); and we have

$$\mathrm{P}_1 \cos\alpha_1 = \frac{-\mathrm{M} + z_2 \Sigma.m\mathrm{V}_x + \Omega(z_2 \Sigma.my - \Sigma.myz)}{z_2 - z_1}, \tag{8}$$

$$\mathrm{P}_1 \cos\beta_1 = \frac{\mathrm{L} + z_2 \Sigma.m\mathrm{V}_y + \Omega(-z_2 \Sigma.mx + \Sigma.mxz)}{z_2 - z_1}, \tag{9}$$

$$\mathrm{P}_2 \cos\alpha_2 = \frac{\mathrm{M} - z_1 \Sigma.m\mathrm{V}_x + \Omega(-z_1 \Sigma.my + \Sigma.myz)}{z_2 - z_1}, \tag{10}$$

$$\mathrm{P}_2 \cos\beta_2 = \frac{-\mathrm{L} - z_1 \Sigma.m\mathrm{V}_y + \Omega(z_1 \Sigma.mx - \Sigma.mxz)}{z_2 - z_1}; \tag{11}$$

whereby the components of the pressures which are parallel to the plane of (x, y) may be determined.

The z-components of P_1 and P_2 enter into only the third equation of (4), and we have

$$\mathrm{P}_1 \cos\gamma_1 + \mathrm{P}_2 \cos\gamma_2 = \Sigma.m\mathrm{V}_z; \tag{12}$$

therefore the sum of these z-components of the pressures is equal to the sum of the z-components of the impressed momenta; but as the sum only is given, each is indeterminate. An explanation of this indeterminateness has been already made in Art. 62, Vol. III: this is the dynamical case, which is therein alluded to. And we are unable to determine the pressure which acts at each fixed point.

To give greater clearness to our ideas, let us suppose the impressed momenta to arise from a single blow, whose momentum is Q, say, and whose line of action is in a plane perpendicular to the rotation-axis. Let us take the rotation-axis to be the z-axis, and the plane perpendicular to it, and containing the line of the blow, to be the plane of (x, y); let the axis of y be parallel to the line of the blow; and let a be the distance between these two lines: then the equations of motion are

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$$\left.\begin{aligned} \Omega\Sigma.my - P_1\cos\alpha_1 - P_2\cos\alpha_2 &= 0, \\ Q - \Omega\Sigma.mx - P_1\cos\beta_1 - P_2\cos\beta_2 &= 0, \\ -P_1\cos\gamma_1 - P_2\cos\gamma_2 &= 0; \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} \Omega\Sigma.mzx + P_1\cos\beta_1 z_1 + P_2\cos\beta_2 z_2 &= 0, \\ \Omega\Sigma.mzy - P_1\cos\alpha_1 z_1 - P_2\cos\alpha_2 z_2 &= 0, \\ Qa - \Omega\Sigma.m(x^2+y^2) &= 0. \end{aligned}\right\} \tag{14}$$

Let M = the mass of the body or system of particles; and let k be the radius of gyration relative to the rotation-axis; then from the last of (14)

$$\Omega = \frac{Qa}{Mk^2}. \tag{15}$$

Let the centre of gravity be $(\bar{x}, \bar{y}, \bar{z})$ when the blow is given; then

$$\Omega\Sigma.my = \frac{Qa\bar{y}}{k^2}, \qquad \Omega\Sigma.mx = \frac{Qa\bar{x}}{k^2};$$

and the values of the axial components of the pressures may be determined.

Now these pressures will compound into a single resultant when $\quad -\Omega M\bar{y}\Sigma.mxz + (Q - \Omega M\bar{x})\Sigma.myz = 0,$

that is, when $\quad -a\bar{y}\Sigma.mxz + (k^2 - a\bar{x})\Sigma.myz = 0; \qquad (16)$

and this condition is satisfied when the rotation-axis is a principal axis, and the line of action of the blow is in its principal plane; and if R is the single pressure,

$$R = \frac{Q}{k^2}\left\{k^4 - 2a\bar{x}k^2 + a^2(\bar{x}^2+\bar{y}^2)\right\}^{\frac{1}{2}};$$

when (16) is satisfied, one point is sufficient to fix the axis.

Hence, if the axis of rotation is a central principal axis, $R = Q$; and evidently acts at the centre of gravity.

Let us apply these results to one or two examples.

Ex. 1. A thin rod of length a revolves with an angular velocity Ω about an axis passing through its end and perpendicular to its length; it is suddenly stopped by a fixed obstacle at its other end; determine the blow which the obstacle receives, and the pressure thereby caused on the fixed points of the axis, these being supposed to be near to each other at the fixed end of the rod.

Let Q = the momentum of the blow with which the obstacle is struck; M = the mass of the rod; then the moment of inertia $= M\frac{a}{3}$; and

$$Q = \frac{aM\Omega}{3};$$

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therefore $$\mathrm{P}_1 + \mathrm{P}_2 = \frac{a\mathrm{M}\Omega}{3};$$

and $\mathrm{P}_2 = \mathrm{P}_1$, since both act close together and perpendicular to the rod;

$$\therefore \quad \mathrm{P}_1 = \mathrm{P}_2 = \frac{a\mathrm{M}\Omega}{6}.$$

Ex. 2. A circular plate of radius a revolving with an angular velocity Ω about an axis passing through its centre and fixed at the extremities of the diameter, is struck with a blow Q at right angles to its plane, at a point in the diameter perpendicular to the rotation-axis at a distance c from the centre; find the pressures on the fixed points of the rotation-axis.

Let the rotation-axis be the z-axis, and the plane of the plate be the plane of (z, x); let M be the mass of the plate; then

$$\Omega = \frac{4\mathrm{Q}c}{\mathrm{M}a^2}; \qquad \mathrm{P}_1 = \mathrm{P}_2 = \frac{\mathrm{Q}}{2}.$$

126.] Let us however further consider certain particular values which the pressures at the fixed points may have. It is evident that they will not generally vanish, whatever is the origin; whatever are the axes; whether they are central principal axes; whether the rotation-axis is a principal axis, and whether the origin is its principal point: yet it may be that they will vanish if the momenta are impressed under certain conditions, and in a certain reference to the constitution of the body. Now we will suppose a single force to act, and to impress a certain momentum in a given direction at a certain point; and we will inquire the point at which and the line along which this force must act, if no pressure is thereby produced on the fixed points.

Let us suppose the momentum impressed by this acting force or blow to be Q; and to be impressed on the body at (ξ, η, ζ), and along the line (α, β, γ): then, since $\mathrm{P}_1 = \mathrm{P}_2 = 0$, (4) become

$$\left.\begin{aligned} \mathrm{Q}\cos\alpha + \Omega\mathfrak{z}.my &= 0, \\ \mathrm{Q}\cos\beta - \Omega\mathfrak{z}.mx &= 0, \\ \mathrm{Q}\cos\gamma \qquad\quad &= 0; \end{aligned}\right\} \tag{17}$$

the last of which shews that $\cos\gamma = 0$; and therefore the line of the impressed momentum must lie in a plane which is perpendicular to the rotation-axis. Thus (5) become

$$\left.\begin{aligned} -\zeta\mathrm{Q}\cos\beta + \Omega\mathfrak{z}.mzx &= 0, \\ \zeta\mathrm{Q}\cos\alpha + \Omega\mathfrak{z}.myz &= 0, \\ \mathrm{Q}(\xi\cos\beta - \eta\cos\alpha) - \Omega\mathfrak{z}.m(x^2+y^2) &= 0. \end{aligned}\right\} \tag{18}$$

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From the first two of (17) and of (18) we have

$$\zeta = \frac{\Sigma.myz}{\Sigma.my} = \frac{\Sigma.mzx}{\Sigma.mx}; \tag{19}$$

whence we have the condition

$$\Sigma.mx\,\Sigma.myz - \Sigma.my\,\Sigma.mzx = 0; \tag{20}$$

and this must be satisfied if the fixed points are free from pressure. Now this expresses a particular constitution of the body relative to the axis of z, and is independent of both the impressed momentum and of the position of the fixed points. It evidently indicates that the axis of z is a principal axis, and (19) gives the distance ζ of its principal point from the origin. Hence we have this first condition. If the fixed points of the rotation-axis are free from pressure, that axis must be a principal axis of the body, and the line of action of the force, or the line of the blow, must be in its principal plane.

Also from the last of (18), in combination with the first two of (17), we have

$$\xi\,\Sigma.mx + \eta\,\Sigma.my - \Sigma.mr^2 = 0; \tag{21}$$

so that if $(\bar{x}, \bar{y}, \bar{z})$ is the centre of gravity, and if k is the radius of gyration relative to the rotation-axis, (21) becomes

$$\bar{x}\xi + \bar{y}\eta - k^2 = 0; \tag{22}$$

which is the equation to the line of action of the blow, in the plane parallel to, and at a distance ζ from, the plane of (x, y). (22) is evidently perpendicular to the line joining the centre of gravity and the rotation-axis; and if h is the distance of the centre of gravity from the axis, and l is the perpendicular distance from the axis on the line of the blow, or the line of percussion, from (22) we have

$$l = \frac{k^2}{h}; \tag{23}$$

hence the line of the blow must be at right angles to the perpendicular from the centre of gravity on the rotation-axis, and at that distance l from the rotation-axis which is given in (23).

Also in this case we have from (6), if M = the whole moving mass,

$$\Omega = \frac{\text{Q}\,l}{\text{M}\,k^2} = \frac{\text{Q}}{\text{M}\,h}; \tag{24}$$

certain special forms of the preceding equations deserve remark.

If the plane of (x, y) is the principal plane of the axis of z, which is the rotation-axis, (20) is satisfied identically; and $\zeta = 0$.

Equation (20) is also satisfied identically if $\Sigma.mx = \Sigma.my = 0$;

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that is, if the rotation-axis passes through the centre of gravity; but in this case $\zeta = \infty$, and therefore $Q = 0$. So that if a body capable of rotation about an axis, passing through the centre of gravity, is struck by a blow, whatever is the direction and the intensity of the blow, certain pressures are always produced at the fixed points of the axis. This result obviously depends on the fact that generally the principal point of an axis passing through the centre of gravity of a body is at an infinite distance.

If at the time when the blow is given the coordinate planes are so chosen that that of (x, z) contains the centre of gravity; then $\Sigma . my = 0$; but as $\Sigma . myz$ evidently vanishes also, ζ has a determinate value.

It appears then that if a body capable of rotation about a fixed axis is struck by a blow and rotates thereby, so that no pressure is produced on those points at which the axis is fixed, it is necessary that (1) the rotation-axis should be a principal axis of the body; (2) the line of the blow should be in the principal plane of this axis, and perpendicular to the plane containing the rotation-axis and the centre of gravity, and at a distance from the axis equal to l, which is defined by (23).

A representation of these circumstances is given in Fig. 22; OP_1P_2 is the fixed rotation-axis, and is the z-axis; P_1, P_2 are the two fixed points which determine it; O is its principal point, and is the origin, so that in this figure $\zeta = 0$; and the plane (x, y) is the principal plane. G is the centre of gravity of the body which is taken to be in the plane of (x, z), so that the line of the blow is parallel to the y-axis. $OK = k$, the radius of gyration of the system relatively to the rotation-axis. $OL = l$; $NG = OM = h$; so that by (23) OL is a third proportional to OM and OK.

If k' is the radius of gyration of the body relatively to MG, by (129), Art. 101,

$$k^2 = h^2 + k'^2;$$

so that

$$l = \frac{h^2 + k'^2}{h};$$

$$= h + \frac{k'^2}{h};$$

$$\therefore \quad h(l-h) = k'^2; \tag{25}$$

$$\therefore \quad OM \times ML = \text{a constant.}$$

Now be it observed that all lines in the plane (x, z), which are parallel to OP_1P_2, are principal axes at some point on them, by reason of Art. 113; whatever therefore the point L is to OP_1P_2,

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so is some point on OP_1P_2 to a line through L parallel to OP_1P_2: these two parallel lines are therefore to some extent reciprocal to each other.

127.] The point L, which has been determined in the preceding Article, is called the Centre of Percussion of the body relative to the given rotation-axis. It determines the line along which a blow must be impressed on a body capable of rotation about a principal axis, when the axis receives no strain thereby; and conversely, if a body rotates about an axis free from all constraint, or if constrained, free from pressure at its bearings, the centre of percussion determines the line in which a blow must be given to the body to reduce it to rest without causing pressure on the bearings; or, in another sense, it determines the positions in which a fixed obstacle may be placed, on which if the body impinges and is brought to rest, the bearings of the axis will suffer no pressure.

It is also evident that as the axis OP_1P_2 is free from pressure at its bearings, it is that axis about which the body continues to rotate; it is therefore a permanent axis. We have hereby then arrived at another property of a permanent axis, and have shewn it to be identical with a principal axis.

It is also evident that if the body is free from all constraint, so that it is capable of translation as well as of rotation, the effect of a blow at L along LQ will cause a rotation about OP_1P_2; for this reason the axis OP_1P_2 is called the Spontaneous Axis of the body relative to the point L. This subject however we shall consider at length in Chapter VII.

I propose now to apply the preceding theory to certain examples, and to exhibit the practical meaning of the results. For this purpose it is often more convenient to express (23) in the following form;

$$l = \frac{M k^2}{M h} = \frac{\text{The moment of inertia}}{M h}. \qquad (26)$$

Ex. 1. Find the centre of percussion of a circular plate, capable of rotation about an axis which touches it.

Let the rotation-axis which touches the plate, and is in its plane, be the axis of z; let the plate be the plane of (z, x); and let the plane passing through the centre of the plate and perpendicular to the rotation-axis be the plane of (x, y); then it is evident that $\Sigma . myz = \Sigma . mzx = 0$; and thus that the rotation-

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axis is a principal axis, and that the point of contact is its principal point. In this case

$$\text{the moment of inertia} = \frac{5\pi\rho\tau a^4}{4};$$

therefore from (26), $$l = \frac{5\pi\rho\tau a^4}{4\pi\rho\tau a^3} = \frac{5a}{4}.$$

Also the line of the blow Q must be perpendicular to the plane of the plate; therefore, by (24),

$$\Omega = \frac{Q}{\pi\rho\tau a^3}.$$

Ex. 2. Find the centre of percussion of a rectangular cube whose rotation-axis is parallel to four parallel edges of the cube, and which is equidistant from the two nearer, as well as from the two farther edges.

Let the rotation-axis be the z-axis; and let the plane passing through it and bisecting the cube be the plane of (x, z); it is evident that $\Sigma.myz = \Sigma.mzx = 0$; so that the rotation-axis is a principal axis, and the line drawn through the centre of the cube perpendicular to it cuts it in its principal point. Let $2a$ be a side of the cube, and let c be the distance of the rotation-axis from its centre of gravity; then

$$k^2 = c^2 + \frac{2a^2}{3},$$

and $$h = c;$$

$$\therefore \quad l = c + \frac{2a^2}{3c};$$

$$\Omega = \frac{Q}{8\rho a^3 c}.$$

Ex. 3. A cylinder is capable of revolving about the diameter of one of its circular ends: find the centre of percussion.

Let a = the length of the cylinder, b = the radius of its circular transverse section. It is evident that the rotation-axis is a principal axis; and that the centre of the circular end is its principal point.

$$l = \frac{3b^2 + 4a^2}{6a}; \qquad \Omega = \frac{2Q}{\pi\rho a^2 b^2}.$$

Hence the centre of percussion will be at the end of the cylinder if $3b^2 = 2a^2$. If b is very small in comparison of a, $l = \frac{2a}{3}$; thus, if a straight rod of small transverse section is held by one

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end in the hand, l gives the point at which it may be struck when the hand will perceive no jar.

Ex. 4. Find the centre of percussion of a sphere revolving about an axis, which touches its surface.

This axis is evidently a principal axis, and the point of contact is its principal point; and we find

$$l = \frac{7a}{5}, \qquad \Omega = \frac{3Q}{4\pi\rho a^4}.$$

128.] Let us now suppose a single blow Q to be applied to the body at a point (ξ, η, ζ), in a line whose direction-angles are λ, μ, ν, relatively to a system of coordinates thus chosen: let the rotation-axis be the z-axis, and let a line perpendicular to it, and passing through the centre of gravity, be the x-axis; so that when the blow is struck the centre of gravity is $(h, 0, 0)$, where h is the distance from the centre of gravity to the rotation-axis. And let us suppose the effect of a blow on the axis to be a single pressure along it, and no pressure at right angles to it; so that the axis may slide in its own direction, if such a motion is possible. In this case we have

$$P_1 \cos\alpha_1 = P_1 \cos\beta_1 = 0; \qquad P_2 \cos\alpha_2 = P_2 \cos\beta_2 = 0;$$

and the equations of motion are

$$\left.\begin{aligned} Q\cos\lambda &= 0, \\ Q\cos\mu - \Omega M h &= 0, \\ Q\cos\nu - P_1 - P_2 &= 0; \end{aligned}\right\} \tag{27}$$

$$\left.\begin{aligned} Q(\eta\cos\nu - \zeta\cos\mu) + \Omega\Sigma.mzx &= 0, \\ Q(\zeta\cos\lambda - \xi\cos\nu) + \Omega\Sigma.myz &= 0, \\ Q(\xi\cos\mu - \eta\cos\lambda) - \Omega\Sigma.m(x^2+y^2) &= 0. \end{aligned}\right\} \tag{28}$$

From the first of (27), $\cos\lambda = 0$; so that the line of blow must be in a plane perpendicular to the line drawn through the centre of gravity at right angles to the rotation-axis.

And if k is the radius of gyration of the body, relatively to the rotation-axis, from the last of (28) and the second of (27) we have

$$\xi = \frac{k^2}{h}; \tag{29}$$

which gives the perpendicular distance from the rotation-axis on the plane which is parallel to it and contains the line of the blow.

Also if, as heretofore, $D = \Sigma.myz$, $E = \Sigma.mzx$, from the first two of (28), since $\cos\nu = \sin\mu$, we have

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$$\text{M}k^2\zeta - \text{D}\eta - \frac{\text{E}k^2}{h} = 0; \tag{30}$$

which is the equation to the line of the blow in the plane given by (29); this makes with the plane of (x, y) an angle $\tan^{-1}\frac{\text{D}}{\text{M}k^2}$; so that (29) and (30) are the equations to the line of the blow. The line just determined is called the Axis of Percussion. If $\text{D} = \text{E} = 0$, that is, if the rotation-axis is a principal axis, of which the origin is the principal point, $\zeta = 0$, and the axis of percussion lies in the principal plane of the rotation-axis; and its intersection with the plane containing the rotation-axis and the centre of gravity is the Centre of Percussion.

Now this axis of percussion may also be arrived at by the following process. At all points on the rotation-axis let the momental ellipsoids be described, and let the planes be drawn which are conjugate to the rotation-axis; these planes shall all intersect in the same straight line; and that line is the axis of percussion.

The equation to the momental ellipsoid at the origin is

$$\text{A}\xi^2 + \text{B}\eta^2 + \text{C}\zeta^2 - 2\text{D}\eta\zeta - 2\text{E}\zeta\xi - 2\text{F}\xi\eta - 1 = 0; \tag{31}$$

and the plane conjugate to the z-axis is

$$\text{C}\zeta - \text{D}\eta - \text{E}\xi = 0;$$

so that for the momental ellipsoid, whose centre is at a distance μ from the origin, the equation to the plane conjugate to the axis of z is

$$\text{C}(\zeta - \mu) - (\text{D} - \mu\,\Sigma.my)\eta - (\text{E} - \mu\,\Sigma.mx)\xi = 0;$$

or

$$\text{C}\zeta - \text{D}\eta - \text{E}\xi - \mu\{\text{C} - \Sigma.my\eta - \Sigma.mx\xi\} = 0. \tag{32}$$

If we take the notation and coordinate-system of the present Article, $\text{C} = \text{M}k^2$, $\Sigma.my = 0$, $\Sigma.mx = \text{M}h$;

so that (32) becomes

$$\text{M}k^2\zeta - \text{D}\eta - \text{E}\xi - \mu\{\text{M}k^2 - \text{M}h\xi\} = 0; \tag{33}$$

which is the equation to a plane, and contains the indeterminate quantity μ; it therefore represents a series of planes, all of which pass through the straight line which is the intersection of the two planes,

$$\left.\begin{aligned} \text{M}k^2\zeta - \text{D}\eta - \text{E}\xi &= 0, \\ k^2 - h\xi &= 0; \end{aligned}\right\} \tag{34}$$

the latter of which is a plane parallel to the plane of (y, z); and by substitution from the latter in the former we have

$$\text{M}k^2\zeta - \text{D}\eta - \frac{\text{E}k^2}{h} = 0;$$

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which is a plane perpendicular to the plane of (η, ζ), and inclined to the plane of (ξ, η), at an angle whose tangent is $\frac{D}{M k^2}$.

As the z-axis and the origin are, relatively to the body, arbitrary, this theorem is true for all lines which traverse the body; and therefore,

If at all points of a straight line which traverses a body the momental ellipsoids are described, the planes of these ellipsoids, which are conjugate to the given line, all pass through one and the same straight line.

Hence also we have this theorem:

If a body is capable of rotation about a certain fixed axis, and at all points of the axis the momental ellipsoids are described, and the planes of them, conjugate to the axis, are drawn; then all these pass through the same straight line; and that straight line is the direction of a blow which will produce no strain on the axis. If the axis is principal at one of its points, this line of blow lies in the corresponding principal plane, and is perpendicular to the plane containing the rotation-axis and the centre of gravity, and there will be no pressure at all on the axis. But if the rotation-axis is not principal at any one of its points, the direction of the blow will be oblique to the plane containing the axis and the centre of gravity, and there will be a pressure acting on the axis in the direction of its length.

In the preceding process we have supposed the body to be initially at rest, and motion to be communicated to it under certain states of pressure on the axis, &c.: the process however may be reversed; we may suppose the body to be moving about a fixed axis with the stated conditions of pressure on it; and the problem which would then have to be solved is, to determine the point of application, &c. of a force, as, for instance, a fixed obstacle, which shall withdraw all the momentum from the body.

SECTION 2.—*Rotation of a body about a fixed axis under the action of finite accelerating forces.*

129.] I PROCEED now to the case of a rigid body rotating about a fixed axis under the action of forces, whereby momenta are continuously impressed. To this case equations (37) and (38), Art. 48, are to be applied.

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Let us take, as in the preceding Articles, the rotation-axis to be the z-axis; and to be fixed at two points whose distances from the origin are respectively z_1 and z_2; let the pressures at these points at the time t be respectively P_1 and P_2; and let the lines of action of these pressures be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

Let $\Sigma.\text{P}\cos\alpha$ ..... $\Sigma.\text{P}z\cos\alpha$ ..... be abridging symbols of the axial components of these pressures, and of their moments relative to the axes; and let L, M, N, as in Art. 77, be the moments of the axial components of the couples of the impressed momentum-increments at the time t; then the equations of motion are

$$\left.\begin{aligned} \Sigma.m\left(\text{X}-\frac{d^2x}{dt^2}\right) - \Sigma.\text{P}\cos\alpha &= 0, \\ \Sigma.m\left(\text{Y}-\frac{d^2y}{dt^2}\right) - \Sigma.\text{P}\cos\beta &= 0, \\ \Sigma.m\left(\text{Z}-\frac{d^2z}{dt^2}\right) - \Sigma.\text{P}\cos\gamma &= 0; \end{aligned}\right\} \tag{35}$$

$$\left.\begin{aligned} \Sigma.m\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right) - \Sigma.\text{P}z\cos\beta &= \text{L}, \\ \Sigma.m\left(z\frac{d^2x}{dt^2}-x\frac{d^2z}{dt^2}\right) + \Sigma.\text{P}z\cos\alpha &= \text{M}, \\ \Sigma.m\left(x\frac{d^2y}{dt^2}-y\frac{d^2x}{dt^2}\right) &= \text{N}. \end{aligned}\right\} \tag{36}$$

As the z-axis is the fixed axis, it is more convenient to transform the last of these into its equivalent in terms of angular velocity by an independent process, than to take the general equations given in (48), (49), (50), Art. 79.

Let r be the distance from the rotation-axis of m, whose place at the time t is (x, y, z); and let θ be the angle between r and the plane of (x, z), which plane is assumed to be fixed in space; let ω be the angular velocity about the fixed z-axis; so that

$$\omega = \frac{d\theta}{dt}, \qquad \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{37}$$

Hence we have,

$$x = r\cos\theta, \qquad y = r\sin\theta;$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2 r\cos\theta - r\sin\theta\frac{d\omega}{dt},$$

$$\frac{d^2y}{dt^2} = -\omega^2 r\sin\theta + r\cos\theta\frac{d\omega}{dt}.$$

so that the last of (36) becomes

$$\Sigma.mr^2\frac{d\omega}{dt} = \text{N};$$

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and as $\frac{d\omega}{dt}$ is the same for all the particles of the system, it may be placed outside the sign of summation, and we have

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{\text{N}}{\Sigma . mr^2}$$

$$= \frac{\text{The moment of the impressed momentum-increments}}{\text{The moment of inertia}}; \quad (38)$$

each of these quantities being estimated relatively to the fixed rotation-axis. The form in which this equation is put shews that it is independent of the particular system of coordinate axes which has been taken. It is indeed identical with (47), Art. 78. By it the angular velocity-increment about the rotation-axis is given; and therefore by integration the angular velocity, and by a subsequent integration the angle described in a given time may be found. Thus the motion of the body about a fixed axis will be determined.

Before however we proceed to examples of this motion, let us shew that (38) may be derived immediately from first principles; for this process will remove any obscurity which may attach to its meaning.

Let m be a type-particle of the body or system; let r be its distance from the rotation-axis of z, so that if θ is the angle between r and the fixed plane of (x, z), the linear velocity of m is $r\frac{d\theta}{dt}$, and the linear velocity-increment is $r\frac{d^2\theta}{dt^2}$; and therefore the moment of the expressed momentum of m is $mr^2\frac{d^2\theta}{dt^2}$; so that relatively to the axis of z the moment of the whole expressed momentum-increment is $\Sigma . mr^2\frac{d^2\theta}{dt^2}$; and therefore if N is the moment relatively to the same axis of the whole impressed momentum-increment in an unit of time, by D'Alembert's principle we have

$$\Sigma . mr^2\frac{d^2\theta}{dt^2} = \text{N};$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{\text{N}}{\Sigma . mr^2}.$$

130.] With respect to this equation, I would in the first place observe, that if the lines of action of all the impressed forces are

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parallel to the axis of z, which is the rotation-axis, $\text{N} = 0$; and that

$$\frac{d^2\theta}{dt^2} = 0;$$

$$\therefore \quad \frac{d\theta}{dt} = \Omega,$$

if Ω is the angular velocity at the time under consideration; so that the system moves with a constant angular velocity. Hence also

$$\theta - a = \Omega t,$$

if a is the value of θ, when $t = 0$; so that equal angles are described in equal times. This is a particular case of the principle of conservation of areas; see Art. 59. Thus, if a heavy body rotates about a vertical axis, the force of gravity has no effect on the angular velocity.

131.] But one of the most important applications of this theorem is the motion of a heavy body rotating about a fixed horizontal axis. Let us take the system of axes delineated in fig. 23; let the axis of z be vertical downwards; let the y-axis be the rotation-axis, and let θ be the angle at which the line from $m(x, y, z)$ to the y-axis is inclined to the vertical plane of (y, z); thus the line of action of gravity, which is the only force acting on m, is parallel to the z-axis. Let G be the centre of gravity of the body, and let the plane passing through it and perpendicular to the axis of y be the plane of (x, z); so that as the body rotates about the axis of y, the line OG moves in the plane of (x, z).

Let M be the mass of the body; $\text{OG} = h =$ the distance of the centre of gravity from the rotation-axis; and let $\text{M}k^2$ be the moment of inertia about the rotation-axis; let $\text{GO}z = \theta$, so that as θ increases the body rotates about the axis of y from the z-axis to the x-axis: also $\frac{d\theta}{dt} = \omega$ is the angular velocity; and is the same for all particles of the body. Now the moment of the impressed momentum on m at (x, y, z) at the time t relatively to the rotation-axis is mgx, and tends to diminish θ; so that the moment of the momenta impressed on all the particles at the time t

$$= -\Sigma . mgx$$

$$= -\text{M}gh\sin\theta;$$

and the moment of the impressed momentum is the same as if the whole mass were collected at its centre of gravity. And thus, from (38), we have

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$$\frac{d^2\theta}{dt} = -\frac{\text{M}gh\sin\theta}{\text{M}k^2}$$

$$= -\frac{gh}{k^2}\sin\theta; \tag{39}$$

which equation gives the expressed angular velocity-increment about the rotation-axis.

Let us multiply both sides of (39) by $d\theta$; and let us suppose the body to be at rest when $\theta = a$; then, integrating (39), we have

$$\frac{d\theta^2}{dt^2} = \frac{2gh}{k^2}(\cos\theta - \cos a); \tag{40}$$

which assigns the angular velocity in terms of θ. From this equation it appears that $\frac{d\theta}{dt} = 0$; that is, that the angular velocity vanishes, and the body is at rest, when

$\theta = a, \quad \theta = -a, \quad \theta = 2\pi + a, \quad \theta = 2\pi - a, \ldots \theta = 2m\pi \pm a$;

so that if (40) expresses the circumstances of the body, the motion of it is oscillatory, the arc of vibration being double of that between the vertical line and the initial position of the line through the axis and the centre of gravity; this latter being the vertical line when the body is at rest. Hence we have the following circumstances of motion of a heavy body capable of oscillation about a horizontal axis. When the body is at rest, the perpendicular from its centre of gravity to the rotation-axis is vertical; let this line be moved through an angle a, and let the body be left to itself; it will oscillate through an angle $2a$, the centre of gravity ascending to equal heights on both sides of the lowest point. Such an oscillating body is called a "compound pendulum."

If the body is moving with an angular velocity Ω when $\theta = a$, the equation of the angular velocity becomes

$$\frac{d\theta^2}{dt^2} - \Omega^2 = \frac{2gh}{k^2}(\cos\theta - \cos a); \tag{41}$$

but as this is of the same form as (40), so far as integration is concerned, we shall inquire into the properties of only (40).

Equations (40) and (41) are evidently those of vis viva.

132.] From (40) we have

$$dt = -\frac{k}{(2gh)^{\frac{1}{2}}}\frac{d\theta}{(\cos\theta - \cos a)^{\frac{1}{2}}}; \tag{42}$$

whence, by integration, the time may be found in terms of θ, and the whole time of an oscillation may be determined.

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This equation however in its present form is an elliptic transcendent, and therefore cannot be (as it is said) integrated. If however the displacement of the body is slight, so that a and θ are both small, then we may expand $\cos\theta$ and $\cos a$, and neglect powers of θ and a above the second; whereby we have

$$\cos\theta = 1 - \frac{\theta^2}{2}, \qquad \cos a = 1 - \frac{a^2}{2};$$

and (42) becomes

$$dt = -\frac{k}{(gh)^{\frac{1}{2}}} \frac{d\theta}{(a^2 - \theta^2)^{\frac{1}{2}}}; \tag{43}$$

$$\therefore \quad t = \frac{k}{(gh)^{\frac{1}{2}}} \cos^{-1}\frac{\theta}{a}; \tag{44}$$

if $t = 0$, when $\theta = a$; and therefore if τ is the time of a small oscillation of a heavy body about a horizontal axis, $t = \tau$ when $\theta = -a$, and

$$\tau = \frac{\pi k}{(gh)^{\frac{1}{2}}}. \tag{45}$$

Now if we consider a heavy particle of infinitesimal dimensions attached to the end of a rigid imponderable rod of length l, and without weight, and vibrating about a horizontal axis perpendicular to its length, to be a *perfect pendulum*, then, as we have shewn in Art. 357, Vol. III, if τ is the time of small oscillation of such a pendulum,

$$\tau = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}};$$

and the time of the heavy oscillating body is identical with this, if

$$l = \frac{k^2}{h}. \tag{46}$$

Thus, the compound pendulum is *isochronous* with a perfect pendulum of the length l, which is given in (46); and l is called the length of the simple isochronous pendulum.

The agreement however in motion between the compound and the simple isochronous pendulum is greater than the preceding investigations lead to. For the general equation of a heavy particle attached to the end of a rigid and imponderable rod of length l, and rotating in a circle, is, see Art. 359, Vol. III,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta;$$

and this equation is identical with (39), which determines the rotation of the heavy body, if

$$l = \frac{k^2}{h};$$

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and hence we conclude that if the whole mass of the rotating body is condensed into a particle at a distance l from the rotation-axis along the line which passes through the centre of gravity, the circumstances of equilibrium and of motion of this particle would be identical with the similar circumstances of the body. And if the body is slightly displaced from its position of stable equilibrium, and oscillates through a small angle, the time of an oscillation

$$= \pi \left(\frac{l}{g}\right)^{\frac{1}{2}};$$

where

$$l = \frac{k^2}{h},$$

k being the radius of gyration of the body about the rotation-axis, and h being the distance of the centre of gravity from the rotation-axis.

The point O, Fig. 23, in which the horizontal rotation-axis pierces the vertical plane containing the centre of gravity, is called the Centre of Suspension; and if OG is produced to C, so that $OC = l$, C is called the Centre of Oscillation, and OC or l is the length of the simple pendulum isochronous with the body; that is, if the whole mass is collected into a particle at C, the circumstances of rotation of the particle thus condensed will be the same as those of the body.

Let k' be the radius of gyration of the body relative to an axis through G, and parallel to the rotation-axis; then, by (129), Art. 101,

$$k^2 = k'^2 + h^2;$$

$$\therefore \quad l = \frac{k'^2 + h^2}{h}$$

$$= h + \frac{k'^2}{h}; \tag{47}$$

$$\therefore \quad (l - h)h = k'^2; \tag{48}$$

and replacing these by the geometrical quantities

$$CG \times GO = k'^2$$

$$= \text{a constant.} \tag{49}$$

Now this equation would be unaltered if the places of O and C were interchanged; whence we infer that if C is the centre of oscillation for an axis Oy through O, O would be the centre of oscillation for a parallel axis through C. This theorem, as it is commonly stated, asserts the convertibility of the centres of suspension and oscillation. As the length of the simple isochronous

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pendulum is the same whether C or O is the centre of oscillation, so the time of oscillation is the same for both parallel axes.

It will be observed that we have the same expressions for the determination of the centre of oscillation and the centre of percussion relative to a given rotation-axis, see (23), Art. 126: in the latter case, however, it is necessary that the rotation-axis should be a principal axis at some point on it, and the centre of percussion should be in its principal plane; here no such restriction as to the nature of the axis is necessary. We have hereby a method by which the centre of percussion may be practically determined. Let the body of which the centre of percussion is to be found be suspended by, and made to vibrate about, the relative rotation-axis; let the number of vibrations in a given time be noted; let, say, n vibrations take place in t; then

$$\frac{t}{n} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad l = \frac{g t^2}{\pi^2 n^2};$$

thus, if t and n are carefully observed, as the other quantities are known, l is also known; and this is the distance of the centre of percussion from the rotation-axis.

133.] Before we enter on other investigations connected with the times, &c. of oscillation of bodies, let us determine l in certain cases; and for this purpose we shall generally find the second of the following forms the more convenient;

$$l = \frac{k^2}{h} = \frac{\text{M}\,k^2}{\text{M}\,h}$$

$$= \frac{\text{The moment of inertia relative to the rotation-axis}}{\text{The mass} \times \text{the distance of the C. of GR. from the axis}}. \quad (50)$$

Ex. 1. A straight heavy wire, of length $2a$, vibrates about an axis passing through its end, and perpendicular to its length: find the length of the simple isochronous pendulum.

$$l = \frac{4a}{3};$$

that is, the length of the simple isochronous pendulum is two thirds of the length of the wire.

Ex. 2. A wire, in the form of the arc of a circle, vibrates about an axis passing through its middle point and perpendicular to its plane; prove that the length of the simple isochronous pen-

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dulum is that of the diameter of the circle, whatever is the length of the wire.

Let ρ and ω be the density and the area of the transverse section of the wire; let a be the radius of the circle; then the origin being the middle point of the wire, the equation to the wire is

$$y^2 + x^2 = 2ax;$$

$$\therefore \quad \text{the moment of inertia} = \rho\omega\int(x^2+y^2)\,ds$$

$$= 2a\rho\omega\int x\,ds.$$

Also $$\text{M}h = \rho\omega\int x\,ds;$$

the limits of integration being the same in both integrals; so that

$$l = 2a.$$

Ex. 3. Compare the times of vibration of a thin circular plate about axes passing through the circumference, and (1) touching the circle and in its plane; (2) at right angles to the plane of the circle.

The moment of inertia relative to a tangent $= \dfrac{5\pi\rho\tau a^4}{4}$;

the moment of inertia relative to a perpendicular axis $= \dfrac{3\pi\rho\tau a^4}{2}$;

and in each case $\qquad \text{M}h = \pi\rho\tau a^3$;

therefore, if l_1 and l_2 are the lengths of the corresponding isochronous pendulums,

$$l_1 = \frac{5a}{4}; \qquad l_2 = \frac{3a}{2}.$$

and if t_1 and t_2 are the corresponding times of small vibration,

$$\frac{t_1}{t_2} = \left(\frac{l_1}{l_2}\right)^{\frac{1}{2}} = \left(\frac{5}{6}\right)^{\frac{1}{2}}.$$

Ex. 4. A right cone oscillates about an axis passing through its vertex and perpendicular to its own axis; it is required to find the length of the simple isochronous pendulum.

Let a = the altitude of the cone; b = the radius of the circular base; then

$$\text{the moment of inertia} = \frac{\pi\rho a b^2}{20}(4a^2+b^2);$$

and $$\text{M}h = \frac{\pi\rho a^2 b^2}{4};$$

therefore $$\therefore \quad l = \frac{4a^2+b^2}{5a}.$$

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If $a = b$, that is, if the cone is right-angled, $l = a$; and the altitude of the cone is the length of the simple isochronous pendulum: thus the centre of oscillation is in the centre of the base; so that the times of oscillation of a right-angled circular cone are equal for axes through the vertex and the centre of the base which are perpendicular to the axis of the cone.

Ex. 5. The mass of the particle at the end of a perfect pendulum of length a is M: to determine the position of another particle m on the rod, so that the time of oscillation of the whole pendulum may be a minimum.

Let $x =$ the distance of m from the centre of suspension; then

$$l = \frac{mx^2 + \text{M}a^2}{mx + \text{M}a};$$

$$\therefore \quad \frac{dl}{dx} = \frac{m^2x^2 + 2m\text{M}ax - m\text{M}a^2}{(mx + \text{M}a)^2} = 0$$

$$\text{if } mx = -\text{M}a \pm a\{\text{M}(\text{M}+m)\}^{\frac{1}{2}};$$

which gives two values for x, of which one is positive and the other is negative.

Let $\bar{x}$ be the distance from the centre of suspension of the centre of gravity of M and m, when m is in its required position; then $(\text{M}+m)\bar{x} = mx + \text{M}a$;

$$\therefore \quad \bar{x} = \pm a\left(\frac{\text{M}}{\text{M}+m}\right)^{\frac{1}{2}};$$

that is, the centre of gravity of M and m is equally distant from the centre of suspension in the two directions along the rod. Also

$$l = 2a\,\frac{\pm\{\text{M}(\text{M}+m)\}^{\frac{1}{2}} - \text{M}}{m}.$$

Ex. 6. A metronome is formed of a heavy rod of given length, having at one end a heavy sphere of radius r and mass m, at a distance a from the rotation-axis, which is perpendicular to the rod; another sphere of radius r' and mass m' slides along the rod: to find the point at which the centre of this latter sphere must be fixed, so that the whole system may oscillate n times in a minute.

Let the metronome be represented in Fig. 24, wherein the rod, which in the position of equilibrium is vertical, is slightly inclined to the vertical.

Let the plane of the paper be the plane of vibration, and let O be the point where the rotation-axis pierces the plane. Let A be the centre of the fixed sphere, $\text{OA} = a$; let P be the centre of

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the sliding sphere, OP $= x$; let OB $= b$; let M = the mass of the rod. Then, relatively to the rotation-axis,

$$\text{the moment of inertia of } m = m\left(\frac{2r^2}{5} + a^2\right)$$

$$\text{..... of } m' = m'\left(\frac{2r'^2}{5} + x^2\right)$$

$$\text{..... of M} = m\left(\frac{a^2 - ab + b^2}{3}\right);$$

and the denominator of (50) in this case

$$= ma - \text{M}\frac{b-a}{2} - m'x;$$

so that

$$l = \frac{2}{15}\,\frac{m(6r^2 + 15a^2) + m'(6r'^2 + 15x^2) + 5\text{M}(a^2 - ab + b^2)}{2ma - \text{M}(b-a) - 2m'x}; \quad (51)$$

As the metronome is to oscillate n times in a minute, we have

$$\frac{60}{n} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}.$$

Let L = the length of the second's pendulum; then

$$1 = \pi\left(\frac{\text{L}}{g}\right)^{\frac{1}{2}};$$

and therefore $$l = \frac{3600}{n^2}\text{L};$$

and if we substitute this in the left hand member of (51), the equation contains x and all known quantities; whence x may be determined; and the rod of the metronome may be graduated so that the system will oscillate in any required time.

If the rod is very thin, as is the case with the ordinary metronomes, M may be neglected; and we have

$$\frac{3600}{n^2}\text{L} = \frac{m(2r^2 + 5a^2) + m'(2r'^2 + 5x^2)}{5(ma - m'x)}.$$

Ex. 7. A pendulum consists of a rod of length a and mass m; at the end of which is a circular plate, Fig. 25, of radius r and mass M, so arranged that the plate is capable of sliding on the rod, and rests on a nut fixed at the end of the rod; the plane of the plate is always in the plane of vibration; find the length of the simple isochronous pendulum;

$$\text{The moment of inertia of the plate} = \text{M}\left\{\frac{r^2}{2} + (a-r)^2\right\};$$

$$\text{The moment of inertia of the rod} = m\frac{a^2}{3};$$

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and $$\text{M}h = \frac{ma}{2} + \text{M}(a-r);$$

$$\therefore \quad l = \frac{3\text{M}(3r^2 - 4ar + 2a^2) + 2ma^2}{3\{ma + 2\text{M}(a-r)\}}. \tag{52}$$

Let us suppose the temperature to vary so that a and r are increased by da and dr respectively; M and m being unaltered; and let us suppose the pendulum to be compensating; so that l remains the same, whatever is the temperature; then, since $\text{D}l = 0$, we have from (52)

$$\{6\text{M}^2(2a^2-4ar+r^2) + m\text{M}(10a^2-8ar-9r^2) + 2m^2a^2\}\,da$$
$$= \text{M}dr\,\{6\text{M}(3r^2-6ar+2a^2) + 2ma(9r-4a)\}. \tag{53}$$

In the most common form of compensating pendulums the straight rod is made of steel, and the weight consists of a cylinder of mercury which is fixed at the end of the rod, the axis of the cylinder coinciding with the rod, and the base of the cylinder resting on a nut at the end of the rod. The amount of expansion of the rod and the mercury having been determined by experiment for an increase of one degree of temperature, and the length of the seconds' pendulum being also known, the quantity of mercury may be determined by a process similar to that which we have just explained*.

134.] The convertibility or the reciprocality of the centres of suspension and oscillation of a pendulum has been applied by Capt. Kater to the determination of its length; and he has hereby obtained means for determining the length of a seconds' pendulum at a given place.

Let the pendulum consist of an ordinary thin straight rod, and a heavy disc, as in Fig. 26. At the points O and C, at the distance l apart, let two knife edges be placed parallel to each other, and at right angles to the rod of the pendulum; so that the pendulum may vibrate on either of them, as in the diagrams of the figure, where it rests on two horizontal and parallel plates. Let a small weight m be capable of sliding on the bar, and of being clamped to it by means of a screw. It is evident that

* For a full account of this pendulum I must refer the reader to a Memoir by Mr. Francis Bailey in the Eighth Volume of the Memoirs of the Royal Astronomical Society of London, for the year 1824: and for a description of various other kinds of compensating pendulums to "Mechanics," by Capt. Kater and Dr. Lardner; Longman and Co., London, 1830.

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whether O or C is the centre of suspension the length of the simple isochronous pendulum will vary according to the place of m; let the place of m be so adjusted that the times of oscillation may be the same, whether the pendulum is suspended by the knife edge at C or by that at O; so that OC ($= l$) is the length of the simple isochronous pendulum; if then this distance OC is carefully measured, the length of a simple pendulum is accurately known: and by means of it the lengths of all other pendulums may be determined.

Thus, suppose the pendulum above described to make n oscillations in a given time, say in t; these quantities can be found by means of an astronomical or any other correct clock, by the method of coincidences: then

$$\frac{t}{n} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}}. \tag{54}$$

Let L be the length of a seconds' pendulum; then

$$1 = \pi \left(\frac{L}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad L = \frac{n^2}{t^2} l; \tag{55}$$

and therefore the length of L is also known.

Now, as we have before remarked, g, which is the velocity-increment of a falling particle due to the earth's attraction in a second of time, varies for different places on the earth's surface, and for several reasons, as we have explained in Arts. 224 and 258, Vol. III. It is important to have means of determining the value of g by observation; as well for the purpose of verifying the theoretical law, as for determining the constants which enter into Clairaut's expression, Art. 258, Vol. III; and the preceding theory of pendulum-motion supplies a method. From (54), we have

$$g = \frac{n^2 \pi^2 l}{t^2}; \tag{56}$$

and thus when n and t have been determined by observation, and l by direct measurement, all the quantities in the right hand member of this equation are known. So that from (55) and (56) the length of the seconds' pendulum, and the velocity-increment due to the earth's attraction, which is usually termed "the force of gravity," may be found at any given place. A table containing the values of L and g for a few places, with their latitudes N or S, is subjoined; the observations are reduced to the

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level of the sea, and to a pendulum vibrating in vacuo, at a temperature 62° of Fahrenheit*.

Name of Place.	Latitude.	Length of Pendulum in Inches.	Gravity in Feet.	Name of Observer.
Spitzbergen	79° 49′ 58″ N	39.2146	32.25294	Sabine.
Drontheim	63° 25′ 54″ N	39.1745	32.2198	Sabine.
London	51° 31′ 8″ N	39.13929	32.1910	Kater, Sabine.
Paris	48° 50′ 14″ N	39.1308	32.1838	Biot, Borda, &c.
New York	40° 42′ 43″ N	39.1016	32.1598	Sabine.
Jamaica	17° 56′ 7″ N	39.0351	32.1052	Sabine.
Sierra Leone	8° 29′ 28″ N	39.0199	32.0933	Sabine.
Cape of Good Hope	33° 55′ 15″ S	39.0787	32.1409	Freycinet.

These results shew that gravity continually increases as we go from the Equator to the Poles. And it is found that the differences between the observed results and the values calculated according to theory are extremely small.

135.] By means of the preceding value for the length of a pendulum which vibrates isochronously with a body relative to a given rotation-axis, we are able to deduce experimentally the radius of gyration of a body relative to an axis; and consequently the central principal radii of gyration, and thus the central ellipsoid of gyration.

About the axis relative to which the radius of gyration of the body is to be determined, let the body make small oscillations: let T be the time of an oscillation, which can be observed by means of a clock; then

$$T = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad T^2 = \frac{\pi^2}{g}\,\frac{k^2}{h}.$$

* For accounts of the process by which General Sabine determined the lengths of the pendulum at those places in the following table to which his name is attached, see "An Account of Experiments to determine the Figure of the Earth by means of Pendulums vibrating seconds in different latitudes, as well as on various other subjects of Philosophical Inquiry," by Edward Sabine, F. R. S., &c., &c.; John Murray, London, 1825; at the expense of the Board of Longitude. See also three other papers by General Sabine in the Philosophical Transactions of 1827.

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Now h, which is the distance of the centre of gravity from the rotation-axis, must be measured; and we have

$$k^2 = \frac{\text{T}^2 h g}{\pi^2}; \tag{57}$$

$$\therefore \quad \text{M}\, k^2 = \frac{\text{T}^2 h}{\pi^2}\, \text{W}, \tag{58}$$

if W is the weight of the body. Thus, (57) gives the radius of gyration, and (58) gives the moment of inertia of a body relative to a given axis.

If the axis passes through the centre of gravity the method fails, because $h = 0$, and therefore $\text{T} = \infty$: in this case let another rotation-axis be taken parallel to the given one through the centre of gravity, and at a distance h from it; then, if k' is the radius of gyration for the axis through the centre of gravity,

$$\text{T}^2 = \frac{\pi^2}{g}\,\frac{h^2 + k'^2}{h};$$

$$\therefore \quad k'^2 = \frac{\text{T}^2 h g}{\pi^2} - h^2, \tag{59}$$

$$\text{M}\, k'^2 = \frac{\text{T}^2 h}{\pi^2}\, \text{W} - \text{M}\, h^2; \tag{60}$$

thus, (59) gives the radius of gyration, and (60) gives the moment of inertia about an axis passing through the centre of gravity. When these have been found for a sufficient number of axes, the central ellipsoid of a body may be constructed. Whenever therefore a body is given, however irregular its bounding surface is, whatever is the law according to which its density or the distribution of its elements varies, its central ellipsoid can always be determined by the preceding method; and consequently every curve or surface connected with it, or which may be derived from it, may always be assumed as known.

136.] Certain general properties of axes of vibration of a body also require investigation. Let us refer the body to the centre of gravity as origin, and to its central principal axes as coordinate axes. Let A, B, C be the three central principal moments of inertia, in the same order of magnitude and about the same axes as we have assumed them to be in the preceding Chapter. So that if $\text{M}\, k'^2$ is the moment of inertia about an axis (α, β, γ) passing through the centre of gravity,

$$\text{M}\, k'^2 = \text{A}\,(\cos\alpha)^2 + \text{B}\,(\cos\beta)^2 + \text{C}\,(\cos\gamma)^2. \tag{61}$$

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Let all axes of a body, relative to which the times of vibration are equal, be called *isochronal;* then for an axis parallel to a line (α, β, γ), which passes through the centre of gravity, and at a distance from it equal to h,

$$l = h + \frac{k'^2}{h} \tag{62}$$

$$= h + \frac{\text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2}{\text{M}h}; \tag{63}$$

and since this is true for all axes parallel to (α, β, γ), and equidistant from it, it follows that all axes lying on the surface of a right circular cylinder whose axis passes through the centre of gravity are isochronal.

Let $l - h = h'$; so that $hh' = k'^2$; then as an axis at a distance h' from the centre of gravity is isochronal with a parallel axis at a distance h, so all axes lying on the surface of a right circular cylinder whose radius is h', and whose axis passes through the centre of gravity, are isochronal; and are isochronal with those which lie on the surface of the coaxal circular cylinder whose radius is h.

From (62) it appears that $l = \infty$ when $h = 0$, and when $h = \infty$; so that there is some value of h between these limits which makes l a minimum. Let us equate to zero the h-differential of (62); then

$$\frac{dl}{dh} = 1 - \frac{k'^2}{h^2} = 0,$$

$$\text{if} \quad h = k';$$

$$\therefore \quad l = 2k'; \tag{64}$$

it appears then that for all parallel axes the time of oscillation is the least for those which are at a distance k' from the centre of gravity of the body, where k' is the radius of gyration of the body relative to a parallel axis through the centre of gravity; and that the length of the corresponding simple isochronous pendulum is $2k'$. In this case, the two coaxal cylinders of isochronal rotation-axes become identical; and from (62) we have

$$l = 2\left\{\frac{\text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2}{\text{M}}\right\}^{\frac{1}{2}}. \tag{65}$$

Since $l = 2k'$, the time of oscillation depends on the central radii of gyration, and is least for an axis parallel to the least radius of gyration; therefore, from (65), l is the least when $\cos\beta = \cos\gamma = 0$: that is, when

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$$l = 2\left(\frac{A}{M}\right)^{\frac{1}{2}}$$
$$= 2a,$$

if a is the least central radius of gyration. And this absolutely gives the least time of oscillation of all axes about which a body can oscillate. And as of all parallel axes that which is at a distance equal to $2k'$ from the parallel central radius of gyration yields the least time of oscillation; so of all, that which is parallel to the axis of the greatest moment of inertia is the maximum minimorum, and that which is parallel to the axis of the least moment of inertia is the minimum minimorum, and the other minima are intermediate to these.

Some examples are subjoined.

Ex. 1. Of all axes passing through and perpendicular to a thin rod of length $2a$, that at a distance $a3^{-\frac{1}{2}}$ from the middle point of the rod is the axis for which the time of oscillation of the rod is the least; and the length of the simple isochronous pendulum is

$$2a\left(\frac{1}{3}\right)^{\frac{1}{2}}.$$

Ex. 2. For a sphere of radius r, all the radii of gyration passing through the centre are equal, and $= r\left(\frac{2}{5}\right)^{\frac{1}{2}}$; so that the axes for which the time of oscillation is the least are at a distance from the centre equal to this quantity; and

$$l = \frac{2^{\frac{3}{2}}r}{5^{\frac{1}{2}}}.$$

Ex. 3. The axis for which an ellipsoid vibrates in the shortest possible time is parallel to its greatest principal axis, and at a distance from it $= \left(\frac{b^2+c^2}{5}\right)^{\frac{1}{2}}$.

137.] Again, since all central equimomental axes lie on the surface of the right cone

$$(\text{H}-\text{A})x^2 + (\text{H}-\text{B})y^2 + (\text{H}-\text{C})z^2 = 0, \qquad (66)$$

where H is the moment of inertia relative to any axis on the cone, this is the locus-surface of all axes of the circular cylinders of equal radius h, all lines lying on the surface of which are isochronal axes; and for which $l = h + \frac{\text{H}}{\text{M}h}$.

It is similarly the locus surface of all axes of the circular

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cylinders of equal radius h', all lines lying on the surfaces of which are isochronal axes; and for which

$$l' = h' + \frac{\mathrm{H}}{\mathrm{M}h'};$$

where $hh' = k'^2$, and the axes lying on the surfaces of all the cylinders are isochronal.

Thus, if two spheres of radii h and h' are described from the vertex of the cone (66) as centre, and cones are described touching them, coaxal with and similar to the given cone, all generating lines of these two cones are isochronal.

Lastly, let us determine the conical surface on which lie all isochronal axes passing through the given point (x_0, y_0, z_0).

Let the equations to one of these isochronal axes referred to the centre of gravity as origin be

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n}$$

$$= \{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}^{\frac{1}{2}} = s \text{ (say)}; \quad (67)$$

$$\therefore \quad h^2 = (ny_0 - mz_0)^2 + (lz_0 - nx_0)^2 + (mx_0 - ly_0)^2;$$

and
$$k'^2 = \frac{\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2}{\mathrm{M}}$$

$$= a^2l^2 + b^2m^2 + c^2n^2,$$

if a, b, c are the principal central radii of gyration. In these equations, replacing l, m, n by their values from (67), we have

$$s^2h^2 = (zy_0 - yz_0)^2 + (xz_0 - zx_0)^2 + (yx_0 - xy_0)^2,$$

$$= (x^2+y^2+z^2)(x_0^2+y_0^2+z_0^2) - (xx_0+yy_0+zz_0)^2;$$

$$s^2k'^2 = a^2(x-x_0)^2 + b^2(y-y_0)^2 + c^2(z-z_0)^2.$$

If therefore l is the length of the simple pendulum, isochronous with the body about each of the axes passing through (x_0, y_0, z_0),

$$lh = h^2 + k'^2;$$

$$l\{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2\}^{\frac{1}{2}}$$
$$\{(zy_0 - yz_0)^2 + (xz_0 - zx_0)^2 + (yx_0 - xy_0)^2\}^{\frac{1}{2}}$$

$$= (zy_0 - yz_0)^2 + (xz_0 - zx_0)^2 + (yx_0 - xy_0)^2 +$$
$$a^2(x-x_0)^2 + b^2(y-y_0)^2 + c^2(z-z_0)^2; \quad (68)$$

which is evidently the equation to a cone of the fourth degree. Let the origin be transferred to the point (x_0, y_0, z_0) which is its vertex; and let

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= r^2 \\ x_0^2 + y_0^2 + z_0^2 &= r_0^2 \end{aligned}\right\};$$

then (68) becomes

$$lr\{r^2r_0^2 - (xx_0 + yy_0 + zz_0)^2\}^{\frac{1}{2}} =$$
$$r^2r_0^2 - (xx_0 + yy_0 + zz_0)^2 + a^2x^2 + b^2y^2 + c^2z^2. \quad (69)$$

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138.] The following problems are in further illustration of the principles contained in the preceding Articles.

Ex. 1. A vertical rod of length $2a$ and of mass M, which turns about a horizontal axis passing through the upper end, is struck by a blow at its centre of percussion, and ascends into its position of unstable equilibrium; determine the force of the blow.

Let P = the momentum of the blow, and let Ω be the angular velocity which is thereby given to the rod about its rotation-axis. Now the centre of percussion is at a distance $\frac{4a}{3}$ from the rotation-axis; so that by (6), Art. 124,

$$\Omega = \frac{P}{aM}.$$

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{4a}\sin\theta;$$

$$\therefore \quad \frac{d\theta^2}{dt^2} - \Omega^2 = \frac{3g}{2a}(\cos\theta - 1),$$

since $\frac{d\theta}{dt} = \Omega$, when $\theta = 0$; and when the rod is in its place of unstable equilibrium, $\theta = \pi$, and $\frac{d\theta}{dt} = 0$; therefore

$$\Omega^2 = \frac{3g}{a}; \qquad \text{and} \qquad P = M(3ag)^{\frac{1}{2}}.$$

Ex. 2. A circular plate of radius a and mass M, capable of rotation about a horizontal axis which is in its plane and touches it, is struck by a blow at its centre of percussion, and ascends into its position of unstable equilibrium; determine the blow.

$$\Omega = \frac{P}{aM};$$

$$\frac{d^2\theta}{dt^2} = -\frac{4g}{5a}\sin\theta;$$

$$\therefore \quad \Omega^2 = \frac{16g}{5a}; \quad \text{and} \quad P = \frac{4M}{5}(5ag)^{\frac{1}{2}}.$$

139.] Another application of these principles has been made by Capt. Robins, to the determination of the velocity of a cannon ball.

A large thick heavy board is suspended by a fixed horizontal axis; a cannon is so placed that a ball projected horizontally from the cannon strikes this board at rest at a certain point; and the board revolves through an angle, which is observed. It

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is required to determine the velocity of the ball. The swinging board with its axis is called a Ballistic Pendulum. A vertical section is given in Fig. 27.

We shall suppose the ball to strike the board at right angles to its plane, and to remain in the board after impact. Let

M = the sum of the masses of the pendulum and ball.

m = the mass of the ball.

$\text{M}k^2$ = the moment of inertia of the pendulum and ball.

v = the velocity of the ball at the instant of impact.

Ω = the angular velocity due to the blow of the ball.

a = the distance of the point of impact from the rotation-axis.

h = the distance of the centre of gravity of the masses of the pendulum and ball from the rotation-axis.

$$\therefore \quad \Omega = \frac{ma\text{v}}{\text{M}k^2};$$

$$\frac{d\theta^2}{dt^2} - \Omega^2 = \frac{2hg}{2}(\cos\theta - 1).$$

Let $\theta = \alpha$, when $\frac{d\theta}{dt} = 0$, that is, when the body comes to rest. then

$$\left(\frac{ma\text{v}}{\text{M}k^2}\right)^2 = \frac{2hg}{k^2}(1 - \cos\alpha)$$

$$\therefore \quad \text{v} = \frac{2\text{M}k}{ma}(hg)^{\frac{1}{2}}\sin\frac{\alpha}{2}; \tag{70}$$

as all the quantities in the right hand member of this equation may be observed, or are known, v is also known.

We may determine α in the following manner. At a point in the board at a distance h from the rotation-axis, let the end of a ribbon be fastened, and let the rest of the ribbon be wound tightly round a reel; as the pendulum ascends, let a length c be unwound from the reel; then c is the chord of the angle α to the radius h, so that

$$c = 2h\sin\frac{\alpha}{2};$$

$$\therefore \quad \text{v} = \frac{\text{M}kc}{ma}\left(\frac{g}{h}\right)^{\frac{1}{2}}; \tag{71}$$

k is determined by the process explained in Art. 135; and if we replace k by the value given in (57) we have

$$\text{v} = \frac{gc\text{M}}{\pi am}\text{T}; \tag{72}$$

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and if M and m are replaced by their weights, say W and w, which are proportional to them, we have

$$\text{V} = \frac{g c \text{W}}{\pi a w} \text{T}. \tag{73}$$

If the mouth of the cannon is placed near to the pendulum, the value of V, given by this formula, must be nearly the velocity of projection. And if the distance of the pendulum from the mouth of the gun be large, so that the velocity of impact on the pendulum is less than that of projection, if the coefficient of resistance of the air is given, we may by the process of Art. 302, Vol. III, estimate the diminution of velocity due to the resistance of the air, and thus determine the velocity of projection.

The velocity however may be determined in the following manner. Let the gun itself be suspended by a horizontal axis, and thus form a pendulum; when the gun is discharged, it will oscillate by reason of the recoil; and by observing the times of these oscillations, and making the required alterations in (73), the velocity of projection will be determined: W will represent in this case the excess of the weight of the gun over that of the ball, and c and a must be similarly altered.

140.] In further illustration of the motion of bodies about fixed axes, I will consider the motion of the parts of certain machines, in which certain rotation-axes are fixed.

Ex. 1. Two weights mg and $m'g$ are connected by a flexible and inextensible string without weight, which passes over a given pulley with a fixed axis and a rough surface; it is required to determine the circumstances of motion of each weight and of the pulley.

The pulley is supposed to be rough, so that as the string moves the pulley moves with it.

Let the weights, &c. be arranged as in Fig. 17; and let the symbols be those of Ex. 1, Art. 46; and let us suppose m and m' to have the initial velocities, &c. of that example. Let M = the mass of the pulley, and a = the radius; then

$$\text{the moment of inertia of the pulley} = \text{M}\frac{a^2}{2}.$$

Let Ω be the initial angular velocity of the pulley due to the instantaneous initial tensions of the string; then, by (6),

$$\frac{\text{M}a^2}{2}\Omega = a(\tau - \tau');$$

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also $v = a\Omega$; so that

$$\left.\begin{aligned} \tau &= mu - ma\Omega, \\ \tau' &= m'u' + m'a\Omega; \end{aligned}\right\} \tag{74}$$

$$\therefore \quad \Omega = \frac{2(mu - m'u')}{a(\mathrm{M} + 2m + 2m')}; \tag{75}$$

whence the initial angular-velocity of the pulley, and the initial tensions of the strings are known.

Again, $$\frac{\mathrm{M}a^2}{2}\frac{d^2\theta}{dt^2} = a(\mathrm{T} - \mathrm{T}');$$

also $$\frac{d^2x}{dt^2} = a\frac{d^2\theta}{dt^2}, \qquad \frac{d^2x'}{dt^2} = -a\frac{d^2\theta}{dt^2}; \tag{76}$$

$$\left.\begin{aligned} \therefore \quad \mathrm{T} &= mg - ma\frac{d^2\theta}{dt^2}, \\ \mathrm{T}' &= m'g + m'a\frac{d^2\theta}{dt^2}; \end{aligned}\right\} \tag{77}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{2(m - m')g}{a(\mathrm{M} + 2m + 2m')}; \tag{78}$$

$$\therefore \quad \frac{d\theta}{dt} - \Omega = \frac{2(m - m')gt}{a(\mathrm{M} + 2m + 2m')}, \tag{79}$$

$$\frac{d\theta^2}{dt^2} - \Omega^2 = \frac{4(m - m')g\theta}{a(\mathrm{M} + 2m + 2m')}; \tag{80}$$

whence by a further integration θ can be determined in terms of t; and thus the space will be known through which m or m' will move in a given time.

If we replace $\frac{d^2\theta}{dt^2}$ in (77) by its value, given in (78), we shall find the tensions of the strings at any time t.

If the weights of the strings are taken into account, the equation of angular motion assumes the following form: Let $\rho =$ the density, $\omega =$ the area of a transverse section of the string; $c =$ the whole length; then, if $\mathrm{M}k^2$ is the moment of inertia of the pulley,

$$(\mathrm{M}k^2 + ma^2 + m'a^2 + \rho\omega ca^2)\frac{d^2\theta}{dt^2} = a(m - m')g + a\rho\omega gx - a\rho\omega gx'$$
$$= a\{m - m' + \rho\omega(x - x')\}g.$$

Ex. 2. To investigate the circumstances of motion of a wheel and axle, the weights of the strings being neglected, and $\mathrm{M}k^2$ being the moment of inertia of the machine relative to its axis.

Let us use the same symbols as in Ex. 2, Art. 46, and those of

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the last example; and let Fig. 18 represent the plan of the wheel and axle when projected on the plane of the paper.

$$\left.\begin{aligned} \tau &= mu - mc\,\Omega, \\ \tau' &= m'u + m'c'\Omega; \end{aligned}\right\} \qquad (81)$$

$$\therefore \quad \Omega = \frac{cmu - c'm'u'}{\mathrm{M}k^2 + mc^2 + m'c'^2}; \qquad (82)$$

whereby the initial angular-velocity of the machine, and the initial tensions of the strings are known.

Again,

$$\left.\begin{aligned} \mathrm{T} &= mg - mc\frac{d^2\theta}{dt^2}, \\ \mathrm{T}' &= m'g + m'c'\frac{d^2\theta}{dt^2}; \end{aligned}\right\} \qquad (83)$$

$$\mathrm{M}k^2\frac{d^2\theta}{dt^2} = c\mathrm{T} - c'\mathrm{T}';$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{(cm - c'm')g}{\mathrm{M}k^2 + mc^2 + m'c'^2}; \qquad (84)$$

whence all the circumstances of motion may be determined.

If P = the pressure on the axis at the time t, it is equal to the weight of the wheel and axle together with the tensions of the strings; therefore

$$\begin{aligned} \mathrm{P} &= \mathrm{M}g + \mathrm{T} + \mathrm{T}' \\ &= (\mathrm{M} + m + m')g - \frac{(mc - m'c')^2}{\mathrm{M}k^2 + mc^2 + m'c'^2}g; \qquad (85) \\ &= \mathrm{M}g + \frac{mm'(c + c')^2 + (m + m')\mathrm{M}k^2}{\mathrm{M}k^2 + mc^2 + m'c'^2}g; \qquad (86) \end{aligned}$$

that is, the pressure on the axis is less than it would be if the machine were at rest; but it can never vanish.

Ex. 3. It is required to determine the motion of a system of wheels and pinions, such as a crane, or the like, the power attached to the first wheel being P, and the weight attached to the last pinion or axle being W.

Whatever is the form of the system, it may always be arranged as in Fig. 28; where we have taken four wheels and pinions: $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3, \mathrm{C}_4$ are the centres of the successive wheels and pinions, C_1 being that of the axle to which the weight W is attached. Let the pressures between the successive wheels and pinions, whether due to the action of teeth or to friction, be $\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3$; let $a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4$, be the radii of the several pinions and wheels in order; and let $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_4$ be their moments of inertia; let T be the tension of the string to which the weight is attached, and

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t the tension of that by which P acts; let M be the mass of the weight W, and let $P = mg$; let us suppose W to descend in the time dt through a space dx, and P to ascend through a space dx'; and let $d\theta_1$, $d\theta_2$, $d\theta_3$, $d\theta_4$ be the angles through which the wheels rotate in that time; then

$$dx = a_1 d\theta_1;\ b_1 d\theta_1 = a_2 d\theta_2;\ b_2 d\theta_2 = a_3 d\theta_3;\ b_3 d\theta_3 = a_4 d\theta_4;\ b_4 d\theta_4 = -dx'. \quad (87)$$

For the translation of W and P, we have

$$\begin{aligned} T &= W - M\frac{d^2x}{dt^2} \\ &= W - a_1 M \frac{d^2\theta_1}{dt^2}; \end{aligned} \quad (88)$$

$$\begin{aligned} t &= P - m\frac{d^2x'}{dt^2} \\ &= P + b_4 m \frac{d^2\theta_4}{dt^2}. \end{aligned} \quad (89)$$

And for the rotation of the pulleys, we have

$$\left.\begin{aligned} H_1 \frac{d^2\theta_1}{dt^2} &= a_1 T - b_1 T_1, \\ H_2 \frac{d^2\theta_2}{dt^2} &= a_2 T_1 - b_2 T_2, \\ H_3 \frac{d^2\theta_3}{dt^2} &= a_3 T_2 - b_3 T_3, \\ H_4 \frac{d^2\theta_4}{dt^2} &= a_4 T_3 - b_4 t; \end{aligned}\right\} \quad (90)$$

whence, by a simple elimination, we have

$$\begin{aligned} \frac{d^2x}{dt^2}\{H_1(a_2a_3a_4)^2 + H_2(a_3a_4b_1)^2 + H_3(a_4b_1b_2)^2 + H_4(b_1b_2b_3)^2 \\ + M(a_1a_2a_3a_4)^2 + m(b_1b_2b_3b_4)^2\} \\ = a_1a_2a_3a_4\{a_1a_2a_3a_4W - b_1b_2b_3b_4P\}; \end{aligned} \quad (91)$$

and, by integration, the space described by W in the time t may be found.

Also, from (87), $$\frac{d^2x'}{dt^2} = -\frac{b_1b_2b_3b_4}{a_1a_2a_3a_4}\frac{d^2x}{dt^2}; \quad (92)$$

and thus the motion of P may be determined.

A similar process may of course be applied, whatever is the number of the wheels and pinions.

If in the preceding example the wheels are all equal, and all the pinions are equal,

$$\frac{d^2x}{dt^2}\{H(a^6 + a^4b^2 + a^2b^4 + b^6) + M a^8 + m b^8\} = a^4\{W a^4 - P b^4\}. \quad (93)$$

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Ex. 4. A heavy flexible and inextensible string of given length a, is wound round a solid cylinder of mass M and radius c, which is capable of rotation about its axis, which is horizontal; a piece of the string of length b hangs down, so that the cylinder begins to rotate; it is required to determine the motion of the string and of the cylinder.

Let the circumstances at the time t be represented in Fig. 29; and let $\text{M}k^2$ be the moment of inertia of the cylinder. In the time t let a chain of length $= x = c\theta$ be unwound from the cylinder; let $\omega =$ the area of a transverse section, $\rho =$ the density of the string; then the weight of the string which hangs vertically at the time $t = \rho\omega g(b+c\theta)$. Let $\text{T} =$ the tension of the string at the point P, $\text{CP} = x$.

$$\therefore \quad \text{T} = \rho\omega(b+c\theta)\left\{g - c\frac{d^2\theta}{dt^2}\right\};$$

and the moment of inertia of the cylinder, and the chain wound round it at the time t,

$$= \text{M}k^2 + \rho\omega(a-b-c\theta)c^2;$$

so that the equation of rotation of the cylinder is

$$\{\text{M}k^2 + \rho\omega(a-b-c\theta)c^2\}\frac{d^2\theta}{dt^2} = c\text{T}$$

$$= \rho\omega c(b+c\theta)\left\{g - c\frac{d^2\theta}{dt^2}\right\};$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{\rho\omega c(b+c\theta)g}{\text{M}k^2+\rho\omega ac^2};$$

$$\therefore \quad \frac{d\theta^2}{dt^2} = \frac{\rho\omega cg}{\text{M}k^2+\rho\omega ac^2}(2b\theta + c\theta^2), \tag{94}$$

since we have assumed $\dfrac{d\theta}{dt} = 0$, when $\theta = 0$; therefore when $c\theta = a-b$, that is, when the whole chain is unwound,

$$\frac{d\theta^2}{dt^2} = \frac{\rho\omega g(a^2-b^2)}{\text{M}k^2+\rho\omega ac^2}.$$

Again, from (94), for the whole time spent in unwinding the string, we have

$$\left\{\frac{\rho\omega cg}{\text{M}k^2+\rho\omega ac^2}\right\}^{\frac{1}{2}} t = \int_0^{\frac{a-b}{c}} \frac{d\theta}{(2b\theta+c\theta^2)^{\frac{1}{2}}}$$

$$= \frac{1}{c^{\frac{1}{2}}}\log\frac{a+(a^2-b^2)^{\frac{1}{2}}}{b}, \tag{95}$$

which gives the time.

By a similar process we may determine the length of string

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which a cylinder, rotating with a given angular velocity, would wind up before it is brought to rest.

Ex. 5. A balance has equal weights in the scales, and oscillates through small angles, the beam and scales moving in a plane which is perpendicular to the axis of vibration; it is required to determine the circumstances of motion.

Let the balance, &c. be represented in Fig. 30, in which the plane of the paper is the plane of motion of the beam and scales, and the axis of vibration is perpendicular to the plane of the paper. Let O be the point where this axis pierces the paper; let G be the centre of gravity of the balance without the weights; let

M = the mass of the balance.

m = the mass of each weight in the scales.

Mk^2 = the moment of inertia of the balance relative to the rotation-axis.

a = the length of each arm = AB = BA′.

OB = b.

OG = h.

Let θ = the angle between OB and the vertical line; which angle, as well as its t-differential, we shall assume to be infinitesimal, so that the squares and higher powers may be neglected. Let $\frac{dx}{dt}$, $\frac{dx'}{dt}$ be the vertical velocities of the weights in the scales at P and P′ respectively; let T and T′ be the tensions of the strings at A and A′ respectively. We shall neglect the oscillations of the scales about the points A and A′.

The perpendicular distances from O on AP and A′P′ respectively are $a\cos\theta - b\sin\theta$, and $a\cos\theta + b\sin\theta$; which quantities, as θ is infinitesimal, are $a - b\theta$ and $a + b\theta$. So that the equation of rotation is

$$Mk^2\frac{d^2\theta}{dt^2} = T(a - b\theta) - T'(a + b\theta) - Mgh\theta. \qquad (96)$$

Now $\quad T = m\left(g - \frac{d^2x}{dt^2}\right), \qquad T' = m\left(g - \frac{d^2x'}{dt^2}\right);$

but $\quad dx = d(a\sin\theta + b\cos\theta), \qquad dx' = d(-a\sin\theta + b\cos\theta);$

$$\frac{dx}{dt} = (a\cos\theta - b\sin\theta)\frac{d\theta}{dt}, \qquad \frac{dx'}{dt} = (-a\cos\theta - b\sin\theta)\frac{d\theta}{dt};$$

$$\frac{d^2x}{dt^2} = (a\cos\theta - b\sin\theta)\frac{d^2\theta}{dt^2}, \qquad \frac{d^2x'}{dt^2} = (-a\cos\theta - b\sin\theta)\frac{d^2\theta}{dt^2};$$

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so that (96) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{2mb + \text{M}h}{2ma^2 + \text{M}k^2} g\theta. \tag{97}$$

Let a be the value of θ when $\frac{d\theta}{dt} = 0$; then

$$\frac{d\theta^2}{dt^2} = \frac{(2mb + \text{M}h)g}{2ma^2 + \text{M}k^2}(a^2 - \theta^2); \tag{98}$$

$$\therefore \quad t = \left\{\frac{2ma^2 + \text{M}k^2}{(2mb + \text{M}h)g}\right\}^{\frac{1}{2}} \cos^{-1}\frac{\theta}{a};$$

and therefore the time of an oscillation

$$= \pi \left\{\frac{2ma^2 + \text{M}k^2}{(2mb + \text{M}h)g}\right\}^{\frac{1}{2}}; \tag{99}$$

and therefore if $l =$ the length of the simple isochronous pendulum

$$l = \frac{2ma^2 + \text{M}k^2}{2mb + \text{M}h}. \tag{100}$$

141.] We must now return to equations (35) and (36), which determine the pressures borne by the fixed ponts of the axis during motion of this kind.

As the z-axis is fixed, the particles of the system have no motion in a direction parallel to that axis, so that for all particles $\frac{d^2z}{dt^2} = 0$; and therefore from the last of (35),

$$\Sigma . \text{P} \cos\gamma = \Sigma . m\text{Z}; \tag{101}$$

and as neither $\text{P}_1 \cos\gamma_1$, nor $\text{P}_2 \cos\gamma_2$, enters into the other equations, this shews that the sum of the components of the pressures along the z-axis is equal to the sum of the similar axial components of the impressed momentum-increments on all the particles; but as the sum only is given each pressure is indeterminate. This case is similar to that of Art. 125, and admits of a similar explanation.

I may observe in passing, that if the axis is capable of sliding in the direction of its length, then the motion of all the particles of the body along that line will be derived from the equation

$$\Sigma . m\left(\text{Z} - \frac{d^2z}{dt^2}\right) = 0;$$

and as $\frac{d^2z}{dt^2}$ will be the same for all the particles, if $\text{M} =$ the mass of the body,

$$\frac{d^2z}{dt^2} = \frac{\Sigma . m\text{Z}}{\text{M}}. \tag{102}$$

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whereby the longitudinal displacement of the axis may be determined.

The components of the pressures at the fixed points, which are perpendicular to the axis, enter into the first two equations of both (35) and (36). In these let us replace $\frac{d^2x}{dt^2}$ and $\frac{d^2y}{dt^2}$ by their equivalents, given in Art. 129, in terms of ω; then these four equations become

$$\left.\begin{aligned} \text{P}_1\cos\alpha_1 + \text{P}_2\cos\alpha_2 &= \omega^2\Sigma.mx + \frac{d\omega}{dt}\Sigma.my + \Sigma.m\text{X}, \\ \text{P}_1\cos\beta_1 + \text{P}_2\cos\beta_2 &= \omega^2\Sigma.my - \frac{d\omega}{dt}\Sigma.mx + \Sigma.m\text{Y}; \end{aligned}\right\} \quad (103)$$

$$\left.\begin{aligned} \text{P}_1 z_1\cos\beta_1 + \text{P}_2 z_2\cos\beta_2 &= \omega^2\Sigma.myz - \frac{d\omega}{dt}\Sigma.mxz - \text{L}, \\ \text{P}_1 z_1\cos\alpha_1 + \text{P}_2 z_2\cos\alpha_2 &= \omega^2\Sigma.mxz + \frac{d\omega}{dt}\Sigma.myz + \text{M}; \end{aligned}\right\} \quad (104)$$

from which equations the components of the pressures perpendicular to the rotation-axis may be determined. It is worth while to consider the forms which the preceding equations take relatively to certain axes of the body.

(1) Let us suppose the rotation-axis to be a principal axis of the body; and let us moreover take the origin at its principal point; then $\Sigma.myz = \Sigma.mzx = 0$; and (104) become

$$\left.\begin{aligned} \text{P}_1 z_1\cos\beta_1 + \text{P}_2 z_2\cos\beta_2 &= -\text{L}, \\ \text{P}_1 z_1\cos\alpha_1 + \text{P}_2 z_2\cos\alpha_2 &= \text{M}; \end{aligned}\right\} \quad (105)$$

from which, with (103), the pressures may be determined.

(2) Let us suppose the rotation-axis to be a central principal axis, and the centre of gravity to be the origin; then

$$\Sigma.myz = \Sigma.mzx = 0; \quad \Sigma.mx = \Sigma.my = 0;$$

then (104) and (103) become respectively (105), and

$$\left.\begin{aligned} \text{P}_1\cos\alpha_1 + \text{P}_2\cos\alpha_2 &= \Sigma.m\text{X}, \\ \text{P}_1\cos\beta_1 + \text{P}_2\cos\beta_2 &= \Sigma.m\text{Y}; \end{aligned}\right\} \quad (106)$$

whence we have

$$\text{P}_1\cos\alpha_1 = \frac{z_2\Sigma.m\text{X} - \text{M}}{z_2 - z_1}, \qquad \text{P}_1\cos\beta_1 = \frac{z_2\Sigma.m\text{Y} + \text{L}}{z_2 - z_1}; \quad (107)$$

$$\text{P}_2\cos\alpha_2 = \frac{-z_1\Sigma.m\text{X} + \text{M}}{z_2 - z_1}, \quad \text{P}_2\cos\beta_2 = \frac{-z_1\Sigma.m\text{Y} - \text{L}}{z_2 - z_1}; \quad (108)$$

If the points of support of the axis are equally distant from the centre of gravity, so that $z_2 = -z_1$, then

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$$\text{P}_1 \cos \alpha_1 = \frac{z_1 \Sigma . m\text{X} + \text{M}}{2z_1}, \qquad \text{P}_1 \cos \beta_1 = \frac{z_1 \Sigma . m\text{Y} - \text{L}}{2z_1}; \quad (109)$$

$$\text{P}_2 \cos \alpha_2 = \frac{z_1 \Sigma . m\text{X} - \text{M}}{2z_1}, \qquad \text{P}_2 \cos \beta_2 = \frac{z_1 \Sigma . m\text{Y} + \text{L}}{2z_1}. \quad (110)$$

(3) If no forces act on the system, so that $\text{X} = \text{Y} = \text{Z} = 0$ for all particles; and $\text{L} = \text{M} = \text{N} = 0$; then the body rotates about the fixed axis with the constant initial velocity n.

And if moreover the rotation-axis passes through the centre of gravity, so that $\Sigma . mx = \Sigma . my = 0$; then, from (103) and from (101), we have

$$\left.\begin{aligned} \text{P}_1 \cos \alpha_1 + \text{P}_2 \cos \alpha_2 &= 0, \\ \text{P}_1 \cos \beta_1 + \text{P}_2 \cos \beta_2 &= 0, \\ \text{P}_1 \cos \gamma_1 + \text{P}_2 \cos \gamma_2 &= 0; \end{aligned}\right\} \quad (111)$$

whence we have

$$\left.\begin{aligned} \text{P}_1 &= -\text{P}_2; \\ \alpha_1 = \alpha_2, \qquad \beta_1 &= \beta_2, \qquad \gamma_1 = \gamma_2; \end{aligned}\right\} \quad (112)$$

so that the pressures at the fixed points are equal and opposite, and act along parallel straight lines; they therefore form a couple, the effect of which would be to alter the rotation-axis of the body, were two points on the axis not fixed.

(4) Moreover, if the rotation-axis is a central principal axis and no forces act on the system, in addition to (111) we have, from (104),

$$\left.\begin{aligned} \text{P}_1 z_1 \cos \beta_1 + \text{P}_2 z_2 \cos \beta_2 &= 0, \\ \text{P}_1 z_1 \cos \alpha_1 + \text{P}_2 z_2 \cos \alpha_2 &= 0; \end{aligned}\right\} \quad (113)$$

and therefore $\text{P}_1 = -\text{P}_2 = 0$; and no pressure exists at the fixed points in the rotation-axis. This result agrees with that of Art. 93, wherein it is proved that the couple of the centrifugal forces vanishes for all points on a central principal axis. Hence it is that such axes are called permanent axes, and are axes of no pressure; they are therefore those axes about which a body will rotate freely, and without fixed points in them, when no forces act.

142.] The following are various applications of the preceding results to certain particular Examples.

Ex. 1. A heavy sphere revolves uniformly about a vertical chord, which is fixed at the two points where it meets the sphere. Determine the pressures at the points.

Let the radius of the sphere $= a$; the mass of the sphere $= \text{M}$; $2c =$ the length of the chord; so that the distance of the chord from the centre $= (a^2 - c^2)^{\frac{1}{2}}$. Let $\omega =$ the angular velo-

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city of the sphere. Let the given chord be the z-axis, and let the plane of (x, y) be that in which the centre of the sphere moves; then

$$\Sigma.m\text{X} = \Sigma.m\text{Y} = 0; \quad \Sigma.m\text{Z} = -\text{M}g; \quad \Sigma.myz = \Sigma.mzx = 0;$$

and at the time t let us suppose the centre of gravity to be in the x-axis; so that $\Sigma.mx = \text{M}(a^2-c^2)^{\frac{1}{2}}$, $\Sigma.my = 0$. Also

$$z_1 = c, \quad z_2 = -c; \quad \text{and} \quad \text{L} = 0, \quad \text{M} = \text{M}g(a^2-c^2)^{\frac{1}{2}};$$

from (103) and (105) we have

$$\text{P}_1 \cos\alpha_1 = \frac{\text{M}}{2}\left(\omega^2 + \frac{g}{c}\right)(a^2-c^2)^{\frac{1}{2}};$$

$$\text{P}_2 \cos\alpha_2 = \frac{\text{M}}{2}\left(\omega^2 - \frac{g}{c}\right)(a^2-c^2)^{\frac{1}{2}};$$

$$\text{P}_1 \cos\beta_1 = -\text{P}_2 \cos\beta_2 = 0;$$

$$\text{P}_1 \cos\gamma_1 + \text{P}_2 \cos\gamma_2 = -\text{M}g.$$

Ex. 2. A heavy cube makes a complete revolution about one of its edges, which is horizontal and is fixed at the two angles of the cube: find the pressures on these points when the centre of gravity of the cube is in its lowest position.

Let a = the length of the edge of the cube; M = the mass; and let the system be referred to a system of coordinate axes which is delineated in Fig. 29; where the edge of the cube, which is the rotation-axis, is the z-axis; A and A′ its extremities are the fixed points, each of which is at a distance $\frac{a}{2}$ from the origin, the origin being taken at the middle point of the edge, and the plane of (x,y) being that in which G, the centre of gravity of the cube, moves; at the time t let the plane passing through the rotation-axis and the centre of gravity be inclined at an angle θ to the plane of (x, z), so that $\text{GO}x = \theta$: now, if h is the distance of the centre of gravity from the rotation-axis, and k is the radius of gyration relative to the rotation-axis,

$$h = \frac{a}{2^{\frac{1}{2}}}, \qquad k^2 = \frac{2a^2}{3};$$

also $\quad \Sigma.m\text{X} = \text{M}g, \quad \Sigma.m\text{Y} = 0, \quad \Sigma.m\text{Z} = 0;$

$$\text{L} = 0, \quad \text{M} = 0, \quad \text{N} = -\text{M}gh\sin\theta;$$

$$\Sigma.mx = \text{M}h\cos\theta, \quad \Sigma.my = \text{M}h\sin\theta, \quad \Sigma.mz = 0;$$

$$\Sigma.myz = 0, \quad \Sigma.mxz = 0;$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = -\frac{\text{M}gh\sin\theta}{\text{M}k^2};$$

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$$\therefore \quad \frac{d\theta^2}{dt^2} = \frac{2gh}{k^2}(\cos\theta + 1);$$

because $\frac{d\theta}{dt} = 0$, when $\theta = \pi$; and therefore at the lowest point, when $\theta = 0$,

$$\frac{d\theta^2}{dt^2} = \omega^2 = \frac{4gh}{k^2}.$$

Also, by (103), we have at the lowest point, when $\theta = 0$,

$$\begin{aligned} \text{P}_1 \cos\alpha_1 + \text{P}_2 \cos\alpha_2 &= \Sigma . m\left(\text{X} - \frac{d^2x}{dt^2}\right) \\ &= \text{M}g + \text{M}h\omega^2; \end{aligned}$$

$$\therefore \quad \left.\begin{aligned} \text{P}_1 \cos\alpha_1 + \text{P}_2 \cos\alpha_2 &= 4\text{M}g, \\ \text{P}_1 \cos\beta_1 + \text{P}_2 \cos\beta_2 &= 0. \end{aligned}\right\}$$

And from (75), $\left.\begin{aligned} \text{P}_1 \cos\beta_1 - \text{P}_2 \cos\beta_2 &= 0, \\ \text{P}_1 \cos\alpha_1 - \text{P}_2 \cos\alpha_2 &= 0; \end{aligned}\right\}$

also $\quad \text{P}_1 \cos\gamma_1 + \text{P}_2 \cos\gamma_2 = 0;$

so that the whole pressure on the axis is equal to four times the weight of the cube.

Ex. 3. A heavy sphere revolves about a horizontal tangent and makes a complete revolution; the rotation-axis is fixed at points equally distant from the point of contact: find the pressure on the axis when the sphere is in the lowest position.

Let us adopt the notation and arrangement of the last example; then

$$h = a, \qquad k^2 = \frac{7a^2}{5};$$

$$\begin{aligned} \frac{d\theta^2}{dt^2} &= \frac{2gh}{k^2}(\cos\theta + 1) \\ &= \frac{10g}{7a}(\cos\theta + 1); \end{aligned}$$

therefore at the lowest position, $\omega^2 = \frac{20g}{7a}$;

$$\therefore \quad \left.\begin{aligned} \text{P}_1 \cos\alpha_1 + \text{P}_2 \cos\alpha_2 &= \frac{27}{7}\text{M}g, \\ \text{P}_1 \cos\beta_1 + \text{P}_2 \cos\beta_2 &= 0; \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{P}_1 \cos\beta_1 - \text{P}_2 \cos\beta_2 &= 0, \\ \text{P}_1 \cos\alpha_1 - \text{P}_2 \cos\alpha_2 &= 0, \\ \text{P}_1 \cos\gamma_1 + \text{P}_2 \cos\gamma_2 &= 0; \end{aligned}\right\}$$

so that the whole pressure on the axis, when the sphere is in its lowest position, is $\frac{27}{7}\text{M}g = \frac{27}{7}\text{W}$, if W = the weight of the sphere.

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Ex. 4. A heavy rod of length a and of mass M is fixed at its two ends in a horizontal position; one support is removed, and the rod turns about the other end: find the pressure at this latter end when the motion begins.

In this case, for the angular motion of the rod at the time t we have

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{2a}\sin\theta;$$

and since $\frac{d\theta}{dt} = 0$, when $\theta = 90°$,

$$\omega^2 = \frac{d\theta^2}{dt^2} = \frac{3g}{a}\cos\theta.$$

Thus, from (103) and (104), when $\theta = 90°$, we have

$$P_1 \cos\alpha_1 + P_2 \cos\alpha_2 = -\frac{3Mg}{4} + Mg$$
$$= \frac{Mg}{4};$$

$$\left.\begin{aligned} P_1 \cos\beta_1 + P_2 \cos\beta_2 &= 0, \\ P_1 \cos\beta_1 - P_2 \cos\beta_2 &= 0, \\ P_1 \cos\alpha_1 - P_2 \cos\alpha_2 &= 0; \end{aligned}\right\}$$

so that while the rod is at rest on its two ends, each support bears one-half of the weight; and if the support is removed, the pressure on the other is immediately diminished to one fourth of the weight of the rod.

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CHAPTER VI.

THE ROTATION OF A RIGID BODY, OR OF AN INVARIABLE SYSTEM, ABOUT A FIXED POINT.

SECTION 1.—*The rotation of a rigid body about a fixed point under the action of instantaneous forces.*

143.] WHEN a rigid body, or any system of particles of invariable form, moves with one point of it fixed, it is evident that it admits only of rotation about an axis passing through that fixed point; generally, the position of this axis will continuously vary, and will describe one cone fixed in the moving body, and another cone fixed in space, which two cones touch each other, and the line of contact of which is the instantaneous axis: it is also evident that any given particle of the system will move on the surface of a sphere whose centre is the fixed point.

We shall suppose the form, matter, and density of every part of the moving system to be given; and therefore the position of the principal axes, and the principal moments of inertia relative to the fixed point, will also be assumed to be known: these latter we shall take to be A, B, C, as in Chap. IV; and we shall assume the order of magnitude to be the same as that of Art. 98; viz.,

$$A < B < C: \qquad (1)$$

we shall also assume the position of the principal axes, as well as the values of the principal moments of inertia, to be given at every point of the system.

Let the fixed point be the origin; and at it let two systems of coordinate axes originate; one of which we assume to be fixed absolutely in space, and the other to be fixed in the body and to move with it: this latter system we will take to be the system of principal axes which originates at the point, because our expressions will be much simplified thereby. The motion of the body will be in the first place referred to this latter system in terms of the angular velocities about the principal axes; and the incidents of its motion in space will be thence inferred by means

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of the connecting equations (103), (104), (105) of Art. 42, or some equivalents of them.

The general investigation will consist of two parts, according as the system is under the action of instantaneous forces or of finite accelerating forces. We shall consider the effects of instantaneous forces in the present section, and in the succeeding section those of finite accelerating forces; and in each case I shall inquire into the resulting angular velocity, the position of the rotation-axis, the pressure on the fixed point, and the other incidents of motion.

144.] For the sake of simplicity, we will suppose the body to be at rest at the time when the instantaneous forces act on it. Let us first refer all the elements to the system of axes fixed in space; and let the axial components of the impressed momenta be $\Sigma.m\mathrm{v}_x$, $\Sigma.m\mathrm{v}_y$, $\Sigma.m\mathrm{v}_z$; and if the force is a single blow which impresses a momentum Q, let Q_x, Q_y, Q_z be its components. Let (x, y, z) be the initial place of m; let P be the pressure at the origin due to the forces, and let λ, μ, ν be the direction-angles of its line of action; then the equations of motion are

$$\left.\begin{aligned}\Sigma.m\left(\mathrm{v}_x-\frac{dx}{dt}\right)-\mathrm{P}\cos\lambda&=0,\\ \Sigma.m\left(\mathrm{v}_y-\frac{dy}{dt}\right)-\mathrm{P}\cos\mu&=0,\\ \Sigma.m\left(\mathrm{v}_z-\frac{dz}{dt}\right)-\mathrm{P}\cos\nu&=0;\end{aligned}\right\}\qquad(2)$$

$$\left.\begin{aligned}\Sigma.m\left\{y\left(\mathrm{v}_z-\frac{dz}{dt}\right)-z\left(\mathrm{v}_y-\frac{dy}{dt}\right)\right\}&=0,\\ \Sigma.m\left\{z\left(\mathrm{v}_x-\frac{dx}{dt}\right)-x\left(\mathrm{v}_z-\frac{dz}{dt}\right)\right\}&=0,\\ \Sigma.m\left\{x\left(\mathrm{v}_y-\frac{dy}{dt}\right)-y\left(\mathrm{v}_x-\frac{dx}{dt}\right)\right\}&=0.\end{aligned}\right\}\qquad(3)$$

Let these equations be transformed into their equivalents in terms of angular velocities, as in Art. 73. Let Ω be the angular velocity which results from the instantaneous forces, and let Ω_x, Ω_y, Ω_z be its axial components; then (2) become

$$\left.\begin{aligned}\Sigma.m\mathrm{v}_x-\Omega_2\Sigma.mz+\Omega_3\Sigma.my-\mathrm{P}\cos\lambda&=0,\\ \Sigma.m\mathrm{v}_y-\Omega_3\Sigma.mx+\Omega_1\Sigma.mz-\mathrm{P}\cos\mu&=0,\\ \Sigma.m\mathrm{v}_z-\Omega_1\Sigma.my+\Omega_2\Sigma.mx-\mathrm{P}\cos\nu&=0;\end{aligned}\right\}\qquad(4)$$

from which, or from (2), the pressure at the fixed point, and the direction-cosines of its line of action, may be determined.

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Let us replace (3) by their equivalents in terms of angular velocities, about the principal axes fixed in the body and moving with it; let G be the moment of the couple of the impressed momenta, or of the blow, if this motion is due to a blow; and let L_1, M_1, N_1, be the axial components of the moment of this couple relative to the three principal axes; and let Ω_1, Ω_2, Ω_3 be the axial components of the instantaneous angular velocity. Then, by the reduction which has already been made in Art. 92, the equations (3) become

$$A\Omega_1 = L_1, \qquad B\Omega_2 = M_1, \qquad C\Omega_3 = N_1: \tag{5}$$

these equations determine the axial components of the angular velocity relative to the principal axes fixed in the body and moving with it. From these the angular velocities relative to the three axes fixed in space may be found, by means of the equations given in (87), Art. 40; and thus the position of the initial rotation-axis is absolutely determined.

The preceding equations admit of dissection, and of deduction from first principles, in a manner similar to that which has been employed in Art. 81 and 82. It is consequently unnecessary to repeat it.

145.] From (5) we have

$$\Omega_1 = \frac{L_1}{A}, \qquad \Omega_2 = \frac{M_1}{B}, \qquad \Omega_3 = \frac{N_1}{C}; \tag{6}$$

$$\therefore \quad \Omega^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2$$

$$= \frac{L_1^2}{A^2} + \frac{M_1^2}{B^2} + \frac{N_1^2}{C^2}; \tag{7}$$

and if α, β, γ are the direction-angles of the instantaneous rotation-axis,

$$\cos\alpha = \frac{\Omega_1}{\Omega}, \qquad \cos\beta = \frac{\Omega_2}{\Omega}, \qquad \cos\gamma = \frac{\Omega_3}{\Omega}, \tag{8}$$

$$= \frac{L_1}{A\Omega}; \qquad = \frac{M_1}{B\Omega}; \qquad = \frac{N_1}{C\Omega}. \tag{9}$$

whence Ω and the direction-angles of its rotation-axis are known.

Hence the equations to the instantaneous and initial rotation-axis are

$$\frac{Ax}{L_1} = \frac{By}{M_1} = \frac{Cz}{N_1}. \tag{10}$$

As L_1, M_1, N_1 are the axial components of the moment of the impressed couple, or, as Poinsot calls it, of "the couple of impulsion," the equations to its axis are

$$\frac{x}{L_1} = \frac{y}{M_1} = \frac{z}{N_1}; \tag{11}$$

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and the equation to its plane is

$$L_1 x + M_1 y + N_1 z = 0. \tag{12}$$

Now these expressions admit of the following interpretation: In reference to the principal axes fixed in the body, the equation to the momental ellipsoid is

$$A x^2 + B y^2 + C z^2 = \mu; \tag{13}$$

relatively to which the equations to the axis conjugate to the plane (12) are, see Art. 14,

$$\frac{A x}{L_1} = \frac{B y}{M_1} = \frac{C z}{N_1}; \tag{14}$$

but these are the equations to the instantaneous initial axis. Hence we have the following theorem:

The instantaneous axis of rotation, due to a given impressed couple, is the axis of the momental ellipsoid which is conjugate to the plane of the couple. Hence, if the momental ellipsoid of the moving system is constructed, and a plane is drawn touching it, and parallel to the plane of the couple of impulsion, the central radius vector of the ellipsoid drawn to the point of contact is the instantaneous axis.

Let the point where the instantaneous axis meets the ellipsoid be called the Instantaneous Pole; then the tangent plane of the ellipsoid at the instantaneous pole is parallel to the plane of the couple of impulsion.

Also, the initial angular velocity varies as the square of the central radius vector of the momental ellipsoid, which coincides with the initial rotation-axis. For if (x, y, z) is the initial instantaneous pole, and R is its distance from the centre of the ellipsoid,

$$\begin{aligned}\frac{x}{\Omega_1} = \frac{y}{\Omega_2} = \frac{z}{\Omega_3} &= \frac{R}{\Omega} \\ &= \left\{\frac{A x^2 + B y^2 + C z^2}{A \Omega_1^2 + B \Omega_2^2 + C \Omega_3^2}\right\}^{\frac{1}{2}} \\ &= \frac{\mu^{\frac{1}{2}}}{\{L_1 \Omega_1 + M_1 \Omega_2 + N_1 \Omega_3\}^{\frac{1}{2}}} \\ &= \frac{\mu^{\frac{1}{2}}}{\{G \Omega \cos\phi\}^{\frac{1}{2}}},\end{aligned}$$

if ϕ is the angle between the initial rotation-axis and the axis of the couple of impulsion; and therefore

$$\Omega = \frac{R^2 G \cos\phi}{\mu}; \tag{15}$$

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that is, the initial angular velocity varies as the square of the central radius vector of the ellipsoid which coincides with the initial rotation-axis, and as the component relative to that axis of the moment of the couple of impulsion. Also, as $\frac{\mu}{R^2}$ is the moment of inertia relative to that axis, the preceding equation is the form which the equation (16), Art. 73, takes in this particular case.

Hence, if a body rotates about an axis passing through a fixed point, the plane of the momental ellipsoid conjugate to that axis is the plane of the couple which instantaneously impressed in an opposite direction will bring the body to rest; and if the axial components of the angular velocity of the body at that instant are ω_1, ω_2, ω_3, and G' is the moment of the couple which reduces the body to rest,

$$G'^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2; \tag{16}$$

and the equation to the plane in which the couple must be impressed is

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = 0; \tag{17}$$

we shall hereafter shew that $G' = G$, and that the position of this plane in space is invariable.

If the plane of the couple of impulsion is a principal plane of the momental ellipsoid, the instantaneous rotation-axis is the axis of the couple; but for no other plane of impulsion will the axis of the couple be also the instantaneous rotation-axis. Hence, if a body is rotating about a principal axis, it may be brought to rest by a couple whose axis is that rotation-axis; but in no other case will the axis of the couple which brings the body to rest coincide with the rotation-axis.

Hence also, if a body is rotating about a given axis, and a blow is given to the body whence a couple of impulsion arises, if the plane of this couple is conjugate to the original axis of rotation, no change of rotation-axis is caused; but if the plane of the couple of the blow is not conjugate to the previous rotation-axis, a change of axis takes place. And therefore if a body is rotating about a principal axis, and a blow is given to the body which produces a couple whose axis is that principal axis, the position of the rotation-axis will be unaltered, and there will only be a change of angular velocity.

I may observe, that if the axes fixed in the body are not principal axes, equations (32), Art. 76, will take the place of

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(5), and the equation of the momental ellipsoid would be (114), Art. 98; it is therefore unnecessary to repeat them here. From these equations however the same geometrical interpretation as that which we have just arrived at may be deduced.

It is to be observed, that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are independent of G, the moment of the momentum of the couple of impulsion; so that if the body is put into motion by a blow, the position of the instantaneous rotation-axis is the same, whatever is the intensity of the blow, provided that its line of action is the same. Not so however the initial angular velocity.

146.] The following examples are in illustration of the preceding.

Ex. 1. A right angled triangular plate, fixed at its centre of gravity, is struck at its right angle by a blow perpendicularly to its plane: it is required to find the position of its initial instantaneous axis.

Let $3a$, $3b$, see Fig. 32, be respectively the side CA, CB of the triangle, $\rho =$ the density, $\tau =$ the thickness of the plate. Let the origin be taken at the centre of gravity; and let the coordinate axes be parallel to the sides. As these axes are not principal, we must recur to (32), Art. 76, for Ω_x, Ω_y, Ω_z. In reference to the origin and axes which we have chosen, the equations to the sides BC, CA, AB respectively are $x = -a$, $y = -b$, $\frac{x}{a} + \frac{y}{b} = 1$; and consequently

$$\text{A} = \Sigma.m(y^2+z^2) = \frac{9\rho\tau ab^3}{4}, \qquad \text{D} = \Sigma.myz = 0,$$

$$\text{B} = \Sigma.m(z^2+x^2) = \frac{9\rho\tau a^3b}{4}, \qquad \text{E} = \Sigma.mzx = 0,$$

$$\text{C} = \Sigma.m(x^2+y^2) = \frac{9\rho\tau ab(a^2+b^2)}{4}, \quad \text{F} = \Sigma.mxy = -\frac{9\rho\tau a^2b^2}{8}.$$

Let Q be the momentum impressed by the blow at C, which is $(-a, -b)$, in a line perpendicular to the plane of the plate, and parallel to the axis of z in a positive direction; so that we have

$$\text{L}_1 = -b\text{Q}, \qquad \text{M}_1 = a\text{Q}, \qquad \text{N}_1 = 0;$$

$$\therefore \quad \Omega_x = -\frac{8\text{Q}}{27\rho\tau ab^2}, \qquad \Omega_y = \frac{8\text{Q}}{27\rho\tau a^2b}, \qquad \Omega_z = 0;$$

and therefore the equation to the instantaneous initial rotation-axis is

$$\frac{x}{a} + \frac{y}{b} = 0;$$

consequently the initial rotation-axis, which of course passes

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through the fixed point, is parallel to the hypothenuse of the triangle.

Ex. 2. Let us consider the general case of a plate of infinitesimal thickness, which has one point fixed, and which is struck at a given point (x_0, y_0) by a blow Q, in a line perpendicular to the plane of the plate.

The plane of the couple of impulsion is evidently that passing through the point of the blow and the fixed point, and perpendicular to the plane of the plate. Thus its equation is

$$\frac{x}{x_0} - \frac{y}{y_0} = 0.$$

Now the axis of the momental ellipsoid (13), which is conjugate to this plane, is

$$\text{A}x_0x + \text{B}y_0y = 0; \tag{18}$$

and this therefore is the initial rotation-axis.

Also, if Q = the momentum impressed by the blow at the point (x_0, y_0) in a line perpendicular to the plane of the plate, and in a direction parallel to the positive direction of the axis of z,

$$\text{L}_1 = y_0\text{Q}, \qquad \text{M}_1 = -x_0\text{Q}, \qquad \text{N}_1 = 0;$$

$$\therefore \quad \Omega_1 = \frac{y_0\text{Q}}{\text{A}}, \qquad \Omega_2 = \frac{-x_0\text{Q}}{\text{B}}, \qquad \Omega_3 = 0;$$

$$\therefore \quad \Omega = \left(\frac{y_0^2}{\text{A}^2} + \frac{x_0^2}{\text{B}^2}\right)^{\frac{1}{2}}\text{Q}.$$

Thus, if the plate is elliptical and fixed at its centre,

$$\text{A} = \frac{\pi\rho\tau b^4}{4}, \qquad \text{B} = \frac{\pi\rho\tau a^4}{4};$$

and consequently the equation of the initial rotation-axis is

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 0;$$

that is, the initial axis is conjugate to the axis passing through the place of the blow.

If the plate is parabolic, and fixed at its vertex, and if a and b are severally the length and the extreme ordinate of the plate,

$$\text{A} = \frac{4\rho\tau ab^3}{15}, \qquad \text{B} = \frac{4\rho\tau a^3b}{7};$$

and the equation to the initial instantaneous axis is

$$\frac{7xx_0}{3a^2} + \frac{5yy_0}{b^2} = 0.$$

Ex. 3. A cube fixed at its centre of gravity is struck by a blow, whose momentum is Q, along an edge: it is required to determine the initial instantaneous axis.

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Let $2a$ = the edge of the cube; let the origin be taken at the centre of gravity of the cube, and let the coordinate axes be parallel to the edges; let the line of the blow Q be parallel to the axis of z; and let its point of application be $(a, a, 0)$; so that

$$L_1 = aQ, \qquad M_1 = -aQ, \qquad N_1 = 0;$$

$$A = B = C = \frac{16\rho\tau a^5}{3};$$

$$\therefore \quad \Omega_1 = \frac{3Q}{16\rho\tau a^4}, \qquad \Omega_2 = -\frac{3Q}{16\rho\tau a^4}, \qquad \Omega_3 = 0;$$

and therefore the equation to the initial rotation-axis is

$$x + y = 0:$$

this result is evident from the theorem, that the instantaneous initial axis is conjugate to the plane of the impulsive couple; for the momental ellipsoid is in this case a sphere, and therefore the instantaneous rotation-axis coincides with the axis of the couple of impulsion.

147.] Let us now consider the equations (2), or their equivalents (4), by means of which the pressure at the fixed point, which is due to the impulsive forces, is to be determined.

Let $\bar{M}$ be the mass of the body; and let its centre of gravity be $(\bar{x}, \bar{y}, \bar{z})$; then

$$\Sigma.mx = \bar{M}\bar{x}, \qquad \Sigma.my = \bar{M}\bar{y}, \qquad \Sigma.mz = \bar{M}\bar{z};$$

and (4) become

$$\left.\begin{aligned} P\cos\lambda &= \Sigma.mv_x + \bar{M}(\Omega_3\bar{y} - \Omega_2\bar{z}), \\ P\cos\mu &= \Sigma.mv_y + \bar{M}(\Omega_1\bar{z} - \Omega_3\bar{x}), \\ P\cos\nu &= \Sigma.mv_z + \bar{M}(\Omega_2\bar{x} - \Omega_1\bar{y}). \end{aligned}\right\} \qquad (20)$$

Now the last terms of these three equations are evidently the axial components of the momentum of the whole moving mass condensed into its centre of gravity; so that the pressure which acts at the fixed point is the sum of the impressed momentum, and the momentum of the whole mass condensed at its centre of gravity, which is due to the initial angular velocity.

To apply these formulæ, let us take Ex. 1 of the preceding Article: in that case

$$\bar{x} = \bar{y} = \bar{z} = 0; \quad \Sigma.mv_x = \Sigma.mv_y = 0, \quad \Sigma.mv_z = Q;$$

$$\therefore \quad P\cos\lambda = P\cos\mu = 0; \qquad P\cos\nu = Q;$$

$$\therefore \quad P = Q;$$

and the line of pressure is perpendicular to the plate.

If there is no pressure at the fixed point, then

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$$\left.\begin{aligned} \Sigma.m\mathrm{v}_x - \bar{\mathrm{M}}(\Omega_2\bar{z} - \Omega_3\bar{y}) &= 0, \\ \Sigma.m\mathrm{v}_y - \bar{\mathrm{M}}(\Omega_3\bar{x} - \Omega_1\bar{z}) &= 0, \\ \Sigma.m\mathrm{v}_z - \bar{\mathrm{M}}(\Omega_1\bar{y} - \Omega_2\bar{x}) &= 0; \end{aligned}\right\} \tag{21}$$

whence we have

$$\bar{x}\Sigma.m\mathrm{v}_x + \bar{y}\Sigma.m\mathrm{v}_y + \bar{z}\Sigma.m\mathrm{v}_z = 0; \tag{22}$$

$$\Omega_1\Sigma.m\mathrm{v}_x + \Omega_2\Sigma.m\mathrm{v}_y + \Omega_3\Sigma.m\mathrm{v}_z = 0; \tag{23}$$

and it appears that the line of action of the resultant of the impressed momenta, or of the blow, if the motion is due to a single blow, is perpendicular to the plane containing the fixed point, the centre of gravity, and the rotation-axis of the initial angular velocity.

If the fixed point is the centre of gravity of the body, then $\bar{x} = \bar{y} = \bar{z} = 0$; and

$$\mathrm{P}\cos\lambda = \Sigma.m\mathrm{v}_x, \quad \mathrm{P}\cos\mu = \Sigma.m\mathrm{v}_y, \quad \mathrm{P}\cos\nu = \Sigma.m\mathrm{v}_z;$$

and the pressure is in intensity, direction, and line of action, equal to the resultant of the impressed momenta.

SECTION 2.—*The rotation of a rigid body about a fixed point under the action of finite accelerating forces.*

148.] LET us, as in the preceding section, refer the motion of the body, or material system, to two sets of coordinate axes, originating at the fixed point: one of which is fixed in space, and the other is fixed in the body, and moves with it: let this latter system be the principal system relative to the fixed point.

Let P be the pressure at the fixed point at the time t, and let λ, μ, ν be the direction-angles of its line of action relatively to the axes fixed in space: relatively to the same axes let (x, y, z) be the place of m, and let X, Y, Z be the axial components of the impressed velocity-increment; then the equations of motion are

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{X} - \frac{d^2x}{dt^2}\right) - \mathrm{P}\cos\lambda &= 0, \\ \Sigma.m\left(\mathrm{Y} - \frac{d^2y}{dt^2}\right) - \mathrm{P}\cos\mu &= 0, \\ \Sigma.m\left(\mathrm{Z} - \frac{d^2z}{dt^2}\right) - \mathrm{P}\cos\nu &= 0; \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{Z} - \frac{d^2z}{dt^2}\right) - z\left(\mathrm{Y} - \frac{d^2y}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{z\left(\mathrm{X} - \frac{d^2x}{dt^2}\right) - x\left(\mathrm{Z} - \frac{d^2z}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{x\left(\mathrm{Y} - \frac{d^2y}{dt^2}\right) - y\left(\mathrm{X} - \frac{d^2x}{dt^2}\right)\right\} &= 0. \end{aligned}\right\} \tag{25}$$

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Let equations (24) be transformed into their equivalents in terms of angular velocities, as in Art. 77; let ω be the angular velocity about the instantaneous axis at the time t, of which let $\omega_x, \omega_y, \omega_z$ be the axial components; then (24) become

$$\left.\begin{aligned}
&\Sigma.m\mathrm{X}-\frac{d\omega_y}{dt}\Sigma.mz+\frac{d\omega_z}{dt}\Sigma.my-\omega_x\Sigma.m\{\omega_x x+\omega_y y+\omega_z z\}+\omega^2\Sigma.mx-\mathrm{P}\cos\lambda=0,\\
&\Sigma.m\mathrm{Y}-\frac{d\omega_z}{dt}\Sigma.mx+\frac{d\omega_x}{dt}\Sigma.mz-\omega_y\Sigma.m\{\omega_x x+\omega_y y+\omega_z z\}+\omega^2\Sigma.my-\mathrm{P}\cos\mu=0,\\
&\Sigma.m\mathrm{Z}-\frac{d\omega_x}{dt}\Sigma.my+\frac{d\omega_y}{dt}\Sigma.mx-\omega_z\Sigma.m\{\omega_x x+\omega_y y+\omega_z z\}+\omega^2\Sigma.mz-\mathrm{P}\cos\nu=0;
\end{aligned}\right\}\quad(26)$$

from which, or from (24), the pressure at the fixed point, and the direction-cosines of its line of action are to be determined.

Let us replace (25) by what they become in terms of angular velocities about the principal axes fixed in the body and moving with it: this reduction has been made in Art. 92; and for the equivalents of (25) we have

$$\left.\begin{aligned}
\mathrm{A}\frac{d\omega_1}{dt}+(\mathrm{C}-\mathrm{B})\omega_2\omega_3&=\mathrm{L},\\
\mathrm{B}\frac{d\omega_2}{dt}+(\mathrm{A}-\mathrm{C})\omega_3\omega_1&=\mathrm{M},\\
\mathrm{C}\frac{d\omega_3}{dt}+(\mathrm{B}-\mathrm{A})\omega_1\omega_2&=\mathrm{N};
\end{aligned}\right\}\quad(27)$$

where L, M, N are the axial components of the moment of the couple of the impressed momentum-increments relative to the principal axes.

As these equations have been already analysed and deduced from first principles in Art. 80—83, it is unnecessary to give further explanation of them. I may observe that the instantaneous angular velocity, and its axial components relative to the moving principal axes, may be (theoretically at least) derived from (27); and thence the axial components relative to the axes fixed in space by means of the equations given in Art. 40; or, as we shall find more convenient, we may determine θ, ϕ, and ψ by means of Art. 42, and thus determine the position of the body in space, as well as the incidents of its motion at the time t.

As the pressure at the fixed point, which is to be determined by means of (26), depends on the instantaneous angular velocity and its axial components, these must be first determined; and consequently I proceed to consider (27) and (25).

The general solution is beyond the present range of Mathematical Analysis; and we can investigate only those more simple cases which can be solved either partially or totally.

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149.] Let us first take the case where $L = M = N = 0$; that is, when the conditions (67), Art. 58, are satisfied; viz., when the forces which act on the body are of the nature explained in the four cases of that Article.

And this problem too is of wider application. It is that of a heavy body rotating about an axis which always passes through its centre of gravity; for as the resultant force of gravity acts through this point, it creates no moment of rotation about an axis passing through it, and therefore so far as gravity affects the motion $L = M = N = 0$. Thus, if a heavy body has motion of both translation and rotation; the principle proved in Section 2, Chapter III, shews that the centre of gravity will most conveniently be taken as the point whose motion of translation is estimated; and the motion of rotation of the body about an axis passing through that point will be estimated by the following process.

In all these cases the equations of motion are

$$\left.\begin{aligned} \Sigma.m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma.m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) &= 0, \\ \Sigma.m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) &= 0; \end{aligned}\right\} \tag{28}$$

and their equivalents relative to principal axes are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + (C-B)\omega_2\omega_3 &= 0, \\ B\frac{d\omega_2}{dt} + (A-C)\omega_3\omega_1 &= 0, \\ C\frac{d\omega_3}{dt} + (B-A)\omega_1\omega_2 &= 0; \end{aligned}\right\} \tag{29}$$

equations (28), it will be remembered, are relative to the axes fixed in space; and (29) refer to the principal axes fixed in the body; but it is desirable to retain both groups of equations as each will give theorems of importance.

Let us consider (29), and let $\Omega_1, \Omega_2, \Omega_3$ be the axial components of the initial angular velocity Ω about the principal axes when $t = 0$; the incidents of such an initial velocity have been considered in the preceding Section.

Let equations (29) be severally multiplied (1) by $\omega_1, \omega_2, \omega_3$; and (2) by $A\omega_1, B\omega_2, C\omega_3$; and let them be added: then we have

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$$\left.\begin{aligned} A\omega_1\frac{d\omega_1}{dt} + B\omega_2\frac{d\omega_2}{dt} + C\omega_3\frac{d\omega_3}{dt} &= 0, \\ A^2\omega_1\frac{d\omega_1}{dt} + B^2\omega_2\frac{d\omega_2}{dt} + C^2\omega_3\frac{d\omega_3}{dt} &= 0; \end{aligned}\right\} \tag{30}$$

then integrating, and taking the limits corresponding to $t = t$, and to $t = 0$, we have

$$A(\omega_1^2 - \Omega_1^2) + B(\omega_2^2 - \Omega_2^2) + C(\omega_3^2 - \Omega_3^2) = 0; \tag{31}$$

$$A^2(\omega_1^2 - \Omega_1^2) + B^2(\omega_2^2 - \Omega_2^2) + C^2(\omega_3^2 - \Omega_3^2) = 0; \tag{32}$$

which are two integrals of (29). Let us interpret these results.

150.] From (31) we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = A\Omega_1^2 + B\Omega_2^2 + C\Omega_3^2. \tag{33}$$

Let α, β, γ be the direction-angles of the instantaneous axis at the time t relative to the principal axis; so that, if ω is the instantaneous angular velocity, and $\Sigma.mr^2$ is the moment of inertia relative to that axis,

$$\begin{aligned} A\omega_1^2 + B\omega_2^2 + C\omega_3^2 &= \omega^2\{A(\cos\alpha)^2 + B(\cos\beta)^2 + C(\cos\gamma)^2\}, \quad (34) \\ &= \omega^2\Sigma.mr^2 \\ &= \Sigma.m\omega^2r^2 \\ &= \Sigma.mv^2 \\ &= \text{the vis viva of the body}; \quad (35) \end{aligned}$$

so that from (33) it appears that the vis viva of the body is constant throughout the motion. Let this constant vis viva be symbolised by k^2; then we have

$$\begin{aligned} A\omega_1^2 + B\omega_2^2 + C\omega_3^2 &= A\Omega_1^2 + B\Omega_2^2 + C\Omega_3^2; \quad (36) \\ &= k^2. \quad (37) \end{aligned}$$

This result is at once evident from the principle of vis viva; see Art. 64; and also may be deduced immediately from (24). For if we apply to the equations the process of virtual velocities, which has been already explained in Art. 63, inasmuch as there is no virtual velocity of the fixed point at which P acts, we have

$$\Sigma.m\left\{\frac{d^2x}{dt^2}dx + \frac{d^2y}{dt^2}dy + \frac{d^2z}{dt^2}dz\right\} = 0; \tag{38}$$

$$\therefore \quad \Sigma.mv^2 = \text{a constant}. \tag{39}$$

151.] Again, from (32) we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = A^2\Omega_1^2 + B^2\Omega_2^2 + C^2\Omega_3^2. \tag{40}$$

Now, suppose L_1, M_1, N_1 to be the axial components of the moment of the couple G_1, which impressed on the body at the time t would reduce it to rest; then, by (6), Art. 144, we have

$$L_1 = A\omega_1, \qquad M_1 = B\omega_2, \qquad N_1 = C\omega_3; \tag{41}$$

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G_1 is called the effective couple at the time t; and suppose L, M, N to be the axial components of the moment of the couple of impulsion G; so that

$$L = A\Omega_1, \qquad M = B\Omega_2, \qquad N = C\Omega_3; \tag{42}$$

then, from (40), we have

$$L_1^2 + M_1^2 + N_1^2 = L^2 + M^2 + N^2;$$

$$\therefore \quad G_1 = G; \tag{43}$$

that is, the moment of the effective couple at the time t is equal to the moment of the couple of impulsion; and therefore, generally, the moment of the effective couple is constant. This is a particular case of the principle of the conservation of the moments of the momenta, which has been proved in Art. 58. Thus (32) becomes

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = G^2. \tag{44}$$

Also the position of the rotation-axis of the effective couple is invariable during the whole motion, being that of the axis of the couple of impulsion. To prove this theorem we must have recourse to the equations (28), which refer to the axes fixed in space. These admit of integration; and we have

$$\left.\begin{aligned} \Sigma.m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) &= h_1, \\ \Sigma.m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) &= h_2, \\ \Sigma.m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) &= h_3; \end{aligned}\right\} \tag{45}$$

Now the left-hand members of (45) are the axial components of the moment of the effective couple at the time t; consequently each of these is constant; and the moment of the effective couple is also constant; which is the theorem just now proved by means of the equations (29); and we have

$$h_1^2 + h_2^2 + h_3^2 = G^2 = G_1^2; \tag{46}$$

and relatively to the axes fixed in space, the direction-cosines of the axis of this constant effective couple are

$$\frac{h_1}{G}, \qquad \frac{h_2}{G}, \qquad \frac{h_3}{G}; \tag{47}$$

this line is called the Invariable Axis.

As the position of the coordinate axes fixed in space is arbitrary, let us assume it to be that which coincides with the principal axes of the body when $t = 0$, and when the angular velocities about the principal axes are Ω_1, Ω_2, Ω_3; then $A\Omega_1$, $B\Omega_2$, $C\Omega_3$

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are the axial components of the moments of the effective couple at that time; and we have

$$h_1 = \mathrm{A}\Omega_1, \qquad h_2 = \mathrm{B}\Omega_2, \qquad h_3 = \mathrm{C}\Omega_3; \tag{48}$$

so that under this supposition the direction-cosines of the fixed axis of the effective couple are

$$\frac{\mathrm{A}\Omega_1}{\mathrm{G}}, \quad \frac{\mathrm{B}\Omega_2}{\mathrm{G}}, \quad \frac{\mathrm{C}\Omega_3}{\mathrm{G}}. \tag{49}$$

Hence also the plane of the effective couple is

$$h_1 x + h_2 y + h_3 z = 0; \tag{50}$$

and this is the invariable plane of Art. 60; and is that in which the sum of the products of each particle and its projected sectorial area is a maximum.

In reference to the axes fixed in the body the direction-cosines of the axis of the effective couple are, see (41),

$$\frac{\mathrm{A}\,\omega_1}{\mathrm{G}}, \quad \frac{\mathrm{B}\,\omega_2}{\mathrm{G}}, \quad \frac{\mathrm{C}\,\omega_3}{\mathrm{G}}; \tag{51}$$

and the equation of the invariable plane is

$$\mathrm{A}\,\omega_1 x + \mathrm{B}\,\omega_2 y + \mathrm{C}\,\omega_3 z = 0. \tag{52}$$

These results give the following equations of condition between the nine direction-cosines of the two systems of reference.

Since $\frac{h_1}{\mathrm{G}}$ is the cosine of the angle between the axis of the couple $\left(\frac{\mathrm{A}\,\omega_1}{\mathrm{G}}, \frac{\mathrm{B}\,\omega_2}{\mathrm{G}}, \frac{\mathrm{C}\,\omega_3}{\mathrm{G}}\right)$ and the fixed x-axis (a_1, b_1, c_1), those direction-cosines referring to the principal axes in the body, we have

$$\left.\begin{aligned} &\mathrm{A}\,a_1\omega_1 + \mathrm{B}\,b_1\omega_2 + \mathrm{C}\,c_1\omega_3 = h_1; \\ \text{similarly,}\quad &\mathrm{A}\,a_2\omega_1 + \mathrm{B}\,b_2\omega_2 + \mathrm{C}\,c_2\omega_3 = h_2, \\ \text{and}\quad &\mathrm{A}\,a_3\omega_1 + \mathrm{B}\,b_3\omega_2 + \mathrm{C}\,c_3\omega_3 = h_3; \end{aligned}\right\} \tag{53}$$

and as six other equations of condition have already been given, the nine direction-cosines may be determined. Thus, if $\omega_1, \omega_2, \omega_3$ are given, we may determine the values of the direction-cosines which fix the principal axes of the body relative to the axes fixed in space, and thus the position of the body will be completely determined.

I may in passing observe, that (53) may be deduced from (45) by transforming the coordinates in (45) to those of the principal axes.

152.] It is however frequently more convenient to determine the position of the body by means of the three angles θ, ϕ, ψ, of Articles 3, and 42.

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Let us take the invariable plane for that of (x, y) fixed in space, the invariable axis being the axis of z; so that $h_1 = h_2 = 0$; $h_3 = \text{G}$: thus, from (53), we have

$$\left.\begin{aligned} \text{A}\,a_1\omega_1 + \text{B}\,b_1\omega_2 + \text{C}\,c_1\omega_3 &= 0, \\ \text{A}\,a_2\omega_1 + \text{B}\,b_2\omega_2 + \text{C}\,c_2\omega_3 &= 0, \\ \text{A}\,a_3\omega_1 + \text{B}\,b_3\omega_2 + \text{C}\,c_3\omega_3 &= \text{G}. \end{aligned}\right\} \tag{54}$$

Let us multiply these equations severally by a_1, a_2, a_3; by b_1, b_2, b_3; by c_1, c_2, c_3; and add in each case; then

$$\text{A}\,\omega_1 = \text{G}\,a_3, \qquad \text{B}\,\omega_2 = \text{G}\,b_3, \qquad \text{C}\,\omega_3 = \text{G}\,c_3; \tag{55}$$

and replacing a_3, b_3, c_3 by their values given in (22), Art. 4, we have

$$\text{A}\,\omega_1 = \text{G}\sin\phi\sin\theta, \quad \text{B}\,\omega_2 = \text{G}\cos\phi\sin\theta, \quad \text{C}\,\omega_3 = \text{G}\cos\theta; \tag{56}$$

$$\therefore \quad \cos\theta = \frac{\text{C}\,\omega_3}{\text{G}}, \qquad \tan\phi = \frac{\text{A}\,\omega_1}{\text{B}\,\omega_2}. \tag{57}$$

ψ does not enter into these equations, because the axis of x in the invariable plane from which ψ is measured is quite arbitrary, and θ and ϕ are sufficient to determine the position of the principal axes at the time t in terms of $\omega_1, \omega_2, \omega_3$, which are functions of t, and are to be determined from the equations (29).

The t-variation of ψ however, that is, the angular velocity of ON, see Fig. 1, about the invariable axis may thus be found. Employing (104), Art. 42, we have

$$\frac{d\psi}{dt} = \frac{\omega_1\sin\phi + \omega_2\cos\phi}{\sin\theta}$$

$$= \frac{\text{A}\,\omega_1^2 + \text{B}\,\omega_2^2}{\text{A}^2\omega_1^2 + \text{B}^2\omega_2^2}\,\text{G}; \tag{58}$$

in which, if ω_1 and ω_2 are replaced by their values in terms of t, and the equation is integrated with limits corresponding to $t = t$, and to $t = 0$, the precessional angle due to that time will be determined. As the right hand member of (58) is necessarily positive, it follows that ψ increases with t; so that the precessional motion in the plane of (x, y) is always from x towards y, that is, is *direct* according to our arrangement of signs and axes.

153.] The instantaneous angular velocity being ω, the direction-cosines of the instantaneous rotation-axis relative to the principal axes are $\frac{\omega_1}{\omega}, \frac{\omega_2}{\omega}, \frac{\omega_3}{\omega}$; and the direction-cosines of the invariable axis are, relatively to the same axes, $\frac{\text{A}\,\omega_1}{\text{G}}, \frac{\text{B}\,\omega_2}{\text{G}}, \frac{\text{C}\,\omega_3}{\text{G}}$:

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thus, if ϕ is the angle between the instantaneous axis and the invariable axis,

$$\cos\phi = \frac{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}{G\omega}$$

$$= \frac{k^2}{G\omega}, \text{ by reason of (37)};$$

$$\therefore \quad \omega\cos\phi = \frac{k^2}{G}, \tag{59}$$

that is, the component of the instantaneous angular velocity about the invariable axis is constant.

154.] The complete solution of the problem requires $\omega_1, \omega_2, \omega_3$ to be expressed in terms of t; the following is the differential equation in terms of $d\omega$ and dt.

From (37) and (44) we have

$$\omega_2^2 = \frac{Ck^2 - G^2 - A(C-A)\omega_1^2}{B(C-B)}; \tag{60}$$

$$\omega_3^2 = \frac{A(B-A)\omega_1^2 - Bk^2 + G^2}{C(C-B)}; \tag{61}$$

and therefore from the first of (29)

$$\frac{d\omega_1}{dt} + \left\{\frac{(B-A)(C-A)}{BC}\left(\omega_1^2 - \frac{Bk^2-G^2}{A(B-A)}\right)\left(\frac{Ck^2-G^2}{A(C-A)} - \omega_1^2\right)\right\}^{\frac{1}{2}} = 0; \tag{62}$$

$$\frac{d\omega_1}{\left\{\left(\omega_1^2 - \frac{Bk^2-G^2}{A(B-A)}\right)\left(\frac{Ck^2-G^2}{A(C-A)} - \omega_1^2\right)\right\}^{\frac{1}{2}}} = \pm\left\{\frac{(B-A)(C-A)}{BC}\right\}^{\frac{1}{2}} dt; \tag{63}$$

the left hand member of this equation is generally an elliptic transcendent, and so does not admit of integration in finite terms. In certain particular cases, as we shall see hereafter, it may be integrated; and in these cases the problem admits of complete solution: similar equations may also be found for $\frac{d\omega_2}{dt}$, and $\frac{d\omega_3}{dt}$; or ω_1 having been determined in terms of t, we may substitute its value in (60) and (61), and we shall thus obtain ω_2 and ω_3 in terms of t. Hereby (57) and (58) will assign the position of the body and the precessional velocity.

155.] The centrifugal forces which are generated by the motion possess some peculiarities which it is desirable to indicate.

Let L'', M'', N'' be the axial components of the moment G'' of the couple of the centrifugal forces; then from (62), Art. 83, or from the equations (29),

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$$L'' = (B-C)\omega_2\omega_3, \quad M'' = (C-A)\omega_3\omega_1, \quad N'' = (A-B)\omega_1\omega_2; \tag{64}$$

$$\therefore \quad L''\omega_1 + M''\omega_2 + N''\omega_3 = 0, \tag{65}$$

$$L''A\omega_1 + M''B\omega_2 + N''C\omega_3 = 0; \tag{66}$$

so that the axis of the couple of the centrifugal forces is at once perpendicular to the instantaneous rotation-axis, and to the invariable axis; and the plane of the couple of the centrifugal forces contains the instantaneous and the invariable axes.

And for the magnitude of G'' we have

$$\begin{aligned} G''^2 &= (B-C)^2\omega_2^2\omega_3^2 + (C-A)^2\omega_3^2\omega_1^2 + (A-B)^2\omega_1^2\omega_2^2 \\ &= \omega^2(A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2) - (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)^2 \\ &= G^2\omega^2 - k^4 \\ &= G^2\omega^2(\sin\phi)^2, \qquad (67) \\ &= k^4(\tan\phi)^2; \qquad (68) \end{aligned}$$

that is, the moment of the couple due to the centrifugal forces varies as the tangent of the angle contained between the instantaneous and the invariable axes; and is represented in magnitude by the area of the parallelogram whose sides are the line-representatives of the instantaneous angular velocity and of the moment of the couple of impulsion.

156.] Now of all these results a perfect geometrical interpretation has been given by M. Poinsot in his admirable "Théorie nouvelle de la Rotation des corps." Liouville's Journal, Vol. XVI; and Paris, Bachelier, 1851. The following Articles are for the most part extracted from that work: he has extended his investigations to a point beyond the scope of our volume; and the student desirous of farther research must have recourse to the original memoir. Indeed the whole of it deserves a most careful study; the reasoning of it is, for the most part, synthetical; and it is a model of such a treatment of a mechanical question; the difficulties with which the problem abounded are cleared away, and the motion of the body from one position to another can be traced in the mind's eye without any interruption.

In the equation to the momental ellipsoid in Art. 98, μ is arbitrary; here however we shall suppose it to be equal to 1, as hereby the formulæ will be simplified. Then the equation to it relatively to the fixed point, and to the principal axes fixed in the body and originating at it, is

$$Ax^2 + By^2 + Cz^2 = 1.$$

Let the point where the instantaneous axis meets this ellipsoid

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be called the Instantaneous Pole; then, if (x, y, z) is this point, and if R is its distance from the fixed point,

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3} = \frac{R}{\omega}, \tag{69}$$

$$= \frac{(Ax^2 + By^2 + Cz^2)^{\frac{1}{2}}}{(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)^{\frac{1}{2}}}$$

$$= \frac{1}{k}; \tag{70}$$

so that the instantaneous pole is $\left(\frac{\omega_1}{k}, \frac{\omega_2}{k}, \frac{\omega_3}{k}\right)$; and as R is the central radius vector of the ellipsoid coinciding with the instantaneous axis,

$$\omega = Rk; \tag{71}$$

so that the instantaneous angular velocity varies as the radius vector of the momental ellipsoid which coincides with the instantaneous axis; and consequently its singular values coincide with the singular values of this radius vector. Of all possible values therefore of this angular velocity the greatest will be when the instantaneous axis is that of least moment, and the least will be when the instantaneous axis is that of greatest moment.

The equation to the tangent plane at the instantaneous pole is

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = k; \tag{72}$$

and this is parallel to the invariable plane (52). The perpendicular distance from the origin on (72)

$$= \frac{k}{(A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2)^{\frac{1}{2}}} = \frac{k}{G}, \tag{73}$$

which is constant; and therefore the plane which touches the momental ellipsoid at the instantaneous pole is absolutely fixed in space, being parallel to the invariable plane, and at a distance $= \frac{k}{G}$ from it. Now this fact supplies us with the following image of the body's motion.

Let us suppose the momental ellipsoid of the body relatively to the fixed point to be described: as the preceding equations involve only the principal moments of inertia of the body, we may imagine the body to be replaced by its momental ellipsoid, and the motion of the latter will be a correct representation of that of the former.

Let the invariable plane be drawn at the fixed point, and let

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a perpendicular be drawn to it equal to $\frac{k}{G}$, (73); through the extremity of this perpendicular let a plane be drawn parallel to the invariable plane; this plane is that whose equation is (72), and the point where the momental ellipsoid touches this plane at any time is the instantaneous pole at that time. If then we imagine the momental ellipsoid, whose centre is fixed at the given point, to roll without sliding on this fixed plane which it always touches, we shall have a true image of the body's motion. Moreover, the point of the contact is the instantaneous pole, and the radius vector to it is the instantaneous rotation-axis; the angular velocity varies as the length of this radius vector, by reason of (71); and the resolved part of the instantaneous angular velocity about the invariable axis is constant. These circumstances are delineated in Fig. 33. O is the fixed point, and the centre of the momental ellipsoid ABC; OG is the invariable axis, and is the axis of the couple of impulsion; along it is taken a distance OG, equal to $\frac{k}{G}$; through G a plane is drawn parallel to the invariable plane, and on it the momental ellipsoid rolls; let I be the point of contact at the time t; I is the instantaneous pole, OI is the instantaneous rotation-axis, and the angular velocity $= k.$OI, and is consequently proportional to the length of OI. GOI is the angle ϕ of Art. 153; and if ω is resolved into two parts, one of which is in the invariable plane, and the other along the invariable axis, the latter $= \omega \cos\phi$ and is always constant.

It will be observed that the elements which determine the geometrical position of these lines and planes depend on the original circumstances of motion; and these latter being given, the positions of the invariable axis and the invariable plane are determined from Section 1 of the present Chapter. The construction mentioned in Art. 145 is only the initial case of that motion of which the present construction gives an interpretation at all times.

As the ellipsoid rolls on the plane GEF, successive points on it come continually into contact with the plane; and these points lie in two curves, one EIF in the plane GEF, and the other IQP on the surface of the ellipsoid; the former is evidently generally a plane curve of an undulating character, as we have indicated in the figure; the latter is a closed curve on the ellipsoid. These

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curves have been named by M. Poinsot the Herpolhode and the Polhode respectively.

157.] The polhode will generally be a curve of double curvature, inasmuch as it is the locus of those points on the ellipsoid, the tangent planes at which are all at the same distance, viz., $\frac{k}{G}$ from the centre; so that the curve is the line of intersection of the two surfaces,

$$A x^2 + B y^2 + C z^2 = 1, \tag{74}$$

$$A^2 x^2 + B^2 y^2 + C^2 z^2 = \frac{G^2}{k^2}, \tag{75}$$

which are the equations to two ellipsoids. For the projections on the principal planes of the momental ellipsoid we have

$$B(B-A) y^2 + C(C-A) z^2 = \frac{G^2}{k^2} - A; \tag{76}$$

$$C(C-B) z^2 - A(B-A) x^2 = \frac{G^2}{k^2} - B; \tag{77}$$

$$A(C-A) x^2 + B(C-B) y^2 = C - \frac{G^2}{k^2}. \tag{78}$$

Now in the general case of an ellipsoid with three unequal axes, the perpendicular from the centre on a tangent plane is greater than the least central radius vector, and is less than the greatest; so that $\frac{G^2}{k^2}$ is greater than A, and is less than C; and consequently (76) and (78) are the equations to ellipses in the planes of (y, z), and of (x, y) respectively; and (77) represents a hyperbola in the plane of (z, x), of which the real axis coincides with the axis of z or x, according as $\frac{G^2}{k^2}$ is greater than or less than B.

Figure 33 represents the case in which $\frac{G^2}{k^2}$ is less than B; and the polhode is a closed and symmetrical curve situated symmetrically relatively to the vertex A of the ellipsoid; and the projection of it on the plane of (y, z) is a complete ellipse: the projection on the plane of (z, x) is an arc of a hyperbola whose real axis coincides with the axis of x; and the projection on the plane of (x, y) is an arc of an ellipse.

If $\frac{G^2}{k^2}$ is greater than B, the polhode is a closed and symmetrical curve situated symmetrically relatively to the vertex C of the

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ellipsoid; and the projection on the plane of (x, y) is a complete ellipse; those on the planes of (y, z) and (z, x) being arcs of an ellipse and a hyperbola respectively.

In each of these cases the curve has evidently four vertices situated in its points of section with the principal planes of the ellipsoid; these planes divide the curve into four equal parts. The radii vectores of the ellipsoid at these vertices are maxima and minima, and the instantaneous angular velocity has maxima and minima values when it coincides with the maxima and minima radii vectores.

If $\frac{G^2}{k^2} = B$, that is, when the distance between the invariable plane, and the plane on which the ellipsoid rolls, is equal to the mean axis of the ellipsoid, (77) represents two planes; and these are equally inclined to the plane of (x, y) at

$$\pm \tan^{-1} \left\{ \frac{A(B-A)}{C(C-B)} \right\}^{\frac{1}{2}}. \tag{79}$$

These planes cut the ellipsoid in two equal ellipses, of which the semi-axis-major and the semi-axis-minor are respectively

$$\left\{ \frac{A+C-B}{AB} \right\}^{\frac{1}{2}}, \quad \text{and} \quad \left(\frac{1}{B}\right)^{\frac{1}{2}}; \tag{80}$$

and the common axis-minor of the two ellipses coincides with the mean axis of the ellipsoid. These ellipses we shall call the Critical Ellipses of the ellipsoid. They are the loci of the instantaneous pole in this particular case; and the polhode becomes a plane curve. The motion of the instantaneous axis in this particular case will be best understood from the peculiarities of the herpolhode.

If $\frac{G^2}{k^2} = A$, and if $\frac{G^2}{k^2} = C$, the polhode is in each case reduced to a point, which is at the extremity of the maximum and the minimum radius vector of the ellipsoid respectively; so that in these cases the momental ellipsoid touches the fixed plane at these points; thus, the instantaneous axis throughout the motion coincides with the invariable axis, and is fixed. The body always rotates about the same axis, which is a principal axis, and is a permanent axis.

It is evident that the motion of the body would be represented

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equally well, if the fixed plane on which the momental ellipsoid rolls were parallel to the invariable plane on the opposite side of the fixed point O to that which is drawn in the figure 33, and at an equal distance from O.

If we eliminate the constant terms from (74) and (75) we have

$$\text{A}\left\{\frac{\text{G}^2}{k^2}-\text{A}\right\}x^2+\text{B}\left\{\frac{\text{G}^2}{k^2}-\text{B}\right\}y^2+\text{C}\left\{\frac{\text{G}^2}{k^2}-\text{C}\right\}z^2=0\,; \quad (81)$$

of which generally the coefficient of the last term is negative, and that of the first term is positive; and that of the middle term may be either positive or negative. Thus (81) will generally represent a cone of the second degree; its principal axes are coincident with those of the momental ellipsoid. This is the cone described in the body by the instantaneous rotation-axis, to which we have alluded in Art. 36.

If $\frac{\text{G}^2}{k^2}$ is less than B, all sections of the cone perpendicular to the axis of x are ellipses, and the x-axis is the internal axis of the cone; if $\frac{\text{G}^2}{k^2}$ is greater than B, all sections perpendicular to the z-axis are ellipses, and the z-axis is the internal axis. If $\frac{\text{G}^2}{k^2}=\text{B}$, (81) represents two planes, into which the cone then degenerates, and which are the planes of the critical ellipses of the ellipsoid.

158.] The herpolhode, which is the curve traced by the instantaneous pole on the fixed plane on which the momental ellipsoid rolls, is evidently generally a plane curve of an undulating character, such as is delineated in the figure, the curves of which are equal and regular, and their maxima and minima radii vectores are respectively equidistant from the pole; so that the curve regularly winds between two concentric circles; it is the locus of the point of contact of the ellipsoid with the fixed plane; which plane touches the ellipsoid along the polhode. Let ρ and r be the radii vectores of the herpolhode and the polhode respectively from G and O as poles; and let $d\sigma$ and ds be length-elements of these curves respectively; then we have

$$\rho^2 = r^2 - \frac{k^2}{\text{G}^2}\,; \quad (82)$$

$$d\sigma = ds\,; \quad (83)$$

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whereby the equation to the herpolhode may (theoretically at least) be found: the equation to the herpolhode will generally be transcendental. In the next Article we shall find, in a particular case, certain properties of it which deserve mention.

The maximum and minimum radii vectores of the herpolhode correspond to those of the polhode; that is, to the vertices of the polhode. These singular values therefore will regularly recur at equal angles on the fixed plane. If the angle between two succeeding maxima radii vectores of the herpolhode is an aliquot part of a right angle, the herpolhode reenters and is a closed curve. If however this angle is incommensurable with a right angle, the curve never reenters; and, although the instantaneous rotation-axis returns periodically to the same place in the body, yet it never returns to a position which it has previously held in space.

The cone, which the instantaneous axis describes in space, and to which we have alluded in Art. 36, is that of which O is the vertex and the herpolhode is the director-curve. This cone therefore will be a kind of circular cone, with its surface worked into ridges and furrows which correspond to the undulations of the herpolhode.

If $\frac{G^2}{k^2} = A$, and if $\frac{G^2}{k^2} = C$, the herpolhode is reduced to a point, viz., G, at which the momental ellipsoid touches the fixed plane; in these cases the rotation-axis of the body has the same position throughout the motion, both in the body and in space; it is one of the principal axes of the body, and is thus a permanent axis.

159.] Let us however consider the form which the herpolhode takes when $\frac{G^2}{k^2} = B$; that is, when the perpendicular from the fixed point to the fixed plane on which the ellipsoid rolls is equal to the mean semi-axis. In this case, as we have just now demonstrated, the polhode is that plane curve in which the momental ellipsoid is intersected by the critical ellipses.

Let r, s, x, y, z refer to the polhode relatively to the principal axes of the momental ellipsoid; then, from (74) and (75) we have

$$\left.\begin{aligned} A\,x^2 + B\,y^2 + C\,z^2 &= 1, \\ A^2x^2 + B^2y^2 + C^2z^2 &= B, \\ x^2 + \quad y^2 + \quad z^2 &= r^2; \end{aligned}\right\} \qquad (84)$$

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$$\left.\begin{aligned} \therefore\quad x^2 &= \frac{C(Br^2-1)}{(B-A)(C-A)}, \\ y^2 &= \frac{C+A-B-ACr^2}{(C-B)(B-A)}, \\ z^2 &= \frac{A(Br^2-1)}{(C-A)(C-B)}. \end{aligned}\right\} \tag{85}$$

Let ρ, σ, ϕ refer to the plane herpolhode, of which G is the pole; then from (82) and (83)

$$\rho^2 = r^2 - \frac{1}{B}; \tag{86}$$

$$d\sigma^2 = ds^2 = d\rho^2 + \rho^2 d\phi^2; \tag{87}$$

$$\left.\begin{aligned} \therefore\quad x^2 &= \frac{BC\rho^2}{(B-A)(C-A)}, \\ y^2 &= \frac{C+A-B-AC\left(\rho^2+\frac{1}{B}\right)}{(C-B)(B-A)}, \\ z^2 &= \frac{AB\rho^2}{(C-A)(C-B)}; \end{aligned}\right\} \tag{88}$$

$$\begin{aligned} \therefore\quad \rho^2 d\phi^2 &= ds^2 - d\rho^2 \\ &= dx^2 + dy^2 + dz^2 - d\rho^2 \\ &= \frac{CA\,d\rho^2}{(C-B)(B-A) - ABC\rho^2}. \end{aligned}$$

Let
$$(C-B)(B-A) = ACn^2; \tag{89}$$

then
$$d\phi = \frac{d\rho}{\rho(n^2 - B\rho^2)^{\frac{1}{2}}};$$

$$\therefore\quad \frac{1}{\rho} = \frac{B^{\frac{1}{2}}}{2n}\left\{\epsilon^{n\phi} + \epsilon^{-n\phi}\right\}; \tag{90}$$

which is the equation to the herpolhode in the fixed plane on which the ellipsoid rolls. It is the same equation as (122), Art. 324, Vol. III. The prime radius is taken so as to coincide with the maximum radius vector of the curve, and also to correspond to the maximum radius of the polhode reckoned from the centre of the ellipsoid; for when $\phi = 0$,

$$\begin{aligned} \rho &= \left\{\frac{(C-B)(B-A)}{ABC}\right\}^{\frac{1}{2}} \\ &= \left\{\frac{C+A-B}{AC} - \frac{1}{B}\right\}^{\frac{1}{2}}, \end{aligned}$$

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and $\frac{C+A-B}{AC}$ is the semi-axis-major of the critical ellipse. The curve lies symmetrically on the two sides of this maximum radius vector; and as ϕ increases ρ decreases, and ultimately $= 0$, when $\phi = \infty$; the curve consequently is a spiral, such as is drawn in Fig. 34; of which G E is the maximum radius, relatively to which the curve is symmetrical. The branches, both in the positive and negative directions, fall into the pole G after an infinite number of convolutions; and this occurs when the instantaneous rotation-axis coincides with the mean axis of the ellipsoid, and consequently with the invariable axis.

In this case, the cone in space, on the surface of which the instantaneous axis lies, consists of a series of sheets arranged spirally in convolutions one within another, and ultimately becoming a straight line, which is the invariable axis.

If then the instantaneous axis is at any time on this cone, it will move along its surface; and although the length of the herpolhode is finite, being equal to the length of the elliptic polhode, yet the time requisite for the passage of the instantaneous pole through the curve is infinite; so that however near the rotation-axis approaches to coincidence with the invariable axis, yet it never coincides with it. If however the instantaneous axis coincides with the invariable axis at any epoch of the motion, it will do so always, unless some other impressed couple acts and produces a change of rotation-axis. Thus the mean axis of the momental ellipsoid is a permanent axis, equally as much as the axes of greatest and least moment.

160.] The following inferences can be drawn as to the change of position of the rotation-axis of a body. Let us suppose a body to be rotating about its mean principal axis; and let us suppose a couple of given moment to be impressed on it; whereby the instantaneous pole is moved along one of the critical ellipses of the body; then the instantaneous pole will continue to move along this ellipse until at last the body is completely overturned; in which case the momental ellipsoid which touched the fixed plane at B will ultimately touch it at B′, and a complete bouleversement of the body will have taken place. This is the greatest derangement which a body rotating about an axis passing through a fixed point can undergo.

If however the instantaneous pole is moved by the action of the new couple from the extremity of the mean axis of the ellip-

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soid to a point which is not in either of the critical ellipses, then the following consequences will occur. Imagine the ellipsoid to be divided into four districts by means of the planes of the critical ellipses; two of these districts will contain the vertices A and A′ of the ellipsoid, and two will contain the vertices C and C′; which are respectively the vertices of the axes of least and of greatest moment. Now if the instantaneous pole is shifted into either of the districts which contain C or C′, the polhode will be a closed curve towards and around C or C′, and the instantaneous pole will perform a complete circuit of this curve, and will periodically return to its first position: the more nearly too to C the instantaneous pole is shifted from B, the less will be the subsequent motion of the instantaneous axis; and should the newly impressed couple be such as just to move the pole from B to C, the instantaneous axis will then be the principal axis of greatest moment, and will become permanent.

But if the instantaneous pole is shifted from B to a position within the districts which include A or A′, the polhode then becomes a closed curve towards and around these points, a complete circuit of which the instantaneous pole describes. And if the instantaneous rotation-axis is shifted, so as to coincide with the principal axis of least moment, its position becomes permanent.

Now these several results depend on the perpendicular distance between the invariable plane and the fixed plane parallel to it, on which the momental ellipsoid rolls. This distance is given in (73), and the least and the greatest values of it are respectively $\frac{1}{C^{\frac{1}{2}}}$ and $\frac{1}{A^{\frac{1}{2}}}$: to adapt these mathematical expressions to the image of the present Article, I shall suppose initially

$$G^2 = B k^2, \tag{91}$$

and the body to be rotating permanently about its axis of mean moment. Let us suppose the position of the rotation-axis to be shifted by the action of a new couple, whereby G becomes G′ and k becomes k'; hereby both the direction-cosines of the invariable axis, and the distance of the plane on which the ellipsoid rolls from the fixed point, will be changed; no change however will be made in A, B, or C, or in the magnitude of the momental ellipsoid, or in the position of the principal axes of the body relatively to the body. If then after the momentum has been impressed by the new couple,

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$$\frac{k'^2}{G'^2} = \frac{k^2}{G^2} = \frac{1}{B}, \tag{92}$$

the instantaneous axis will move along the plane of one of the critical ellipses, and ultimately a complete bouleversement of the body will take place. If however

$$\frac{k'^2}{G'^2} \text{ is greater than } \frac{k^2}{G^2}, \tag{93}$$

the plane on which the ellipsoid rolls is moved to a greater distance from the fixed point; whereby the instantaneous pole is shifted into one or other of the two districts of the surface of the momental ellipsoid in which is A or A′, and the instantaneous pole moves in a closed curve about A or A′. And if $\frac{k'^2}{G'^2}$ takes the greatest value which it admits of, viz., $\frac{1}{A}$, the rotation-axis becomes the principal axis of least moment of the ellipsoid, and is permanent.

Again, if

$$\frac{k'^2}{G'^2} \text{ is less than } \frac{k^2}{G^2}, \tag{94}$$

the instantaneous pole is shifted into one or other of the districts which contain C or C′, and moves on a closed curve about these vertices. And if the impressed couple is such that $\frac{k'^2}{G'^2}$ takes the least possible value, viz., $\frac{1}{C}$, the rotation-axis becomes the principal axis of greatest moment, and is permanent.

The angles which are determined by (79) may be taken as the measures of the stability of rotation of the body relatively to the axes of greatest and least moment. Thus the larger the angles defined by (79) are, the larger is the district surrounding the axis of least moment within which, if the instantaneous pole is, the centre of the polhode will be A or A′; and consequently the smaller will be the districts within which the polhodes will have C or C′ for their centres. And thus we say that the larger the angles are, the more stable is the body relatively to the axis of least moment; and the smaller the angles are, the greater is the stability of rotation relatively to the axis of greatest moment.

Hence, the axes of greatest and of least moments are stable axes; and the axis of mean moment is an unstable axis.

161.] In Art. 154 we have determined the most general differential equation which connects the time and the angular ve-

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locities about the principal axes; and we have observed that the equation is an elliptic transcendent. In the particular cases however in which the forces are such that the distance from the fixed point to the plane on which the ellipsoid rolls is equal to one of the principal semi-axes of the ellipsoid, (63) admits of further simplification.

If the axis of least moment is the rotation-axis, it is a permanent axis, and coincides with the invariable axis, and $G^2 = A k^2$; thus from (37) and (44) we have

$$B(B-A)\omega_2^2 + C(C-A)\omega_3^2 = 0; \tag{95}$$

and as the coefficients of ω_2^2 and ω_3^2 are positive quantities, this equation is satisfied only when $\omega_2 = \omega_3 = 0$; in which case (29) gives

$$\frac{d\omega_1}{dt} = 0;$$

$$\therefore \quad \omega_1 = \text{a constant} = \Omega_1 = \frac{L}{A}. \tag{96}$$

Similarly, if the axis of greatest moment is the rotation-axis, it is also a permanent axis, and coincides with the invariable axis, and $G^2 = C k^2$; and by a process similar to that just explained,

$$\frac{d\omega_3}{dt} = 0;$$

$$\therefore \quad \omega_3 = \text{a constant} = \Omega_3 = \frac{N}{C}. \tag{97}$$

If however the plane on which the ellipsoid rolls is at a distance from the fixed point equal to the semi-axis of mean moment, then $G^2 = B k^2$; and (37) and (44) become

$$\left.\begin{aligned} A\,\omega_1^2 + B\,\omega_2^2 + C\,\omega_3^2 &= k^2, \\ A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 &= G^2 = B\,k^2; \end{aligned}\right\} \tag{98}$$

$$\therefore \quad \frac{A\,\omega_1^2}{C-B} = \frac{k^2 - B\,\omega_2^2}{C-A} = \frac{C\,\omega_3^2}{B-A}; \tag{99}$$

thus the second equation of (29), employing the substitution (89), becomes

$$\frac{B\,d\omega_2}{k^2 - B\,\omega_2^2} = \frac{B^2\,d\omega_2}{G^2 - B^2\omega_2^2} = n\,dt; \tag{100}$$

$$\therefore \quad \log K\,\frac{G + B\,\omega_2}{G - B\,\omega_2} = \frac{2Gn}{B}\,t, \tag{101}$$

where K is an undetermined constant dependent on the circumstances of the body when $t = 0$. Thus, for instance, if when $t = 0$, $\omega_2 = \Omega_2$, and β is the angle between the invariable axis and the axis of mean moment, then $\Omega_2 B = G \cos\beta$; and

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$$\text{K} = \frac{\text{G}-\text{B}\,\Omega_2}{\text{G}+\text{B}\,\Omega_2} = \frac{1-\cos\beta}{1+\cos\beta} = \left(\tan\frac{\beta}{2}\right)^2; \tag{102}$$

and if $\frac{2\text{G}n}{\text{B}} = m$, then from (101)

$$\omega_2 = \frac{\text{G}}{\text{B}}\,\frac{\epsilon^{mt}-\text{K}}{\epsilon^{mt}+\text{K}}; \tag{103}$$

$$\therefore\quad \omega_1^2 = \frac{4\,\text{K}\,\text{G}^2(\text{C}-\text{B})}{\text{A}\,\text{B}(\text{C}-\text{A})}\,\frac{\epsilon^{mt}}{(\epsilon^{mt}+\text{K})^2}, \tag{104}$$

$$\omega_3^2 = \frac{4\,\text{K}\,\text{G}^2(\text{B}-\text{A})}{\text{B}\,\text{C}(\text{C}-\text{A})}\,\frac{\epsilon^{mt}}{(\epsilon^{mt}+\text{K})^2}. \tag{105}$$

If we take the invariable plane to be that of (x, y), and the invariable axis for the axis of z; then, as in Art. 152,

$$\cos\theta = \frac{2}{\epsilon^{mt}+\text{K}}\left\{\frac{\text{K}\,\text{C}(\text{B}-\text{A})\,\epsilon^{mt}}{\text{B}(\text{C}-\text{A})}\right\}^{\frac{1}{2}}, \tag{106}$$

$$\tan\phi = \frac{2}{\epsilon^{mt}-\text{K}}\left\{\frac{\text{K}\,\text{A}(\text{C}-\text{B})\,\epsilon^{mt}}{\text{B}(\text{C}-\text{A})}\right\}^{\frac{1}{2}}, \tag{107}$$

$$\frac{d\psi}{dt} = \text{G}\,\frac{(\text{C}-\text{B})(\epsilon^{mt}+\text{K})^2+(\text{B}-\text{A})(\epsilon^{mt}-\text{K})^2}{\text{A}(\text{C}-\text{B})(\epsilon^{mt}+\text{K})^2+\text{C}(\text{B}-\text{A})(\epsilon^{mt}-\text{K})^2}; \tag{108}$$

whence, by integration, ψ will be given in terms of t, and the problem will be completely solved.

From (103) it appears, that $\omega_2 = \frac{\text{G}}{\text{B}}$ when $t = \infty$; so that $\frac{\text{G}}{\text{B}}$, which in that case is equal to Ω_2, is the angular velocity to which ω_2 approaches and ultimately becomes equal. In this limit $\omega_1 = \omega_3 = 0$, and the body revolves permanently about the mean axis. The polhode is the critical ellipse through which the instantaneous pole travels, and the rotation-axis ultimately coincides with the mean axis of the ellipsoid. The herpolhode is the spiral which has already been described in Art. 159; and the point of contact of the ellipsoid with the fixed plane coincides with the pole of this spiral only when $t = \infty$.

162.] The special cases which have been just discussed depend on G and k, which are functions of the forces and couples by which the body at first receives its motion. But there are other varieties which depend on the constitution of the body itself. Let us briefly examine these; and let us first consider the case wherein two principal moments are equal; these we will take to be the least and the mean; that is, let $\text{B} = \text{A}$; so that the greatest moment is the unequal moment; then, from (29), we have

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$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + (C-A)\omega_2\omega_3 &= 0, \\ A\frac{d\omega_2}{dt} - (C-A)\omega_3\omega_1 &= 0, \\ C\frac{d\omega_3}{dt} \qquad\qquad\qquad &= 0; \end{aligned}\right\} \tag{109}$$

from the last of which,

$$\begin{aligned} \omega_3 &= \text{a constant} \\ &= \Omega_3 = n\,(\text{say}); \end{aligned} \tag{110}$$

and from the first two we have

$$\omega_1\,d\omega_1 + \omega_2\,d\omega_2 = 0; \tag{111}$$

$$\begin{aligned} \therefore\quad \omega_1^2 + \omega_2^2 &= \Omega_1^2 + \Omega_2^2, \\ &= m^2\,(\text{say}); \end{aligned} \tag{112}$$

$$\therefore\quad \omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2 = m^2 + n^2, \tag{113}$$

$$= \Omega^2; \tag{114}$$

so that the angular velocity is constant about the instantaneous axis; and also about the axis of greatest moment. And if γ is the ζ-direction-angle of the instantaneous rotation-axis, relatively to the principal axes fixed in the body,

$$\cos\gamma = \frac{\omega_3}{\omega} = \frac{n}{(m^2+n^2)^{\frac{1}{2}}}, \tag{115}$$

which is constant: so that the instantaneous axis moves in the body in the surface of a right circular cone whose semi-vertical angle is γ, and whose axis is the principal axis of greatest moment.

Again, from (112), and the first of (109), we have

$$\begin{aligned} \frac{d\omega_1}{(m^2-\omega_1^2)^{\frac{1}{2}}} &= \frac{C-A}{A}\,n\,dt \\ &= \mu n\,dt, \end{aligned}$$

if $\dfrac{C-A}{A} = \mu$; (116)

so that

$$\left.\begin{aligned} \omega_1 &= m\sin(\mu n t + a); \\ \therefore\quad \omega_2 &= m\cos(\mu n t + a); \end{aligned}\right\} \tag{117}$$

where a is a constant, such that, when $t = 0$,

$$\tan a = \frac{\Omega_1}{\Omega_2}. \tag{118}$$

Thus, the axial components of the angular velocity relatively to the principal axes of the body are expressed in terms of t; and from these values it follows that the instantaneous axis moves

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over the surface of the cone in a retrograde direction, with a constant angular velocity equal to μn.

If we refer these to the invariable plane as the plane of (x, y), we have, from Art. 152,

$$\cos\theta = \frac{\text{C}n}{\text{G}} = \frac{\text{C}n}{(\text{A}^2m^2 + \text{C}^2n^2)^{\frac{1}{2}}}; \tag{119}$$

$$\tan\phi = \tan(\mu n t + a);$$

$$\therefore \quad \phi = \mu n t + a; \tag{120}$$

so that $\phi = a$, when $t = 0$; and thus a is the angle between the line of nodes and the principal axis of ξ, when $t = 0$.

Also
$$\frac{d\psi}{dt} = \frac{\text{G}}{\text{A}};$$

$$\therefore \quad \psi - \psi_0 = \frac{\text{G}}{\text{A}}\, t. \tag{121}$$

These equations completely determine the motion of the body, and its position at the time t. From (119) it appears that θ is constant; thus, the axis of greatest moment is always inclined at the same angle to the invariable axis; and therefore describes a right circular cone in space, the axis of which is the invariable axis. And since $\frac{d\psi}{dt}$ is constant, the precessional velocity of the line of intersection of the plane of axes of equal moment with the invariable plane is constant, so that the axis of greatest moment describes the circular cone uniformly. And from (120) it appears that the right ascension of the principal axes of equal moment advances uniformly.

In this case the moving body is such that its momental ellipsoid is an oblate spheroid. All the polhodes are circles parallel to the equator of the spheroid; and all the herpolhodes are also concentric circles whose centre is at G; see Fig. 33. The critical ellipses in this case unite and become the circle of the equator. Thus, if the rotation-axis is ever in the equator, it will pass throughout it, and a complete bouleversement will take place again and again; this will take place when $\text{G}^2 = \text{A}k^2$, and the plane on which the spheroid rolls is at a distance from the fixed point equal to the equatorial radius. Although in this case the rotation-axis moves in the body through a complete circle, yet in space it is coincident with the invariable axis, and is fixed. Thus, the axis of greatest moment is the only rotation-axis which has stability and permanence.

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If the two equal principal axes are the greatest and the mean, that is, if $C = B$, the equations of motion are

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} &= 0, \\ C\frac{d\omega_2}{dt} - (C-A)\,\omega_3\,\omega_1 &= 0, \\ C\frac{d\omega_3}{dt} + (C-A)\,\omega_1\,\omega_2 &= 0; \end{aligned}\right\} \qquad (122)$$

the results of which are so exactly similar to those of (109) that it is unnecessary to explain them at length. In this case however the momental ellipsoid is a prolate spheroid, of which the principal axis of least moment is the axis of revolution. The polhodes are circles in planes perpendicular to this axis, and the herpolhodes are also concentric circles of which G is the centre. The critical ellipses unite into the circle BCB′C′. If the rotation-axis is ever in this circle, it will move throughout it, and a complete bouleversement will take place; this is the case when $G^2 = Ck^2$: and although the rotation-axis moves in the body through this circle, yet it is fixed in space, being coincident with the invariable axis. The axis OA, which is that of least moment, is the only axis in this system which has stability and permanence.

163.] If $A = B = C$, that is, if the three principal moments are equal,

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0;$$

$$\therefore\ \frac{\omega_1}{\Omega_1} = \frac{\omega_2}{\Omega_2} = \frac{\omega_3}{\Omega_3} = \frac{\omega}{\Omega} = 1,$$

and the angular velocity is constant; and the rotation-axis is fixed both relatively to the moving body and to space.

In this case the momental ellipsoid is a sphere, and the distance of the fixed plane on which it rolls is at a distance from its centre equal to its radius. The polhode and herpolhode are only points; and whatever new couple is impressed, and whatever consequently is the displacement of the rotation-axis, the axis in its new place is stable and permanent.

And this case is indeed the only one in which generally it is possible for the angular velocity to be constant throughout the motion; for, if ω is constant, the solution depends on the three equations

$$\left.\begin{aligned} A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 &= G^2, \\ A\,\omega_1^2 + B\,\omega_2^2 + C\,\omega_3^2 &= k^2, \\ \omega_1^2 + \omega_2^2 + \omega_3^2 &= \omega^2; \end{aligned}\right\}$$

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whence $\omega_1, \omega_2, \omega_3$ are evidently constant; and consequently

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0;$$

and therefore from (29)

$$(\mathrm{C}-\mathrm{B})\,\omega_2\omega_3 = (\mathrm{A}-\mathrm{C})\,\omega_3\omega_1 = (\mathrm{B}-\mathrm{A})\,\omega_1\omega_2 = 0;$$

which are satisfied (1) when $\mathrm{A} = \mathrm{B} = \mathrm{C}$, that is, when the principal moments are equal, and every axis is permanent and stable; (2) when two of the three quantities $\omega_1, \omega_2, \omega_3$ are equal to zero; that is, when the body rotates about a principal axis; but in this last case on any shifting of the axis a change of angular velocity takes place.

164.] The differential equations (29) are integrable also, at least approximately, when the angle between the rotation-axis of the system and one of the principal axes, say that of the greatest moment, is always small; so that the angles between the rotation-axis and the other principal axes are almost right angles, and thereby their cosines are very small. In this case ω_1 and ω_2 are so small that their squares and their products may be neglected in linear equations which involve their first powers, and ω_3 may be replaced by the resultant instantaneous angular velocity; then, from the third of (29), we have

$$\frac{d\omega_3}{dt} = 0;$$

$$\therefore \quad \omega_3 = \text{a constant}$$
$$= n\,(\text{say});$$

and the first and second of (29) become

$$\left.\begin{aligned} \mathrm{A}\frac{d\omega_1}{dt} + (\mathrm{C}-\mathrm{B})\,n\omega_2 &= 0, \\ \mathrm{B}\frac{d\omega_2}{dt} + (\mathrm{A}-\mathrm{C})\,n\omega_1 &= 0; \end{aligned}\right\}$$

whence we have

$$\frac{d^2\omega_1}{dt^2} + \frac{(\mathrm{C}-\mathrm{A})(\mathrm{C}-\mathrm{B})}{\mathrm{AB}}\,n^2\omega_1 = 0; \tag{123}$$

$$\left.\begin{aligned} \omega_1 &= \Omega_1 \cos\left\{\frac{(\mathrm{C}-\mathrm{A})(\mathrm{C}-\mathrm{B})}{\mathrm{AB}}\right\}^{\frac{1}{2}} nt, \\ \omega_2 &= \Omega_1 \left\{\frac{\mathrm{A}(\mathrm{C}-\mathrm{A})}{\mathrm{B}(\mathrm{C}-\mathrm{B})}\right\}^{\frac{1}{2}} \sin\left\{\frac{(\mathrm{C}-\mathrm{A})(\mathrm{C}-\mathrm{B})}{\mathrm{AB}}\right\}^{\frac{1}{2}} nt; \end{aligned}\right\} \tag{124}$$

the limits of integration being such that $\omega_1 = \Omega_1$, when $t = 0$,

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at which time $\omega_2 = 0$; so that the initial rotation-axis is in the plane of (ξ, ζ); and if Ω is the initial angular velocity

$$\Omega^2 = \Omega_1^2 + n^2.$$

Since Ω_1 is very small, (124) shew that ω_1 and ω_2 are always small, so long as $C-A$ and $C-B$ have the same sign; that is, so long as the principal axis, near to which the rotation-axis is, is the axis of either the greatest or the least moment. If however $C-A$ and $C-B$ are of different signs, the integral of (123) involves exponential expressions; and ω_1 and ω_2 will increase indefinitely with the time. Whence we infer that if a body, free from the action of forces producing rotation, rotates at any time about an axis nearly coinciding with the principal axis of greatest or least moment, the rotation-axis will always nearly coincide with that principal axis. But if the principal axis, with which the rotation-axis nearly coincides is the principal axis of mean moment, the rotation-axis will deviate more and more from that axis. Hereby we have another conception of the stability and instability of principal axes; those of greatest and least moment are stable; that of mean moment is unstable.

165.] We now return to the general case; although the invariable axis is fixed in space, yet in the body it describes a cone of the second degree, the equation to which is thus found.

In reference to the principal axes of the body the equations of the invariable axis are

$$\frac{x}{A\omega_1} = \frac{y}{B\omega_2} = \frac{z}{C\omega_3}; \tag{125}$$

but from (37) and (44) we have

$$A(Ak^2 - G^2)\omega_1^2 + B(Bk^2 - G^2)\omega_2^2 + C(Ck^2 - G^2)\omega_3^2 = 0;$$

$$\therefore \quad \left(k^2 - \frac{G^2}{A}\right)x^2 + \left(k^2 - \frac{G^2}{B}\right)y^2 + \left(k^2 - \frac{G^2}{C}\right)z^2 = 0; \tag{126}$$

which is the equation to a cone of the second degree coaxal with the momental ellipsoid; and is a circular cone if two principal moments are equal; and becomes two planes passing through the axis of y, if $G^2 = Bk^2$, of which the equations are

$$\{C(B-A)\}^{\frac{1}{2}}x \pm \{A(C-B)\}^{\frac{1}{2}}z = 0. \tag{127}$$

Hence also we have the following image of the body's motion. Let the instantaneous angular velocity ω at the time t be resolved into two components, the axis of one of which is the invariable

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axis, and the axis of the other is the line perpendicular to it in the invariable plane. Now if ϕ is the angle between the instantaneous and the invariable axes, $\omega \cos \phi$ is the former component, and is, by reason of (59), Art. 153, constant: the latter component is $\omega \sin \phi$; about the axis of which the body rotates in the time dt through an angle $\omega \sin \phi \, dt$; and thus the invariable axis moves over a surface element of the cone (126); and as this discomposition may be continued, so will the invariable axis describe a conical surface. Also the axis of the latter component will continuously change its position in the body, although it is always in the invariable plane; and as it is always perpendicular to a generating line of the cone (126), so will it in its successive positions generate a cone which is the reciprocal of (126); and of which consequently the equation is

$$\frac{\mathrm{A}x^2}{\mathrm{A}k^2 - \mathrm{G}^2} + \frac{\mathrm{B}y^2}{\mathrm{B}k^2 - \mathrm{G}^2} + \frac{\mathrm{C}z^2}{\mathrm{C}k^2 - \mathrm{G}^2} = 0; \tag{128}$$

and this is evidently a cone coaxal with (126), having the same internal axis, and whose major and minor external axes are respectively the minor and major external axes of (126). If $\mathrm{B}k^2 = \mathrm{G}^2$, $y = 0$; in which case (128) represents a straight line.

Now in the motion of the body the surface of the cone (126) always contains the invariable axis; and the surface of the second is always in contact with the invariable plane. Thus the motion of the body may be represented by the rolling of the cone (128) on the invariable plane.

166.] Lastly, let us consider, as shortly as possible, certain properties of the principal axes of the body; and for this purpose let us refer them to the invariable plane and to the invariable axis: let α, β, γ be the direction-angles of the invariable axis, and let (x, y, z) be the instantaneous pole at the time t, relative to the principal axes of the body; then, from (51) and (70), we have

$$\left.\begin{aligned} \cos \alpha &= \frac{\mathrm{A}\omega_1}{\mathrm{G}} = \frac{\mathrm{A}kx}{\mathrm{G}}, \\ \cos \beta &= \frac{\mathrm{B}\omega_2}{\mathrm{G}} = \frac{\mathrm{B}ky}{\mathrm{G}}, \\ \cos \gamma &= \frac{\mathrm{C}\omega_3}{\mathrm{G}} = \frac{\mathrm{C}kz}{\mathrm{G}}. \end{aligned}\right\} \tag{129}$$

From which equations we have

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$$\frac{(\cos a)^2}{A} + \frac{(\cos \beta)^2}{B} + \frac{(\cos \gamma)^2}{C} = \frac{A\omega_1^2 + B\omega_2^2 + C\omega_3^2}{G^2}$$
$$= \frac{k^2}{G^2}; \tag{130}$$

that is, the sum of the squares of the projections of the three principal axes of the momental ellipsoid on the invariable axis is constant, and is equal to the square of the perpendicular from the centre of the ellipsoid to the plane on which the ellipsoid rolls.

Again, let the three principal axes be produced until they meet the plane on which the ellipsoid rolls, as we have imagined in Art. 156; and let ρ_1, ρ_2, ρ_3 be the lengths of these three lines thus produced; then, as the perpendicular from the centre on this plane is equal to $\frac{k}{G}$, so

$$\rho_1 \cos a = \rho_2 \cos \beta = \rho_3 \cos \gamma = \frac{k}{G}; \tag{131}$$

$$\therefore \quad \frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} + \frac{1}{\rho_3^2} = \frac{G^2}{k^2}; \tag{132}$$

and from (129)

$$\frac{1}{\rho_1} = Ax, \qquad \frac{1}{\rho_2} = By, \qquad \frac{1}{\rho_3} = Cz; \tag{133}$$

$$\therefore \quad \frac{1}{A\rho_1^2} + \frac{1}{B\rho_2^2} + \frac{1}{C\rho_3^2} = Ax^2 + By^2 + Cz^2$$
$$= 1. \tag{134}$$

Again, let the three principal axes of the momental ellipsoid be projected on the invariable plane; and as the body rotates, let $\omega_a, \omega_b, \omega_c$ be the angular velocities of the projections of these axes respectively in this plane. Then it is evident that the area described on this plane by the projection of OA

$$= \frac{(\sin a)^2}{2A} \omega_a dt; \tag{135}$$

but this area is evidently equal to the projection on the same plane of the sectorial area described in space by OA itself. OA however has no motion by reason of ω_1; but, by reason of ω_2 and ω_3, OA describes the sectorial areas

$$\frac{\omega_2 dt}{2A}, \text{ and } \frac{\omega_3 dt}{2A},$$

which are perpendicular respectively to OB and to OC; and of which therefore the projections on the invariable plane are respectively

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$$\frac{\omega_2\,dt}{2\text{A}}\cos\beta, \quad \text{and} \quad \frac{\omega_3\,dt}{2\text{A}}\cos\gamma. \tag{136}$$

Thus, equating these to (135), we have

$$(\sin a)^2\omega_a = \omega_2\cos\beta + \omega_3\cos\gamma\,;$$

$$\left.\begin{aligned} \therefore\quad \omega_a &= \frac{k^2(1-\text{A}x^2)}{\text{G}(\sin a)^2}, \\ \omega_b &= \frac{k^2(1-\text{B}y^2)}{\text{G}(\sin\beta)^2}, \\ \omega_c &= \frac{k^2(1-\text{C}z^2)}{\text{G}(\sin\gamma)^2}; \end{aligned}\right\} \tag{137}$$

$$\begin{aligned} \therefore\quad \omega_a(\sin a)^2 + \omega_b(\sin\beta)^2 + \omega_c(\sin\gamma)^2 &= \frac{k^2}{\text{G}}(3-\text{A}x^2-\text{B}y^2-\text{C}z^2) \\ &= \frac{2k^2}{\text{G}}; \end{aligned} \tag{138}$$

that is, is equal to twice the component of the instantaneous angular velocity about the invariable axis; see (59), Art. 153.

Also, from (137) we have

$$\begin{aligned} \text{A}\omega_a(\sin a)^2 + \text{B}\omega_b(\sin\beta)^2 + \text{C}\omega_c(\sin\gamma)^2 \\ &= \frac{k^2}{\text{G}}\{\text{A}+\text{B}+\text{C}-\text{A}^2x^2-\text{B}^2y^2-\text{C}^2z^2\} \\ &= \frac{k^2}{\text{G}}\left\{\text{A}+\text{B}+\text{C}-\frac{\text{G}^2}{k^2}\right\}; \end{aligned} \tag{139}$$

but A, B, C are proportional respectively to the squares of the radii of gyration which lie along the corresponding axes: and therefore it appears that the sum of the areas described on the invariable plane, by the projections on that plane of the principal radii of gyration, is proportional to the time.

Other theorems of the same kind, relative to principal axes and to the projections of their extremities on the invariable plane, have been investigated by Poinsot in the Memoir to which reference has so frequently been made, but it is beyond my purpose to enter into the subject further in this place; and the student desirous of other information must have recourse to the original Memoir.

167.] The problem which has been solved in the preceding Articles of this Section is that of a rigid body, or a system of material particles of invariable form, rotating about an axis passing through a fixed point, when $\text{L}=\text{M}=\text{N}=0$; and is also that of a heavy body about an axis passing through its centre of gravity, whether this latter point is fixed or not; because through

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it the resultant of the acting forces, viz., the weight of the body, passes, and thus produces no moment to effect a change of either the position of the rotation-axis, or the angular velocity.

Let us now investigate, as far as is possible, the rotation of a heavy body, of which two principal moments are equal, and a point is fixed in the axis of unequal moment, so that the body rotates about an axis passing through this point. Let us moreover assume the centre of gravity to be in the axis of unequal moment, and not to be at the fixed point. It is evident that of many systems of particles satisfying these conditions, one is a heavy homogeneous solid of revolution, which is capable of rotation about a point fixed in its axis of figure. Such is a top, the point of whose peg keeps the same place during the rotation, and the friction at which is neglected.

Let us take the coordinates, and other symbols of Arts. 2, 4, and 42. Thus, let the fixed point be the origin; and let the axis of unequal moment be the axis of ζ, the axes of ξ and η being in a plane perpendicular to that axis, so that these coordinate axes are the principal axes of the body at the given point. In reference to these axes let the centre of gravity be $(0, 0, h)$; let m = the mass of the body; and let $\text{A} = \text{B}$; and let C be the unequal principal moment. Let the fixed system of axes at the origin be so arranged that the axis of z is vertical, and in a direction contrary to that of gravity, and the plane of (x, y) is horizontal. At the time t let θ, ϕ, ψ be those angles of connection between the two systems of rectangular coordinates which are given in Arts. 3 and 4: then, since the weight acting on any particle dm in a line parallel to the axis of z is equal to $-g\,dm$, for the components of this force along the principal axes of the body we have

$$\left.\begin{aligned} \text{X} &= -g\,dm \sin\theta \sin\phi, \\ \text{Y} &= -g\,dm \sin\theta \cos\phi, \\ \text{Z} &= -g\,dm \cos\theta. \end{aligned}\right\} \qquad (140)$$

Since the axis of ζ passes through the centre of gravity,

$$\Sigma.\xi\,dm = \Sigma.\eta\,dm = 0; \qquad \Sigma.\zeta\,dm = mh; \qquad (141)$$

and consequently in reference to the principal axes fixed in the body

$$\left.\begin{aligned} \text{L} &= \Sigma.dm\,(\text{Z}\eta - \text{Y}\zeta) = mhg \sin\theta \cos\phi, \\ \text{M} &= \Sigma.dm\,(\text{X}\zeta - \text{Z}\xi) = -\,mhg \sin\theta \sin\phi, \\ \text{N} &= \Sigma.dm\,(\text{Y}\xi - \text{X}\eta) = 0; \end{aligned}\right\} \qquad (142)$$

and the three equations of motion become

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$$\text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})\omega_2\omega_3 = \quad mhg\sin\theta\cos\phi, \tag{143}$$

$$\text{A}\frac{d\omega_2}{dt} + (\text{A}-\text{C})\omega_3\omega_1 = -mhg\sin\theta\sin\phi, \tag{144}$$

$$\text{C}\frac{d\omega_3}{dt} = 0; \tag{145}$$

from the third we have

$$\begin{aligned}\omega_3 &= \text{a constant}\\ &= n\,(\text{say});\end{aligned} \tag{146}$$

so that the angular velocity about the axis of unequal moment is constant; and therefore substituting in (143) and (144) we have

$$\left.\begin{aligned}\text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})n\omega_2 &= \quad mhg\sin\theta\cos\phi,\\ \text{A}\frac{d\omega_2}{dt} + (\text{A}-\text{C})n\omega_1 &= -mhg\sin\theta\sin\phi;\end{aligned}\right\} \tag{147}$$

$$\begin{aligned}\therefore\ \text{A}\left(\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt}\right) &= mhg\sin\theta\,(\omega_1\cos\phi - \omega_2\sin\phi)\\ &= mhg\sin\theta\frac{d\theta}{dt},\end{aligned} \tag{148}$$

by reason of (103), Art. 42. Consequently, if k^2 is the initial vis viva of the body, and ϕ_0 is the initial value of ϕ, the integral of (148) gives

$$\text{A}(\omega_1^2 + \omega_2^2) + \text{C}n^2 = k^2 + 2mhg(\cos\theta_0 - \cos\theta); \tag{149}$$

which is indeed the equation of vis viva, and might have been immediately inferred from the principle given in Article 63. Whenever $\theta = \theta_0$, that is, whenever the angle between the vertical line and the axis of unequal moment has its initial value, the vis viva of the system is equal to the initial vis viva; and the vis viva is increased or diminished according as the position of the centre of gravity is lower or higher than its initial position. This is in accordance with the conservation of vis viva.

From (100) and (101), Art. 42,

$$\omega_1^2 + \omega_2^2 = \left(\frac{d\theta}{dt}\right)^2 + \left(\sin\theta\frac{d\psi}{dt}\right)^2; \tag{150}$$

so that (149) becomes

$$\text{A}\left(\frac{d\theta}{dt}\right)^2 + \text{A}(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 + \text{C}n^2 = k^2 + 2mhg(\cos\theta_0 - \cos\theta), \tag{151}$$

which is the equation of vis viva connecting θ, ψ, and t.

Again, from (147) we have

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$$\text{A}\left(\sin\phi\frac{d\omega_1}{dt}+\cos\phi\frac{d\omega_3}{dt}\right)+(\text{C}-\text{A})n(\omega_2\sin\phi-\omega_1\cos\phi)=0; \quad (152)$$

but from (104), Art. 42,

$$\omega_1\sin\phi+\omega_2\cos\phi = \sin\theta\frac{d\psi}{dt};$$

therefore

$$\sin\phi\frac{d\omega_1}{dt}+\cos\phi\frac{d\omega_2}{dt}=\frac{d}{dt}\left(\sin\theta\frac{d\psi}{dt}\right)-(\omega_1\cos\phi-\omega_2\sin\phi)\frac{d\phi}{dt}$$

$$=\frac{d}{dt}\left(\sin\theta\frac{d\psi}{dt}\right)-\frac{d\theta}{dt}\left(n-\cos\theta\frac{d\psi}{dt}\right),$$

by reason of (102), Art. 42; and substituting these in (152), we have

$$\text{A}\left\{2\cos\theta\frac{d\theta}{dt}\frac{d\psi}{dt}+\sin\theta\frac{d^2\psi}{dt^2}\right\}-\text{C}n\frac{d\theta}{dt}=0; \quad (153)$$

and multiplying by $\sin\theta$, and integrating, we have

$$\text{A}(\sin\theta)^2\frac{d\psi}{dt}+\text{C}n\cos\theta=h'; \quad (154)$$

where h' is a constant introduced in integration, and is indeed the sum of twice the product of each particle and the projection on the horizontal plane of (x, y) of the sectorial area described by its radius vector in an unit of time. For although the principle of conservation of moments or of areas, see Art. 58, is not true of this motion relatively to any plane, yet it is true for the plane of (x, y), because the axis of z is parallel to the line of gravity, and the weight consequently does not produce any moment relatively to that axis. Thus, of equations (70) in Art. 58, the third is true, and (154) is the form of its integral in this case. For $\text{A}\omega_1 dt$, $\text{A}\omega_2 dt$, $\text{C}n\,dt$ are respectively the sum of twice the products of each particle and the sectorial area described by its radius vector in dt in the planes of (η, ζ), (ζ, ξ), (ξ, η) respectively; and the areas described on the plane of (x, y) are the sums of the projections of these on that plane; and therefore

$$=\{\text{A}\omega_1\sin\theta\sin\phi+\text{A}\omega_2\sin\theta\cos\phi+\text{C}n\cos\theta\}\,dt$$

$$=\left\{\text{A}(\sin\theta)^2\frac{d\psi}{dt}+\text{C}n\cos\theta\right\}dt;$$

and this is constant by the principle of conservation of areas; so that if h' is the sum of twice the product of each particle, and the protection on the horizontal plane of the sectorial area, described by its radius vector in an unit of time, (154) is the particular form assumed by the principle in this problem.

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If we eliminate $\frac{d\psi}{dt}$ from (151) and (154), an equation will result of the form

$$dt = f(\theta)\,d\theta\,; \tag{155}$$

whence θ may be determined in terms of t; and if we substitute this value of θ in (154), we shall have another differential expression of the form

$$dt = f(\psi)\,d\psi\,; \tag{156}$$

whereby ψ may be expressed in terms of t. And lastly, by means of (102), Art. 42, we have

$$\frac{d\phi}{dt} = n - \cos\theta\,\frac{d\psi}{dt}\,; \tag{157}$$

which will give us ϕ in terms of t. The problem will thus be completely solved. All these differential expressions, although reduced to simple quadratures, involve elliptic transcendents, and are functions whose properties we have not discussed in this treatise. I propose therefore to consider only one or two simple cases of the problem.

I would observe however that the three equations (151), (154), and (157), on which the solution depends, are the same as those found by Lagrange after his own process, in the Mécanique Analytique, Part II, Section IX, 35. It does not appear that the problem had been solved before that time. Poisson afterwards solved it, and gave the solution in Cahier XVI, Journal de l'Ecole Polytechnique, published in 1815. He does not refer to Lagrange. Poisson has also given the solution in the second volume of the Traité de Mécanique.

168.] Let us suppose the axis of the angular velocity which is impressed on the body initially to be the axis of unequal moment; so that initially $\omega_1 = \omega_2 = 0$; and also $\frac{d\theta}{dt} = \frac{d\psi}{dt} = 0$; and therefore $k^2 = \mathrm{c}n^2$, and $h' = \mathrm{c}n\cos\theta_0$; so that (151) and (154) become

$$\left.\begin{aligned} \mathrm{A}\left(\frac{d\theta}{dt}\right)^2 + \mathrm{A}(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 &= 2mhg(\cos\theta_0 - \cos\theta), \\ \mathrm{A}(\sin\theta)^2\frac{d\psi}{dt} &= \mathrm{c}n(\cos\theta_0 - \cos\theta)\,; \end{aligned}\right\} \tag{158}$$

the former of these shews that h and $(\cos\theta_0 - \cos\theta)$ are always of the same sign; and that consequently if h is positive, θ is greater than θ_0; and if h is negative, θ is less than θ_0; thus, the inclination of the axis of unequal moment to the vertical in-

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creases or decreases according as the centre of gravity is initially above or below the fixed point. And from the latter it appears that the precessional motion, of which $\frac{d\psi}{dt}$ is the symbol and the measure, is direct or retrograde, according as the centre of gravity is initially above or below the fixed point.

From (158) we have, by the elimination of $\frac{d\psi}{dt}$,

$$dt = \pm \frac{\text{A} \sin\theta \, d\theta}{\{\cos\theta_0 - \cos\theta\}^{\frac{1}{2}} \{2mh\text{A}(\sin\theta)^2 g - \text{c}^2 n^2 (\cos\theta_0 - \cos\theta)\}^{\frac{1}{2}}}. \quad (159)$$

To fix our thoughts in the discussion of this equation, we will suppose h to be positive; so that the centre of gravity of the body is initially above the horizontal plane which contains the fixed point, and the system is in unstable equilibrium when the axis of unequal moment is vertical, that is, when $\theta_0 = 0$. Thus, the second radical in the denominator of (159) is positive when the motion begins; consequently the first radical is also positive, and thus θ must be greater than θ_0; the inclination of the axis of unequal moment to the vertical increases at first, and $\frac{d\theta}{dt}$ is positive. The angle θ continues to increase, until it reaches a value at which the second radical in the denominator of (159) vanishes, when $\frac{d\theta}{dt} = 0$, and there is no variation of inclination; this value of θ is less than π; for when $\theta = \pi$, (159) is imaginary; let it be θ_1; and let T be the time in which the axis of unequal moment will attain this position; then

$$\text{T} = \int_{\theta_0}^{\theta_1} \frac{\text{A} \sin\theta \, d\theta}{\{\cos\theta_0 - \cos\theta\}^{\frac{1}{2}} \{2mh\text{A}(\sin\theta)^2 g - \text{c}^2 n^2 (\cos\theta_0 - \cos\theta)\}^{\frac{1}{2}}}. \quad (160)$$

Now T is finite, although the elements of the integral are infinite at both limits; to prove this, let

$$\cos\theta = s, \qquad \cos\theta_1 = s_1, \qquad \cos\theta_0 = s_0; \quad (161)$$

then

$$\text{T} = \int_{s_1}^{s_0} \frac{\text{A} \, ds}{\{s_0 - s\}^{\frac{1}{2}} \{2mh\text{A} g (1 - s^2) - \text{c}^2 n^2 (s_0 - s)\}^{\frac{1}{2}}}. \quad (162)$$

The form of the denominator of the quantity under the integral sign, shew that the factors $s_0 - s$ and $s_1 - s$ do not enter to any power higher than $-\frac{1}{2}$; and generally, if e is an infinitesimal, and $f(s)$ is continuous when $s = a$,

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$$\int_a^{a+\epsilon} \frac{f(s)\,ds}{(s-a)^{\frac{1}{2}}} = f(a)\int_a^{a+\epsilon} \frac{ds}{(s-a)^{\frac{1}{2}}} = 2f(a)\left[(s-a)^{\frac{1}{2}}\right]_a^{a+\epsilon} = 0;$$

so that the time within which θ passes from θ_0 to θ_1 is finite.

At this instant, when $\theta = \theta_1$, the first radical in (159) is still positive, and thus the second must also be positive, and therefore θ must decrease; and will continue to do so until θ becomes θ_0. This diminution will continue during the same time that has been occupied in the increase from θ_0 to θ_1; and thus the inclination of the axis to the vertical will be the same as before, and another change of position will take place similar to the former. Thus $\text{O}\zeta$, the axis of unequal moment, will make isochronal oscillations in the vertical plane $z\text{O}\zeta$ passing through $\text{O}z$, see Fig. 1, as that plane revolves about the vertical $\text{O}z$.

The angular velocity however of this vertical plane about $\text{O}z$ is not uniform, but the variations of its angular velocity are periodic, the period of which is 2T; and is the time in which the nutational oscillations of the axis of greatest moment takes place; for $\frac{d\psi}{dt}$ is this precessional velocity; and from the second equation of (158), we have

$$\frac{d\psi}{dt} = \frac{\text{C}n(\cos\theta_0 - \cos\theta)}{\text{A}(\sin\theta)^2}; \tag{163}$$

so that the precessional velocity vanishes when $\theta = \theta_0$, and attains its maximum value when $\theta = \theta_1$; afterwards it begins to decrease, and vanishes again when $\theta = \theta_0$; and it continues to make these periodical oscillations. If we project on the plane of (x, y) the curve described by the centre of gravity, it is evident that it is contained between two concentric circles whose radii are respectively $h \sin\theta_0$ and $h \sin\theta_1$; that it consists of a series of arcs which touch the outer circle, and are at right angles to the inner circle at the points where they meet it. This curve is delineated in Fig. 35, where $\text{AC} = \text{CA}' = \ldots\ldots =$ the precession which takes place in the time T. I may observe that this curve is a graphic interpretation of the oscillatory motion of the axis of a spinning top, in which its inclination to the vertical periodically varies.

If ω is the instantaneous angular velocity,

$$\omega^2 = \omega_1^2 + \omega_2^2 + n^2;$$

and ω is a minimum when $\omega_1^2 + \omega_2^2 = 0$, that is, when $t = 0$ and when $\theta = \theta_0$; and ω is a maximum when $\omega_1^2 + \omega_2^2$ is a maximum; and since

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$$\omega_1^2 + \omega_2^2 = \left(\frac{d\theta}{dt}\right)^2 + (\sin\theta)^2 \left(\frac{d\psi}{dt}\right)^2$$
$$= 2mhg(\cos\theta_0 - \cos\theta), \tag{164}$$

the angular velocity is a maximum when θ has its greatest value; that is, when $\theta = \theta_1$, and when the angle of inclination of the axis of unequal moment to the vertical is the greatest. This is evidently tbe case by reason of the principle of vis viva.

If h is negative, similar results follow, except that θ_1 will be less than θ_0; thus, the principal axis of unequal moment will come nearer to the vertical line than it was in its initial position, and will make periodical ascents and descents.

169.] If the initial angular velocity of the system about its axis of unequal moment is very great, n is very great; and since θ_1 is the value of θ determined by the equation,

$$2mh\text{A}g(\sin\theta)^2 - \text{c}^2n^2(\cos\theta_0 - \cos\theta) = 0;$$

θ_1 is very little greater than θ_0; so that the value of θ is confined within very narrow limits. In this case the requisite integrations may be effected approximately. Let u be a very small angle of which the cubes and higher powers may be neglected; and let

$$\theta = \theta_0 + u; \tag{165}$$

then

$$\cos\theta_0 - \cos\theta = \cos\theta_0 - \cos(\theta_0 + u)$$
$$= u\sin\theta_0 + \frac{u^2}{2}\cos\theta_0;$$

$$\frac{\cos\theta_0 - \cos\theta}{(\sin\theta)^2} = \frac{u}{\sin\theta_0} - \frac{u^2\cos\theta_0}{2(\sin\theta_0)^2};$$

and thus (159) becomes

$$dt = \frac{\text{A}\,du}{\{2mh\text{A}g\sin\theta_0 u - (\text{c}^2n^2 - mh\text{A}g\cos\theta_0)u^2\}^{\frac{1}{2}}}; \tag{166}$$

$$\left.\begin{aligned} &\text{let} \quad \frac{\text{c}^2n^2 - mh\text{A}g\cos\theta_0}{\text{A}^2} = k^2, \\ &\text{and} \quad \frac{mh\text{A}g\sin\theta_0}{\text{c}^2n^2 - mh\text{A}g\cos\theta_0} = b; \end{aligned}\right\} \tag{167}$$

$$\therefore \quad k\,dt = \frac{du}{(2bu - u^2)^{\frac{1}{2}}};$$

$$u = b\,\text{versin}\,kt, \tag{168}$$

the limits of integration being such that $u = 0$, when $t = 0$;

$$\therefore \quad \theta = \theta_0 + b\,\text{versin}\,kt$$
$$= \theta_0 + b - b\cos kt. \tag{169}$$

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According to our hypothesis u is always very small; and consequently b must also be small; and therefore either θ_0 is always very small, and the axis of unequal moment is always nearly vertical; or n is very large; and as this is the assumption which we have made, we have approximately

$$b = \frac{mh\text{A}g\sin\theta_0}{\text{C}^2n^2}, \qquad k = \frac{\text{C}n}{\text{A}}.$$

Again, from (163),

$$\frac{d\psi}{dt} = \frac{\text{C}nu}{\text{A}\sin\theta_0}$$

$$= \frac{kb}{\sin\theta_0}(1-\cos kt), \tag{170}$$

$$\therefore \quad \psi = \frac{kb}{\sin\theta_0}t - \frac{b}{\sin\theta_0}\sin kt, \tag{171}$$

the axes of x and y being so placed that $\psi = 0$ when $t = 0$.

And from (157)

$$\frac{d\phi}{dt} = n - \cos\theta\frac{d\psi}{dt}$$

$$= n - kb\cot\theta_0(1-\cos kt);$$

$$\therefore \quad \phi = (n - kb\cot\theta_0)t + b\cot\theta_0\sin kt, \tag{172}$$

the axes of ξ and η being so placed that $\phi = 0$, when $t = 0$. This equation completes the solution of the problem.

Now these equations, (169), (171), and (172), enable us to indicate the motion of the axis of unequal moment with great geometrical exactness. Imagine an axis for which $\theta = \theta_0 + b$, and $\psi = kb\operatorname{cosec}\theta_0 t$; then this axis will describe the surface of a circular cone round the vertical axis with a constant angular velocity. Let this axis be called the mean axis; and any point of it describes, with a constant angular velocity $kb\operatorname{cosec}\theta_0$, a circle whose centre is in the axis of z, and whose plane is perpendicular to that axis. Now the true axis revolves about this mean axis, with displacements depending on the two periodical terms, $\quad -b\cos kt$, and $-b\operatorname{cosec}\theta_0\sin kt$;

so that the true axis is sometimes before, and sometimes behind the mean axis; sometimes nearer to, and sometimes farther from the vertical axis; but as b is a very small quantity, these periodical terms are always very small.

The path which the true axis describes relatively to the mean axis may be imagined by the following figure. Imagine a sphere of radius unity to be described about the fixed point as centre;

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then the mean axis intersects this sphere along a parallel circle of latitude, whose angular distance from the pole is $\theta_0 + b$; and the greatest and the least angular distances of the true axis from the vertical axis are respectively $\theta_0 + 2b$ and θ_0; so that the true axis intersects the sphere in an undulating path contained between two parallels of latitude at angular distances of θ_0 and of $\theta_0 + 2b$ from the polar axis.

These motions of the true and the mean axis are delineated in Fig. 36; in which O is the centre of a sphere whose radius is unity; $OA = OB = OC = 1$; $COR = \theta_0$, $COS = \theta_0 + b$, $COT = \theta_0 + 2b$; RR′, SS′, TT′ are three parallels of latitude, SS′ being that along which the mean axis moves, and RR′ and TT′ being those which limit the displacement of the true axis. The true axis evidently describes the wavy line contained between these two latter parallels; and ON, the projection of OQ on the plane of (x, y), is the line which moves uniformly in that plane with the angular velocity $kb \operatorname{cosec} \theta_0$, which is the mean precessional angular velocity.

Now suppose the point where the mean axis pierces the surface at the time t to be an origin, at which let coordinate axes of ξ and η be taken along the tangent to the meridian, and the tangent to the parallel of latitude respectively; then, if (ξ, η) is the place of the point of intersection of the true axis with the sphere at the time t,

$$\left.\begin{aligned} \xi &= -b \cos kt, \\ \eta &= -\sin\theta_0 \times b \operatorname{cosec}\theta_0 \sin kt = -b \sin kt; \end{aligned}\right\} \qquad (173)$$

the second being multiplied by $\sin\theta_0$, because $\sin\theta_0$ is the approximate value of the radius of the parallel of latitude; therefore squaring and adding,

$$\xi^2 + \eta^2 = b^2, \qquad (174)$$

which is the equation to a circle; and thus the true axis describes uniformly a circular cone of small angle about the mean axis, and intersects the surface of the sphere in a small circle whose centre is the point of intersection of the mean axis with the sphere; this motion of the true axis relatively to the mean axis is called Nutation; its amplitude is always the same, viz., b, but varies in the two directions of latitude and longitude; that is, of the axes of ξ and η.

Since $$\frac{\xi d\eta - \eta d\xi}{dt} = b^2 k = \frac{b^2 \mathrm{C} n}{\mathrm{A}}, \qquad (175)$$

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it appears that the angular velocity of the true axis about the mean has the same sign as n; and therefore the direction of the nutational motion about the mean axis is the same as that of the body about its axis of unequal moment.

Also, the periodic time of the nutation $= \dfrac{2\pi}{k}$

$$= \frac{2\pi A}{nC}; \qquad (176)$$

and therefore the greater n is, the less is the periodic time of the nutation; and consequently the greater the angular velocity of the body about the axis of unequal moment is, the less is the periodic time of the true axis relatively to the mean axis.

The angular velocity of the body, and the place of the instantaneous axis at the time t may thus be found. From (169), (171), and (172), we have

$$\left.\begin{aligned} \theta &= \theta_0 + b - b\cos kt, \\ \psi &= b\operatorname{cosec}\theta_0 (kt - \sin kt), \\ \phi &= nt - b\cot\theta_0 (kt - \sin kt). \end{aligned}\right\} \qquad (177)$$

Let us substitute from these in (100), (101), (102), of Art. 42; and, omitting terms containing squares and higher powers of b, we have

$$\left.\begin{aligned} \omega_1 &= bk\{\sin nt + \sin(k-n)t\}, \\ \omega_2 &= bk\{\cos nt - \cos(k-n)t\}, \\ \omega_3 &= n; \end{aligned}\right\} \qquad (178)$$

$$\therefore \quad \omega^2 = 4b^2k^2\left(\sin\frac{kt}{2}\right)^2 + n^2; \qquad (179)$$

and the direction-cosines of the instantaneous axis are also determined.

170.] And now we will investigate the problem in a still more special form: in the case wherein the angle of inclination of the axis of unequal moment to the vertical is constant throughout the motion. Let θ_0 be the constant value of θ; then

$$\frac{d\theta}{dt} = 0;$$

and, taking the most general forms, from (151) we have

$$\frac{d\psi}{dt} = \left\{\frac{k^2 - Cn^2}{A}\right\}^{\frac{1}{2}} \operatorname{cosec}\theta_0, \qquad (180)$$

$$= a \text{ (say)};$$

$$\therefore \quad \psi = at; \qquad (181)$$

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the position of the axes of x and y being such that $\psi = 0$, when $t = 0$; so that the precessional velocity is constant; and the axis of unequal moment describes uniformly the surface of a right cone whose axis is the vertical axis of z.

And from (157) we have

$$\frac{d\phi}{dt} = n - \cos\theta_0 \frac{d\psi}{dt};$$

$$\therefore \quad \phi = nt - \cos\theta_0 \psi, \tag{182}$$

the position of the axes of ξ and η being such that $\phi = 0$, when $t = 0$;

$$\begin{aligned} \therefore \quad \phi &= nt - a\cos\theta_0 t \\ &= (n - a\cos\theta_0)\, t \\ &= \beta t \text{ (say)}; \end{aligned} \tag{183}$$

$$\text{if} \quad \beta = n - a\cos\theta_0; \tag{184}$$

and thus ϕ varies directly as t; and thus, using the language of astronomy, the angular velocity β of the right ascension of the ξ-axis is constant. And thus the motion of the principal axis of unequal moment is completely determined.

The angular velocity of the body, and the position of the instantaneous axis at the time t, may thus be found. From Art. 42 we have

$$\left.\begin{aligned} \omega_1 &= \frac{d\psi}{dt}\sin\theta_0 \sin\phi = a\sin\theta_0 \sin\beta t, \\ \omega_2 &= \frac{d\psi}{dt}\sin\theta_0 \cos\phi = a\sin\theta_0 \cos\beta t, \\ \omega_3 &= \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta_0 = \beta + a\cos\theta_0 = n; \end{aligned}\right\} \tag{185}$$

$$\left.\begin{aligned} \therefore \quad \omega^2 &= a^2 + 2a\beta\cos\theta_0 + \beta^2 \\ &= a^2(\sin\theta_0)^2 + n^2; \end{aligned}\right\} \tag{186}$$

and hereby the position of the instantaneous axis at the time t is also known.

Hence also we can deduce the conditions under which the invariability of the angle of inclination of the axis of unequal moment to the vertical is possible. Equations (185) are the three final integrals of (143), (144), and (145); let us substitute the former in the latter; then the first two become

$$\left.\begin{aligned} \sin\theta_0 \{\text{A}a\beta + (\text{C}-\text{A})na - mhg\} \cos\beta t &= 0, \\ \sin\theta_0 \{\text{A}a\beta + (\text{C}-\text{A})na - mhg\} \sin\beta t &= 0; \end{aligned}\right\} \tag{187}$$

but these are to be satisfied for all values of t; and therefore

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$$\sin\theta_0 \{\text{A}\alpha\beta + (\text{C}-\text{A})n\alpha - mhg\} = 0; \tag{188}$$

which is an equation to be satisfied by the elements of the body, the angle θ_0, and the velocities of precession and right ascension, when the axis of unequal moment is, throughout the motion, inclined at the same angle to the vertical.

This is satisfied, firstly, when $\sin\theta_0 = 0$, that is when $\theta_0 = 0$, and $\theta_0 = 180°$; in both cases the axis of unequal moment is vertical; and in the former case the centre of gravity is above, and in the latter case is below the fixed point. In both cases the velocities of precession and of right ascension are arbitrary, for α and β may have any values. In the former case $\omega = \alpha + \beta$, and in the latter $\omega = \alpha - \beta$; $\omega_1 = \omega_2 = 0$, and the rotation-axis is vertical, and the body rotates about it with an uniform angular velocity.

171.] Again, (188) is satisfied when

$$\text{A}\alpha\beta + (\text{C}-\text{A})n\alpha - mhg = 0;$$

and if we replace n by its value from (183), we have

$$\alpha^2(\text{C}-\text{A})\cos\theta_0 + \text{C}\beta\alpha - mhg = 0; \tag{189}$$

which is a quadratic equation in terms of α, but only a simple equation in terms of β; consequently for a given value of β we have generally two values of α; but for a given value of α, only one value of β.

There is however only one value of α, when (1) $\theta_0 = 90°$; (2) $\text{C}-\text{A} = 0$. In the former of these two cases the centre of gravity of the moving body, and the principal axis of unequal moment, move in the horizontal plane which contains the fixed point, and we have, what at first view appears an impossibility, a heavy body rotating about a horizontal axis, passing through its centre of gravity, and supported at one point, when that one point is not the centre of gravity. In the latter case, the momental ellipsoid of the body at the fixed point is a sphere; and the body must be constituted in accordance with the conditions investigated in Art. 110; and if A, B, C are the principal central moments,

$$h^2 = \frac{\text{C}-\text{A}}{m}.$$

In both these cases

$$\alpha\beta = \frac{mhg}{\text{C}}; \tag{190}$$

and thus α and β vary inversely as each other; and the product of them varies directly as the distance of the centre of gravity

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from the fixed point, and inversely as the moment of inertia about the axis of greatest moment. And a and β have the same sign if h is positive, and opposite signs if h is negative; thus, if β is given and is positive, a is greater, the greater the distance is between the centre of gravity and the fixed point, and the greater the weight (mg) of the body; and is less the greater C is; and is positive or negative, that is, the precession is direct or retrograde, according as the centre of gravity is above or below the fixed point, when $\theta = 0$. In the former of these two cases

$$\omega_1 = a \sin nt, \qquad \omega_2 = a \cos nt, \qquad \omega_3 = n.$$

$$\therefore \quad \omega^2 = a^2 + n^2.$$

Also, if $h = 0$, that is, if the centre of gravity is at the fixed point, then, from (189), we have

$$(1) \quad a = 0; \qquad (2) \quad a = -\frac{C\beta}{(C-A)\cos\theta_0}; \tag{191}$$

thus, when β is positive, the precession is direct or retrograde, according as θ_0 is an obtuse or an acute angle.

Let us return to the general equation of condition; the two values of a given by (189) are real, equal, or imaginary, according as

$$4mhg(C-A)\cos\theta_0 + C^2\beta^2 \text{ is } >, =, \text{ or } < 0; \tag{192}$$

thus, so long as h and $\cos\theta_0$ are both positive, the two values of a are both real, and unequal and of opposite signs; so that the precession may be either direct or retrograde. If however h is negative, or, what amounts to the same configuration, θ_0 is obtuse, then the two values of a may be equal; and may be imaginary.

Lastly, let us suppose the body to have no motion in right ascension, so that $\beta = 0$; and consequently from (189),

$$a = \frac{d\psi}{dt} = \pm \left\{\frac{mhg}{(C-A)\cos\theta_0}\right\}^{\frac{1}{2}}. \tag{193}$$

This is the case of the common conical pendulum of such a form, that relatively to the fixed point all moments of inertia about axes perpendicular to the rod of the pendulum are equal, and that relative to the rod of the pendulum is C; thus, in the common form of this pendulum, $C-A$ is negative and θ_0 is obtuse; and thus (193) is possible, and indicates the angular velocity of the pendulum about a vertical line through its fixed point of attachment; hence we have, taking the positive sign,

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$$\psi = \left\{\frac{mhg}{(\text{C}-\text{A})\cos\theta_0}\right\}^{\frac{1}{2}} t; \tag{194}$$

and the periodic time of the pendulum

$$= 2\pi\left\{\frac{(\text{C}-\text{A})\cos\theta_0}{mhg}\right\}^{\frac{1}{2}}. \tag{195}$$

For a heavy particle m, placed at the end of a rod of small thickness whose length is h, C may be neglected, and $\text{A} = mh^2$; and if a is the inclination of the rod to the vertical, taken downwards, then

$$\psi = \left(\frac{g}{h\cos a}\right)^{\frac{1}{2}}; \tag{196}$$

and the periodic time $= 2\pi\left(\frac{h\cos a}{g}\right)^{\frac{1}{2}}$. (197)

This solution completes the theory of small oscillations of heavy bodies. Huyghens first gave a solution of the problem of cycloidal and circular pendulums. Clairaut gave the solution of conical pendulums*, which we have already explained after Lagrange and Bravais, in Art. 370, Vol. III; this kind of oscillation taking place when the pendulum, drawn out of its place of rest, receives an impulsion, the line of which is not in the plane containing its line of rest. If however the pendulum at the same time receives a motion of rotation about its axis, the oscillations are deranged by reason of the centrifugal forces thereby developed; and this most general problem of small oscillations is that which we have discussed in the immediately preceding Articles.

172.] Now many machines have been devised for the purpose of exhibiting the phænomena which are expressed in the preceding equations; the construction of some is so curious that they are for the most part found only in collections of mechanical apparatus; others are so simple in form that they are the toys of children. Of the latter kind is the common spinning top, of the several motions of which the preceding Articles give explanations, provided that the point of its peg continues in the same place, and the friction of the point is neglected. Of the former kind is, in the first place, Bohnenberger's machine, which we have already described in Art. 28; it is delineated in Fig. 8, and the first account of it is given in Gilbert's Annalen, Bande

* Memoirs of the French Academy of Sciences of the year 1735.

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60, Leipzig, 1819. The rotating body in the middle is in our figure a sphere, but any other body may be substituted for that; and if the centre of gravity of it coincides with the centre of the three several rings, then, according to the notation in the preceding Articles, $h = 0$. Let us suppose the central body whether it is a sphere, an oblate spheroid, a cone, a cylinder, or any other body such that $A = B$, which rotates about the axis AA′, to be capable of removal from the ring ABA′B′; and to it when so removed let a rapid rotation be given by means of a suitable machine; let it be replaced with its pivots in the holes at A and A′; then the construction of the machine allows the several movements consequent on the rotation of the body to be exhibited, when $h = 0$; for throughout the centre of gravity will remain in the centre of the rings, and be unmoved. And if the pivots at B and B′ are fastened so that no rotation takes place about the axis BB′, the inclination of the axis AA′ to the vertical CC′ is constant throughout: this is the case wherein $\theta = \theta_0$ and $h = 0$, &c. Thus, if the central body is a sphere of radius a,

$$A = B = C = \frac{8\pi\rho a^5}{15},$$

and therefore, from (191), $\alpha = 0$, and $a = \infty$; and therefore the equatorial plane of the sphere always intersects the horizontal plane along the same line; and no rotation can be given to the sphere whereby its axis will describe a conical surface about the vertical CC′.

Again, let the body which rotates about AA′ be a right cone whose altitude is a, and the radius of whose base is b; then

$$A = B = \frac{\pi\rho a b^2}{80}(a^2 + 4b^2), \qquad C = \frac{\pi\rho a b^4}{10};$$

and from (191)

$$\alpha = 0, \quad \text{and} \quad a = -\frac{8b^2\beta}{(4b^2 - a^2)\cos\theta_0};$$

the latter of which by its sign shews that the precessional motion is retrograde.

Another machine of the latter kind is that devised by Fessel, which is described in Poggendorff's Annalen, Bande 90, Leipzig, 1853, and which is delineated in Fig. 37. Q is a heavy fixed stand, the vertical shaft of which is a cylinder bored smoothly, in which works a vertical rod CC′, as far as possible without friction, carrying at its upper end a small frame BB′. In BB′ a horizontal axis works, at right angles to which is a small cylinder

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D, with a tightening screw H, through which passes a long rod GG′, to one end of which is affixed a large ring AA′, and along which slides a small cylinder carrying a weight W, which is capable of being fixed at any point of the rod; and so that it may act as a counterpoise to the ring, or to the ring and any weight attached to it. An axis AA′ works on pivots in the ring, in the same straight line with GG′; to AA′ a disc, or sphere, or cone, or any other body can be attached, and thus can rotate about AA′ as its axis; to the body thus attached to AA′ a rapid rotation can be given, either by means of a string wound round AA′, or by a machine contrived for the purpose when AA′ and its attached body are applied to it. It is evident that the counterpoise W can be so adjusted that the centre of gravity of the rod, the ring, the attached body, and the counterpoise, should be in the axis BB′; or at any point on either side of it; that is, h may be positive, or be equal to 0, or may be negative. Also by fixing BB′ in the arm of CC′ which carries it, the inclination of the rod GG′ to the vertical may be made constant, that is, θ may be equal to θ_0 throughout the motion. When the counterpoise is so adjusted that the centre of gravity of the rod GG′ and its appendages is in CC′, then $h=0$, or, what is equivalent, $mg=0$.

If the counterpoise is adjusted so that the centre of gravity of the rod GG′, of the ring, and of W, without AA′ and its attached body, is in BB′, then the weight of the body will produce its full effect, and the results indicated in the foregoing Articles will be exhibited.

173.] In application of the general equations of rotatory motion we may here insert another problem which is of great interest and importance, although perhaps it more properly comes into the following Chapter.

When a body has motion of both translation and rotation, the investigation into these several motions may be conducted separately, by virtue of those fundamental theorems which have been proved in Section 2 of Chap. III, and the rotation may be considered relative to the centre of gravity and an axis passing through it; just as if the centre of gravity was a fixed point and had no motion of translation. This is precisely what I propose to do now: I propose to consider the rotatory phænomena of the earth, having its centre of gravity fixed at least hypothetically, under the action of the attracting forces of the sun and

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the moon; I shall indeed consider it as merely a mathematical problem; but it will have its application to these three bodies: and as the resulting differential equations will not admit of integration exactly in their general form, I shall make those hypotheses as to small quantities which are given to us by the circumstances of these bodies. Our inquiry too will be general, and will include the action of all bodies by which the rotation of the earth is affected; that is, of not only the sun and the moon, if there are others whose influence affects the earth's motion of rotation. The law of action of these bodies on the earth is of course that of gravitation. The attraction varies directly as the product of the masses, and inversely as the square of the distance.

Let m be the mass of the body whose action on the earth we are considering; let the centre of gravity of the earth be the origin, and let the central principal axes of the earth be, as heretofore, the axes of ξ, η, ζ; the ζ-axis being the geometrical polar axis; and let A, B, C be the central principal moments of the earth relative to these axes respectively; let dm' be a mass-element of the earth, of which the density is ρ, and let its place be (ξ, η, ζ), and of these coordinates ρ is a function; let (x, y, z) be the centre of gravity of m; $r' =$ the distance of (x, y, z) from (ξ, η, ζ); and let r be the distance of m from the centre of gravity of the earth; and let the attraction which two unit-particles at an unit-distance exert on each other be the unit of attraction, and be unity; then

$$\left.\begin{aligned} r^2 &= x^2 + y^2 + z^2, \\ r'^2 &= (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2. \end{aligned}\right\} \qquad (196)$$

Now, for two reasons, we consider the attraction of m on the earth to be the same as if m were condensed into a particle of mass m at its centre of gravity; (1) because the distance between m and the earth is supposed to be very great, and consequently the theorem proved in Art. 191, Vol. III, is applicable to its action; (2) because the bounding surface of m is nearly spherical, and m is supposed to consist of a series of concentric spherical shells, the attraction of each of which on an external particle dm' is the same as if it were condensed into its centre of gravity.

Let X, Y, Z be, relatively to the earth's principal axes, the axial components of the attraction of m on the earth; then

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$$\left.\begin{aligned} \text{X} &= m\iiint \frac{(x-\xi)\rho\, d\xi\, d\eta\, d\zeta}{r'^3}, \\ \text{Y} &= m\iiint \frac{(y-\eta)\rho\, d\xi\, d\eta\, d\zeta}{r'^3}, \\ \text{Z} &= m\iiint \frac{(z-\zeta)\rho\, d\xi\, d\eta\, d\zeta}{r'^3}; \end{aligned}\right\} \tag{197}$$

$$\therefore \quad \left.\begin{aligned} \text{L} &= \text{Z}y-\text{Y}z = m\iiint \frac{\rho(z\eta-y\zeta)\, d\xi\, d\eta\, d\zeta}{r'^3}, \\ \text{M} &= \text{X}z-\text{Z}x = m\iiint \frac{\rho(x\zeta-z\xi)\, d\xi\, d\eta\, d\zeta}{r'^3}, \\ \text{N} &= \text{Y}x-\text{X}y = m\iiint \frac{\rho(y\xi-x\eta)\, d\xi\, d\eta\, d\zeta}{r'^3}; \end{aligned}\right\} \tag{198}$$

the integrations in each equation being such that all the elements of the earth are included.

As the distance of the centre of m from the centre of the earth is very great in comparison with the mean radius of the earth, and consequently with the coordinates of any element of the earth, even when m is the moon; so the quantities under the signs of integration in the right hand members of the preceding equations may be expressed as series of terms rapidly converging in powers of $\frac{\xi}{r}, \frac{\eta}{r}, \frac{\zeta}{r}$: the greatest value of either of these quantities is, in the case of the moon, $\frac{1}{59.96}$; and in the case of the sun, $\frac{1}{23984}$; in the following expansion therefore I shall omit all powers of these quantities above the second.

For the effect of subsequent terms in the series, the student may consult a Memoir, having for its title, "Théorie du mouvement de la Terre autour de son Centre de Gravité," by M. J. A. Serret; and contained in Vol. V of "Annales de l'Observatoire Impérial de Paris," 1859. He will there find the mode of calculating the terms which arise on the hypothesis, that the oblateness of the northern and southern hemispheres of the earth is different; and on the hypothesis, that the earth is not symmetrical relatively to the polar axis of figure.

Now, from (196),

$$\begin{aligned} \frac{1}{r'^3} &= \{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\}^{-\frac{3}{2}} \\ &= \{x^2+y^2+z^2-2(x\xi+y\eta+z\zeta)+\xi^2+\eta^2+\zeta^2\}^{-\frac{3}{2}} \end{aligned}$$

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$$\frac{1}{r'^3} = \frac{1}{r^3}\left\{1 - \frac{2(x\xi + y\eta + z\zeta)}{r^2} + \frac{\xi^2+\eta^2+\zeta^2}{r^2}\right\}^{-\frac{3}{2}}$$

$$= \frac{1}{r^3}\left\{1 + \frac{3(x\xi + y\eta + z\zeta)}{r^2} - \frac{3}{2}\frac{\xi^2+\eta^2+\zeta^2}{r^2} + \frac{15}{2}\frac{(x\xi + y\eta + z\zeta)^2}{r^4}\right\}; \quad (199)$$

let us substitute this value in (198); then, since the centre of gravity of the earth is the origin, and the central principal axes are the coordinate axes,

$$\iiint \rho\xi\, d\xi\, d\eta\, d\zeta = \iiint \rho\eta\, d\xi\, d\eta\, d\zeta = \iiint \rho\zeta\, d\xi\, d\eta\, d\zeta = 0; \quad (200)$$

$$\iiint \rho\eta\zeta\, d\xi\, d\eta\, d\zeta = \iiint \rho\zeta\xi\, d\xi\, d\eta\, d\zeta = \iiint \rho\xi\eta\, d\xi\, d\eta\, d\zeta = 0; \quad (201)$$

and consequently, omitting all powers of small quantities above the second, (198) become

$$\left.\begin{aligned} \text{L} &= 3m(\text{C}-\text{B})\frac{yz}{r^5}, \\ \text{M} &= 3m(\text{A}-\text{C})\frac{zx}{r^5}, \\ \text{N} &= 3m(\text{B}-\text{A})\frac{xy}{r^5}. \end{aligned}\right\} \quad (202)$$

With regard to the last two terms of (199) which do not appear in these equations, having been omitted on account of the smallness of the quantities, I would observe, that they disappear of themselves in the integration, if the earth is supposed to be symmetrical in the distribution of its elements in the northern and southern hemispheres, and with respect to its polar axis of figure. So that under this hypothesis the equations (202) are much more approximate than they appear to be at first sight.

Since $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$, are the direction-cosines of the line joining the centres of the earth and the attracting body, it appears that L, M, and N vary directly as the mass of the attracting body, and inversely as the cube of the distance of its centre from the centre of the earth. Hence, if we calculate, from a synoptic table of the elements of the moon and of the planets, this quantity, it will at once be seen that the sun and the moon are the only bodies which produce any sensible effect on the rotation of the earth; the effect of the sun is due to its very large

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mass; and the effect of the moon, which is much greater, to its nearer distance.

174.] Equations (202) admit of further simplification; and let us first consider them with respect to the sun.

Let n' = the mean angular velocity of the earth about the sun; let E = the mass of the earth; then, as the eccentricity of the earth's orbit is very small, r may be taken as the mean distance of the earth from the sun; and, equating the earth's periodic time in terms of n' with that given in Vol. III, Art. 332, (154), we have

$$\frac{2\pi}{n'} = \frac{2\pi r^{\frac{3}{2}}}{(m+\text{E})^{\frac{1}{2}}}; \tag{203}$$

now $\frac{\text{E}}{m} = \frac{1}{389551}$, according to Encke, quoted by sir John Herschel; and this quantity being small may be neglected, so that

$$\frac{m}{r^3} = n'^2; \tag{204}$$

and therefore for the action of the sun, the equations (202) become

$$\left.\begin{aligned} \text{L} &= 3n'^2(\text{C}-\text{B})\frac{yz}{r^2}, \\ \text{M} &= 3n'^2(\text{A}-\text{C})\frac{zx}{r^2}, \\ \text{N} &= 3n'^2(\text{B}-\text{A})\frac{xy}{r^2}. \end{aligned}\right\} \tag{205}$$

Again, let us consider (202) with respect to the moon; and let all the quantities which refer to the sun receive an accent, and thus refer to the moon.

Let n'' be the mean angular velocity of the moon about the earth; then, if we neglect the eccentricity of the moon's orbit, and take r' to be the mean distance, by the same theorem as that which we have just now applied to the sun,

$$\frac{2\pi}{n''} = \frac{2\pi r'^{\frac{3}{2}}}{(m'+\text{E})^{\frac{1}{2}}}; \tag{206}$$

but $\frac{\text{E}}{m'}$ = 81.84 nearly, = e (say); so that

$$\frac{m'}{r'^3} = \frac{n''^2}{1+e}; \tag{207}$$

and substituting this value in (202) we have

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$$\left.\begin{aligned} \text{L}' &= \frac{3n''^2}{1+e}(\text{C}-\text{B})\frac{y'z'}{r'^2}, \\ \text{M}' &= \frac{3n''^2}{1+e}(\text{A}-\text{C})\frac{z'x'}{r'^2}, \\ \text{N}' &= \frac{3n''^2}{1+e}(\text{B}-\text{A})\frac{x'y'}{r'^2}; \end{aligned}\right\} \tag{208}$$

and the equations of rotation of the earth become

$$\left.\begin{aligned} \frac{d\omega_1}{dt} + \frac{\text{C}-\text{B}}{\text{A}}\omega_2\omega_3 &= \frac{3(\text{C}-\text{B})}{\text{A}}\left\{\frac{n'^2yz}{r^2} + \frac{n''^2y'z'}{(1+e)r'^2}\right\}, \\ \frac{d\omega_2}{dt} + \frac{\text{A}-\text{C}}{\text{B}}\omega_3\omega_1 &= \frac{3(\text{A}-\text{C})}{\text{B}}\left\{\frac{n'^2zx}{r^2} + \frac{n''^2z'x'}{(1+e)r'^2}\right\}, \\ \frac{d\omega_3}{dt} + \frac{\text{B}-\text{A}}{\text{C}}\omega_1\omega_2 &= \frac{3(\text{B}-\text{A})}{\text{C}}\left\{\frac{n'^2xy}{r^2} + \frac{n''^2x'y'}{(1+e)r'^2}\right\}. \end{aligned}\right\} \tag{209}$$

The complete integration of these equations is beyond the power of analysis; and we are obliged to have recourse to methods of approximation, taking advantage of those circumstances as to small quantities which the relations of the sun, earth, and moon offer to us: these we proceed to explain.

175.] In the first place, geodetic measurements shew that the figure of the earth is nearly that of a solid of revolution, whose axis is the polar axis of figure; and as there is no reason to suppose any great want of symmetry in the distribution of the material elements in the interior of the earth, we may suppose the two principal moments in the plane of the equator, and consequently all the moments of inertia in that plane, to be equal; thus, $\text{B} = \text{A}$; and this equality exists whatever are the positions of the axes of x and y in the plane of the equator.

I may moreover observe, that the most profound calculations*, based on the hypothesis of an unsymmetrical distribution of material within the earth, lead to the conclusion that ω_3 is constant to a first approximation, and that consequently $\text{B} = \text{A}$; this result follows from the fact that the action of the sun and moon is very small in comparision of the actual vis viva of the earth.

Observations made with the pendulum are in accordance with direct measurement, and shew the earth to be a solid of revolution, whose polar axis is shorter than the equatorial; and that

* See the Memoir of Serret quoted above; also Le Verrier, Annales de l'Observatoire Impérial de Paris, Tome II.

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its figure is approximately an oblate spheroid; and thus C, which is the central principal moment relative to the axis of revolution, is the greatest of all moments. Now, putting $B = A$ in (209), it is plain that C and A enter into the equations of motion only in the form $\frac{C - A}{A}$: the value of this quantity cannot be determined by direct observation, because we are ignorant of the law of density of the matter of the earth, and we are obliged to have recourse to indirect methods. The observed values of precession and nutation give it a value of nearly $\frac{1}{306}$*, which is beyond doubt almost correct; also a hypothesis of Laplace, discussed in the Mécanique Céleste, Livre XI, gives a result nearly identical; this value we shall take. Since the physical constitution of the earth enters into the equations of motion only by means of these quantities, it is evident that the phænomena of precession and nutation would be the same, whatever change took place in the earth, so long as the ratio $\frac{C - A}{A}$ was unaltered.

Again, the actual axis of rotation of the earth is almost fixed in it, and is almost identical with the axis of figure; that is, the poles of the earth are almost fixed points on its surface. Were they not so, geographical latitudes would vary from time to time; whereas no variation has been indicated by observation, so far as I know. Moreover, as the true rotation-axis of the earth in all its positions nearly coincides with the axis of figure, the true angular velocity ω, which is the resultant of ω_1, ω_2, ω_3, is nearly equal to ω_3, which is the angular velocity about the earth's axis of figure, and is constant; and thus ω_1 and ω_2 are very small quantities. Thus, if we image the actual rotation of the earth by the rolling of one cone on another, that cone which the earth's axis describes in itself has a very small vertical angle, the cone fixed in space having a vertical angle a little greater than 46°.

This is the information which observation gives as to the circumstances of the constitution and the figure of the earth, and as to the approximate invariability of its angular velocity, and of the position of its rotation-axis.

Under these circumstances the equations of motion become

* See the Memoirs of Serret and Le Verrier.

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$$\left.\begin{aligned}
\frac{d\omega_1}{dt} + \frac{C-A}{A}\omega_2\omega_3 &= \frac{3(C-A)}{A}\left\{\frac{n'^2yz}{r^3} + \frac{n''^2y'z'}{(1+e)r'^2}\right\},\\
\frac{d\omega_2}{dt} - \frac{C-A}{A}\omega_3\omega_1 &= -\frac{3(C-A)}{A}\left\{\frac{n'^2zx}{r^3} + \frac{n''^2z'x'}{(1+e)r'^2}\right\},\\
\frac{d\omega_3}{dt} &= 0;
\end{aligned}\right\} \quad (210)$$

from the third of these $\omega_3 = n$, (211)

if n is the angular velocity of the earth about its polar axis of figure.

Also, for convenience of expression, let

$$\frac{C-A}{A} = a;$$

then the first two of (210) become

$$\left.\begin{aligned}
\frac{d\omega_1}{dt} + an\omega_2 &= 3a\left\{\frac{n'^2yz}{r^3} + \frac{n''^2y'z'}{(1+e)r'^2}\right\},\\
\frac{d\omega_2}{dt} - an\omega_1 &= -3a\left\{\frac{n'^2zx}{r^3} + \frac{n''^2z'x'}{(1+e)r'^2}\right\};
\end{aligned}\right\} \quad (212)$$

and from these equations all the phænomena of the rotation are to be deduced.

Equations (210) shew that the action of both the sun and the moon on the earth is due to the physical constitution of the earth itself. If $C = A$, that is, if all the principal central moments of the earth were equal, $\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0$; and thus the angular velocity would be constant, and the earth's rotation-axis would be fixed in itself, and would be absolutely fixed in space; the protuberant matter at the earth's equator which causes the inequality of the central principal moments, is thus the indirect cause of the peculiar motion of the earth's rotation-axis, which we are about to investigate.

176.] The arrangement of the bodies which is convenient for our system of symbols and equations is exhibited in Fig. 38.

O is the centre of the earth; and the plane XOY is the fixed plane of the ecliptic; OX being the line of the vernal equinox when $t = 0$. About O as a centre a sphere is described whose radius is equal to unity; and the several curved lines of the figure are the intersections of the surface of this sphere by various planes and lines drawn through O, and all refer to the configuration of the system at the time t; yxN is the plane of the earth's equator,

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so that ON is the line of the vernal equinox, and XON is the precession; Oz is the earth's polar axis about which the angular velocity is n, and Ox and Oy are the earth's principal axes in the plane of the equator, Ox being so chosen that it coincides at the same time with OX, OS, and ON; OS is the radius vector of the sun, which is always in the plane of the ecliptic; OM is the radius vector of the moon, MN′I being the plane of the moon's orbit; ON′ is the line of intersection of that plane with the plane of the ecliptic, and is the line of the moon's nodes; OI is the line of intersection of the plane of the moon's orbit with the plane of the earth's equator. Let i be the angle of inclination of the plane of the moon's orbit to the ecliptic; then i is nearly constant, and has a mean value of 5°8′48″; we shall take it to be constant. Now the line of nodes of the moon revolves in the plane of the ecliptic, and performs a complete revolution in about 6793 days. Thus ON′ revolves about OZ: let β be its angular velocity; then, if $n = 1$, we have approximately,

$$n' = \frac{1}{365.25}, \quad n'' = \frac{1}{27.32}, \quad \beta = \frac{1}{6793};$$

so that β is much less than the other quantities; the small fraction a is also a factor of all the terms into which these quantities enter.

As the angular motion of the line of equinoxes is very small, the angle XON is very small compared with NOx, or XOS; so that approximately NOS = XOS = $n't$; and NOx = nt. We shall also in calculating small terms neglect variations of θ. From this arrangement we have

$$\left.\begin{aligned} \frac{x}{r} &= \cos xOS = \quad \cos nt \cos n't + \sin nt \sin n't \cos\theta, \\ \frac{y}{r} &= \cos yOS = -\sin nt \cos n't + \cos nt \sin n't \cos\theta, \\ \frac{z}{r} &= \cos zOS = -\sin n't \sin\theta; \end{aligned}\right\} \quad (218)$$

which are thus expressed in terms of t and of constants.

Again, as to the moon; let us in the first place refer it to the ecliptic; then, if ψ'_0 is the longitude of the moon's node at the vernal equinox, that is, when $t = 0$, NON′ = $\psi'_0 + \beta t = \psi'$, say; and if ϕ'_0 is the moon's right ascension at the vernal equinox, N′OM = $\phi'_0 + n''t = \phi'$, say; then, as i is very small, N′OM and its projection on the plane of the ecliptic may be considered to be equal; so that the longitude of M is $\phi' + \psi'$; and, if we replace $\sin i$ by i, $\cos MOZ = i \sin\phi'$;

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$$\left.\begin{aligned}
\frac{x'}{r'} &= \cos xOM = \{\sin(\phi'+\psi')\cos\theta + i\sin\phi'\sin\theta\}\sin nt + \cos(\phi'+\psi')\cos nt,\\
\frac{y'}{r'} &= \cos yOM = \{\sin(\phi'+\psi')\cos\theta + i\sin\phi'\sin\theta\}\cos nt - \cos(\phi'+\psi')\sin nt,\\
\frac{z'}{r'} &= \cos zOM = i\sin\phi'\cos\theta - \sin(\phi'+\psi')\sin\theta\,;
\end{aligned}\right\}\quad(214)$$

and as $\phi'+\psi' = \phi'_0 + \psi'_0 + (\beta + n'')t$, these quantities are expressed in terms of t, and of known quantities; they are to be substituted in the equations (212); which are then to be integrated.

The linear form of the equations (212) shews that the effects of the action of the sun and the moon may be calculated separately; and that the whole effect is the sum of the two separate effects. We shall consequently calculate each by itself. Instead of determining ω_1 and ω_2 by means of these equations, it will be more convenient to calculate θ and ψ directly, as the position of the earth will hereby be determined with reference to fixed lines.

The equations which connect these angles with the principal angular velocities are given in Art. 42; and are

$$\frac{d\theta}{dt} = \omega_1\cos\phi - \omega_2\sin\phi,$$

$$\sin\theta\frac{d\psi}{dt} = \omega_1\sin\phi + \omega_2\cos\phi.$$

177.] Our object in this inquiry is not to calculate accurately the motion of the earth's rotation-axis, and the earth's angular velocity which determines the length of a day; but to trace roughly, and to indicate in their salient points the results of the action of the sun and moon. We shall therefore retain as far as possible only finite quantities, and small quantities of the first order; and we shall only notice the kind of change which is produced, with a view rather to the general effect of such action than to numerical calculations.

We will first consider the terms in (212) which refer to the sun, and which will be replaced by their values in (213).

Now the earth's axis is inclined to the normal of the ecliptic at an angle which is nearly constant; let ι be its mean value, which is about 23°27′32″; this angle being that between the earth's equator and the ecliptic is called the obliquity of the ecliptic. It is the mean value of θ according to our arrange-

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ment, and we shall replace θ by it in terms involving small quantities. Also, as the earth rotates uniformly about its polar axis with the angular velocity n, and as the angular velocity of ON is very small, $\phi = nt$, omitting small quantities; and thus the equations of the last Article become

$$\left.\begin{aligned} \frac{d\theta}{dt} &= \omega_1 \cos nt - \omega_2 \sin nt, \\ \sin \mathrm{I} \frac{d\psi}{dt} &= \omega_1 \sin nt + \omega_2 \cos nt; \end{aligned}\right\} \tag{215}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \cos nt \frac{d\omega_1}{dt} - \sin nt \frac{d\omega_2}{dt} - n(\omega_1 \sin nt + \omega_2 \cos nt)$$

$$= \cos nt \frac{d\omega_1}{dt} - \sin nt \frac{d\omega_2}{dt} - n \sin \mathrm{I} \frac{d\psi}{dt}; \tag{216}$$

$$\sin \mathrm{I} \frac{d^2\psi}{dt^2} = \sin nt \frac{d\omega_1}{dt} + \cos nt \frac{d\omega_2}{dt} + n \frac{d\theta}{dt}; \tag{217}$$

substituting for $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt}$ their values given in (212) taking only the terms which depend on the sun's action, we have

$$\frac{d^2\theta}{dt^2} + (1+a)n \sin \mathrm{I} \frac{d\psi}{dt} + 3an'^2 (\sin n't)^2 \sin \mathrm{I} \cos \mathrm{I} = 0; \tag{218}$$

$$\sin \mathrm{I} \frac{d^2\psi}{dt^2} - (1+a)n \frac{d\theta}{dt} - 3an'^2 \sin n't \cos n't \sin \mathrm{I} = 0. \tag{219}$$

Integrating (218) we have

$$\frac{d\theta}{dt} + (1+a)n \sin \mathrm{I}\, \psi + \frac{3an'^2}{2} \sin \mathrm{I} \cos \mathrm{I} \left(t - \frac{\sin 2n't}{2n'}\right) = 0; \tag{220}$$

no constant being added, because if no disturbing force acts, that is, if $n' = 0$, $\frac{d\theta}{dt} = 0$; and $\psi = 0$, when $t = 0$. Substitute for $\frac{d\theta}{dt}$ in (219), and we have

$$\frac{d^2\psi}{dt^2} + (1+a)^2 n^2 \psi = -\frac{3a(1+a)n'^2 n \cos \mathrm{I}}{2} t$$

$$+ \frac{3an'}{4} \{(1+a)n \cos \mathrm{I} + 2n'\} \sin 2n't; \tag{221}$$

$$\therefore \quad \psi = -\frac{3an'^2 \cos \mathrm{I}}{2(1+a)n} t + \frac{3an'}{4} \frac{(1+a)n \cos \mathrm{I} + 2n'}{(1+a)^2 n^2 - 4n'^2} \sin 2n't$$

$$+ c' \sin \{(1+a)nt + \gamma\}; \tag{222}$$

where c' and γ are constants introduced in integration; but since $\psi = 0$ when $t = 0$, and $\frac{d\psi}{dt}$ is independent of t when $n' = 0$, $c' = 0$, and $\gamma = 0$.

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Also, in the coefficient of $\sin 2n't$ we may omit $\frac{n'^2}{n^2}$, on account of its smallness; and, since $a = \frac{C-A}{A}$,

$$\frac{a}{1+a} = \frac{C-A}{C}; \tag{223}$$

and thus (222) becomes

$$\psi = -\frac{3n'^2 \cos I}{2n} \frac{C-A}{C} t + \frac{3n' \cos I}{4n} \frac{C-A}{C} \sin 2n't. \tag{224}$$

Replacing ψ in (220) by this value we have an identity; which shews that the terms herein retained destroy each other in the variation of θ, although they give a finite result in the value of ψ. We must therefore replace ψ by the value which it has before small terms are omitted; that is, we must substitute for ψ the value given in (222), putting however $c' = 0$; then (220) becomes

$$\frac{d\theta}{dt} = -\frac{3n'^2 \sin I}{2n} \frac{C-A}{C} \sin 2n't; \tag{225}$$

$$\therefore \quad \theta = I + \frac{3n' \sin I}{4n} \frac{C-A}{C} \cos 2n't; \tag{226}$$

where I is the constant of integration and is the mean value of θ.

Equations (224) and (226) exhibit the effects of the sun's action on the rotation of the earth. ψ is the angle through which the line of equinoxes, ON in Fig. 38, moves in the time t, and is called the Solar Precession of the equinoxes; (224) shews that it consists of two terms, from the former of which it appears that ψ increases directly as the time; and from the latter, that this continual motion is accompanied by a periodical variation, of which the periodic time is $\frac{\pi}{n'}$, that is, is half a year. This periodical quantity is called the Solar Nutation of the Earth's Axis in Longitude, or, the Nutation of the Equinoxes. Thus, the line of equinoxes is sometimes a little in advance of, and sometimes a little behind, its mean place; and coincides with its mean place every half year; but as the coefficient of this periodical part is very small, so does the term scarcely ever acquire a sensible magnitude.

From (226) it appears that θ has a mean value I; but that the earth's axis has a small oscillatory motion, depending on the second term of which the period is also half a year; and this second term is always very small because its coefficient is small. It is called the Solar Nutation of the Earth's Axis in Latitude,

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or, the Nutation of the Obliquity. Thus, the rotation-axis of the earth would have a very slow progressive motion in space, inclined at a constant angle 1 to the normal of the ecliptic, if it were disturbed by only the sun's action.

178.] The effect of the moon on the rotation of the earth is expressed by the latter terms in the right hand members of (212); these we now proceed to inquire into, and by a process similar to that by which we have investigated the action of the sun.

For abridgment of notation let the moon's longitude $= \mu + \nu t$; so that

$$\mu = \phi_0 + \psi_0; \qquad \nu = \beta + n''; \tag{227}$$

and let us replace θ by 1; (214) become

$$\left.\begin{aligned}
\frac{x'}{r'} &= \{\sin(\mu+\nu t)\cos 1 + i\sin(\phi'_0 + n''t)\sin 1\}\sin nt + \cos(\mu+\nu t)\cos nt,\\
\frac{y'}{r'} &= \{\sin(\mu+\nu t)\cos 1 + i\sin(\phi'_0 + n''t)\sin 1\}\cos nt - \cos(\mu+\nu t)\sin nt,\\
\frac{z'}{r'} &= i\sin(\phi'_0 + n''t)\cos 1 - \sin(\mu+\nu t)\sin 1;
\end{aligned}\right\} \tag{228}$$

as i is a small angle, the squares and higher powers of it will be omitted. Substituting these quantities in (212) and in (216), we have

$$\begin{aligned}
\frac{d^2\theta}{dt^2} &= -(1+a)n\sin 1\frac{d\psi}{dt} + \frac{3an''^2}{1+e}\frac{z'}{r'}\{\sin(\mu+\nu t)\cos 1 + i\sin(\phi'_0 + n''t)\sin 1\}\\
&= -(1+a)n\sin 1\frac{d\psi}{dt} + \frac{3an''^2}{1+e}\left\{-\frac{\sin 2 1}{4}\{1-\cos 2(\mu+\nu t)\}\right.\\
&\qquad \left. + \frac{i\cos 2 1}{2}(\cos\{\psi'_0 + \beta t\} - \cos\{2\phi'_0 + \psi'_0 + (\beta + 2n'')t\})\right\};
\end{aligned} \tag{229}$$

and substituting again in (217),

$$\begin{aligned}
\sin 1\frac{d^2\psi}{dt^2} &= (1+a)n\frac{d\theta}{dt} + \frac{3an''^2}{1+e}\left\{\frac{\sin 1}{2}\sin 2(\mu+\nu t)\right.\\
&\qquad \left. + \frac{i\cos 1}{2}(\sin\{\psi'_0 + \beta t\} - \sin\{2\phi'_0 + \psi'_0 + (\beta + 2n'')t\})\right\};
\end{aligned} \tag{230}$$

From (229) we have

$$\begin{aligned}
\frac{d\theta}{dt} &= -(1+a)n\sin 1\,\psi + \frac{3an''^2}{1+e}\left\{-\frac{\sin 2 1}{4}\left(t - \frac{\sin 2(\mu+\nu t)}{2\nu}\right)\right.\\
&\qquad \left. + \frac{i\cos 2 1}{2}\left(\frac{\sin(\psi'_0 + \beta t)}{\beta} - \frac{\sin\{2\phi'_0 + \psi'_0 + (\beta + 2n'')t\}}{\beta + 2n''}\right)\right\}.
\end{aligned} \tag{231}$$

Now substituting this in (230), we have

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$$\frac{d^2\psi}{dt^2}+(1+a)^2n^2\psi=-\frac{3an''^2(1+a)n\cos\text{I}}{2(1+e)}t$$
$$+\frac{3an''^2}{(1+e)}\left\{\frac{(1+a)n\cos\text{I}+2\nu}{4\nu}\sin 2(\mu+\nu t)\right.$$
$$+\frac{(1+a)n\cos 2\text{I}+\beta\cos\text{I}}{2\beta\sin\text{I}}i\sin(\psi'_0+\beta t)$$
$$\left.-\frac{(1+a)n\cos 2\text{I}+(\beta+2n'')\cos\text{I}}{2(\beta+2n'')\sin\text{I}}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\};\ (232)$$

$$\therefore\ \psi=-\frac{3an''^2\cos\text{I}}{2(1+e)(1+a)n}t+\frac{3an''^2}{1+e}\left\{\frac{(1+a)n\cos\text{I}+2\nu}{4\nu\{(1+a)^2n^2-4\nu^2\}}\sin 2(\mu+\nu t)\right.$$
$$+\frac{(1+a)n\cos 2\text{I}+\beta\cos\text{I}}{2\beta\sin\text{I}\{(1+a)^2n^2-\beta^2\}}i\sin(\psi'_0+\beta t)$$
$$\left.-\frac{(1+a)n\cos 2\text{I}+(\beta+2n'')\cos\text{I}}{2(\beta+2n'')\sin\text{I}\{(1+a)^2n^2-(\beta+2n'')^2\}}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\};\ (233)$$

the constants being omitted for the same reason as they are omitted in (222).

Now β is very small compared with n'', and thus ν may be replaced by n''; and the squares and higher powers of $\frac{n''}{n}$ may be omitted; so that after all reductions (233) becomes

$$\psi=-\frac{3an''^2\cos\text{I}}{2(1+e)(1+a)n}t+\frac{3an''\cos\text{I}}{4n(1+e)(1+a)}\sin 2\{\phi'_0+\psi'_0+(\beta+n'')t\}$$
$$+\frac{3an''^2\cos 2\text{I}}{2\beta n(1+e)(1+a)\sin\text{I}}i\sin(\psi'_0+\beta t)$$
$$-\frac{3an''\cos 2\text{I}}{4n(1+a)(1+e)\sin\text{I}}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}.\ (234)$$

If we substitute this value for ψ in (231) it leads to an identity, and thus it appears that the terms which are herein retained cancel each other in the variation of θ; we must therefore replace ψ in (231) by its more approximate value, which is given in (233); and we have

$$\frac{d\theta}{dt}=\frac{3an''^2}{(1+e)(1+a)n}\left\{\frac{-\sin\text{I}}{2}\sin 2(\mu+\nu t)-\frac{\cos\text{I}}{2}i\sin(\psi'_0+\beta t)\right.$$
$$\left.+\frac{\cos\text{I}}{2}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\};\ (235)$$

$$\theta=\text{I}+\frac{3an''^2}{2(1+e)(1+a)n}\left\{\frac{\sin\text{I}}{2n''}\cos 2\{\phi'_0+\psi'_0+(\beta+n'')t\}+\frac{\cos\text{I}}{\beta}i\cos(\psi'_0+\beta t)\right.$$
$$\left.-\frac{\cos\text{I}}{2n''}i\cos\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\};\ (236)$$

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where I is the mean value of θ, and is the constant introduced in integration.

In (234) and (236) the last terms which involve the angle $2\phi'_0+\psi'_0+(\beta+2n'')t$ are to be omitted, because of the smallness of the coefficient in which i is a factor; the next preceding terms in each however must be retained, because β, which is a very small quantity, is in the denominator of the coefficient, and this brings it into importance, although it contains i as a factor; thus, if we replace $\frac{a}{1+a}$ by its value, given in (223), we have from (234) and (236),

$$\psi = -\frac{3n''^2\cos \mathrm{I}}{2(1+e)n}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}t+\frac{3n''\cos \mathrm{I}}{4n(1+e)}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\left\{\sin 2\{\phi'_0+\psi'_0+(\beta+n'')t\}\right.$$
$$\left.+\frac{4n''\cos 2\mathrm{I}}{\sin 2\mathrm{I}}\frac{i}{\beta}\sin(\psi'_0+\beta t)\right\}; \quad (237)$$

$$\theta = \mathrm{I}+\frac{3n''}{2n(1+e)}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\left\{\frac{\sin \mathrm{I}}{2}\cos 2\{\phi'_0+\psi'_0+(\beta+n'')t\}\right.$$
$$\left.+n''\cos \mathrm{I}\frac{i}{\beta}\cos(\psi'_0+\beta t)\right\}; \quad (238)$$

On comparing these values with (224) and (226), which express the sun's action, it is evident that they produce effects on the earth's axis of precisely the same kind; so that what has there been said of solar precession and nutation, may here, mutatis mutandis, be said of lunar precession and nutation; but the effect of the terms in these latter expressions is much greater than that of those in the former, because n'' is much greater than n'.

179.] The whole precession and nutation is the sum of the two separate effects; but before we add, we must make a remark or two on the signs of our quantities. We have taken all the angular velocities to be positive; that is, we have supposed the bodies to revolve from the axis of X towards that of Y in Fig. 38; and this hypothesis is in accordance with the convention of signs which has been adopted throughout the volume; it is not however necessarily that of the actual motion of the earth and moon, of the moon's line of nodes, and of the apparent motion of the sun: let OX be east on the ecliptic, and let OZ be the normal to the ecliptic towards the north: now all the bodies revolve in their orbits, as well as about their axes, from west to east; so that the signs of n and of n'' are to be changed; that of n' is correct, because the sun's motion is apparent only, being

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due to the actual motion of the earth. The line of the moon's nodes also retrogrades, that is, goes from east to west, so that the sign of β is correct.

Also let Ω be the longitude of the moon's line of nodes at the time; let $\odot$ and ☾ be the longitudes of the sun and moon respectively*; then

$$\Omega = \psi'_0 + \beta t, \quad \odot = n't, \quad ☾ = \phi'_0 + \psi'_0 + (\beta - n'')t; \quad (239)$$

so that for the whole precession and nutation we have

$$\psi = \frac{3\cos 1}{2n}\frac{C-A}{C}\left(n'^2 + \frac{n''^2}{1+e}\right)t$$
$$+ \frac{3\cos 1}{4n}\frac{C-A}{C}\left\{\frac{4n''^2\cos 2 1}{(1+e)\sin 2 1}\frac{i}{\beta}\sin\Omega - \frac{n''}{1+e}\sin 2☾ - n'\sin 2\odot\right\}; \quad (240)$$

$$\theta = 1 - \frac{3}{2n}\frac{C-A}{C}\left\{\frac{n''^2\cos 1}{1+e}\frac{i}{\beta}\cos\Omega - \frac{n''\sin 1}{2(1+e)}\cos 2☾ - \frac{n'\sin 1}{2}\cos 2\odot\right\}; \quad (241)$$

the terms involving n'' and n' arise from the action of the moon and sun respectively.

The second of these equations shews that the earth's axis is inclined to the normal of the ecliptic at an angle which is nearly constant; yet that there are small variations of the angle which are expressed by the latter terms of (241); these terms are periodic, and are very small because their coefficients are small; they depend on the longitude of the moon's ascending node on the ecliptic, on the longitude of the sun, and on the longitude of the moon; they constitute the luni-solar nutation in latitude or in obliquity.

Equation (240) shews that the line of equinoxes has a general retrograde motion along the ecliptic, with an angular velocity

$$= \frac{3\cos 1}{2n}\frac{C-A}{C}\left(n'^2 + \frac{n''^2}{1+e}\right) = \Psi, \text{ say}; \quad (242)$$

this quantity is called the luni-solar precession of the equinoxes; yet that this retrograde motion is not uniform, but is subject to slight variations, which are periodic, and are expressed by the last three terms of the right hand member of (240); that these periodical quantities are very small, because their coefficients are small: they likewise depend on the longitude of the moon's line of nodes, and on the longitude of the sun and the moon; and they constitute the luni-solar nutation in longitude.

* It will be observed that no distinction has been made between true and mean longitude, true and mean ecliptic, &c.; our calculations have not been carried far enough for such accurate positions.

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The motion therefore of the earth's axis in space will be well represented by the Fig. 36; in which O is the centre of the earth and is supposed to be fixed, and the radius of the sphere is unity. The axis whose motion is defined by the equations

$$\theta = \text{I}, \qquad \psi = \Psi t, \tag{243}$$

may be called the mean axis of the earth, I and Ψt being respectively the mean obliquity and the mean precession. And the axis which is defined by the complete expressions (240) and (241) will be the true axis. Let COR = I; then the circle RS will be that along which the mean axis will intersect the surface of the sphere; and if OQ is the mean axis at the time t, and OP is the true axis, the angle POQ will be small; and as t varies OP will be sometimes before, and sometimes behind OQ; and sometimes nearer to, and sometimes farther from the pole of the ecliptic. Thus, the true axis of the earth will intersect the sphere in a wavy line contained between two parallels of the sphere at distances from RS, determined by the greatest positive and negative values of the periodic terms of θ given in (241).

The motion of the true axis relatively to the mean axis may, as to its principal and its most important terms, be exhibited in the following way. Suppose the point of intersection of the mean axis with the sphere to be an origin, at which two axes originate, say of ξ and η, in the plane touching the sphere; that of η being a tangent to the parallel along which the mean axis moves, and the ξ-axis being perpendicular to it, and thus being a tangent to the meridian through the place of the mean axis. Now the most important periodic terms in (240) and (241) are those which depend on the longitude of the moon's ascending node, on account of the smallness of β, as we have before observed; let these principal terms in the directions of the two axes of ξ and η be represented by ξ and η, so that

$$\xi = \frac{3}{2n}\,\frac{\text{C}-\text{A}}{\text{C}}\,\frac{n'^2\cos \text{I}}{1+e}\,\frac{i}{\beta}\cos\Omega,$$

$$\eta = -\frac{3}{2n}\,\frac{\text{C}-\text{A}}{\text{C}}\,\frac{n'^2\cos 2\text{I}}{1+e}\,\frac{i}{\beta}\sin\Omega;$$

$$\therefore\quad \frac{\xi^2}{(\cos \text{I})^2} + \frac{\eta^2}{(\cos 2\text{I})^2} = \left(\frac{3}{2n}\,\frac{\text{C}-\text{A}}{\text{C}}\,\frac{n'^2}{1+e}\,\frac{i}{\beta}\right)^2; \tag{244}$$

which is the equation to an ellipse whose axes are in the ratio of cos I to cos 2I, and of which that directed towards the pole of the ecliptic is the greatest; and thus it follows that so far as the most important terms affect the motion, the true axis de-

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scribes a small ellipse on the surface of the sphere relatively to the mean axis which passes through the centre of the ellipse; This ellipse is called the Ellipse of Nutation.

In the preceding image of the motion of the earth's axis, we have assumed the earth's centre to be fixed, and the radius of the sphere to whose surface we have referred the motion of the axis, has also been assumed to be unity. The earth's centre however is not fixed; yet the image is a correct representation of the facts, because we refer the motion of the earth's axis to the sidereal vault, of which we may say the radius is so great that, in comparision of it, the distance through which the earth's centre moves is infinitesimal. Thus, the mean axis describes a circle about the pole of the ecliptic, the angular radius of which circle is 23°27′32″; and the true axis describes a wavy line symmetrically situated with reference to this circle; and if the mean axis is considered fixed, the true axis describes an ellipse on the sidereal vault, the centre of which is the place where it is pierced by the mean axis.

The periodic time in which the mean pole describes its circle is $\frac{2\pi}{\Psi}$; and the true pole will describe its ellipse about the mean pole in the same time as that in which the moon's line of nodes describes a complete revolution.

The value of the annual luni-solar precession is found as follows:

$$\Psi = \frac{3\cos\text{I}}{2n}\,\frac{\text{C}-\text{A}}{\text{C}}\left(n'^2 + \frac{n''^2}{1+e}\right)$$

$$= 3\cos\text{I}\,\frac{\text{C}-\text{A}}{\text{C}}\,\frac{n'}{n}\left\{1 + \frac{n''^2}{(1+e)n'^2}\right\}\frac{n'}{2};$$

so that the annual luni-solar precession

$$= 3\cos\text{I}\,\frac{\text{C}-\text{A}}{\text{C}}\,\frac{n'}{n}\left\{1 + \frac{n''^2}{(1+e)n'^2}\right\}180°.$$

Now if we take the epoch to be Jan. 1, 1850*, $\text{I} = 23°27'32''$,

$$\frac{\text{C}-\text{A}}{\text{C}} = \frac{1}{306},\quad \frac{n'}{n} = \frac{1}{365.25},\quad \frac{n''}{n'} = \frac{365.25}{27.32},\ \text{and}\ e = 81.84;$$

and therefore

$$1 + \frac{n''^2}{(1+e)n'^2} = 3.15764;$$

also

$$\cos 23°27'32'' = .91735;$$

therefore the annual luni-solar precession

* See the Memoir of M. Serret in Vol. V of "Annales de l'Observatoire Impérial de Paris," page 321.

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$$= \frac{.91735}{102} \times \frac{315.764}{36525} \times 180 \times 60 \times 60'',$$

$$= 50''.3828. \qquad (245)$$

The observed value of the luni-solar precession is 50″.37140; so that our result is very nearly correct, although it is only approximate. I may* in passing remark, that the coefficients of $\sin \Omega$ and of $\cos \Omega$ in (240) and (241) respectively, are $-17''.251$, and $+9''.223$; the former being the largest value of the principal term of the nutation of the equinoxes, and the latter being the largest value of the principal term of the nutation of the obliquity. Also the mean axis describes a complete circle in the heavens in 25724 years.

180.] Of the problem of precession and nutation an approximate solution has also been given by M. Poinsot in the Additions to the "Connaissance des Temps" for 1858. The principal terms only are found by it; but it exhibits the problem in such an elementary form, and dissects the results of the action of the sun and moon into the several phænomena so distinctly, that it is peculiarly fitted for an elementary treatise. We shall employ the symbols of the preceding Articles, and shall make use of the couples of the impressed momenta which have been therein determined.

We consider all quantities at the time t, and investigate the effects which accrue during the infinitesimal time dt. If n is the angular velocity of the earth about its rotation-axis, and G is the moment of the effective couple and C is the moment of inertia relative to that axis; then

$$\text{G} = n\text{C}. \qquad (246)$$

Let L and M be the moments of the impressed couples relative to the axes of x and y in the plane of the earth's equator; as the position of these axes in that plane is indeterminate, and as we shall consider the effects for only the time dt, we may suppose the axis of x to lie along the line of equinoxes, and the axis of y to be perpendicular to it. Thus (213) and (214) become

$$\frac{x}{r} = \cos n't, \quad \frac{y}{r} = \cos \text{I} \sin n't, \quad \frac{z}{r} = -\sin \text{I} \sin n't; \qquad (247)$$

* On this subject see the Memoir entitled "Numerus constans Nutationis ex ascensionibus rectis stellæ polaris in Speculâ Dorpatensis annis 1822 ad 1838 observatis deductus," by C. F. Peters, and contained in "Mémoirs de l'Académie Impérial des Sciences de Saint Pétersbourg, 6e série, premiere partie, Sciences mathématiques et physiques, Tome III, Saint Pétersbourg, 1844."

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$$\left.\begin{aligned}\frac{x'}{r'} &= \cos(\mu+\nu t),\\ \frac{y'}{r'} &= \cos\text{I}\sin(\mu+\nu t)+i\sin\text{I}\sin(\phi'_0+n''t),\\ \frac{z'}{r'} &= -\sin\text{I}\sin(\mu+\nu t)+i\cos\text{I}\sin(\phi'_0+n''t);\end{aligned}\right\} \quad (248)$$

and therefore

$$\text{L} = \frac{-3n'^2}{2}(\text{C}-\text{A})\sin\text{I}\cos\text{I}(1-\cos 2n't), \quad (249)$$

$$\text{M} = \frac{3n'^2}{2}(\text{C}-\text{A})\sin\text{I}\sin 2n't; \quad (250)$$

and if we omit the terms in L′ and M′, which involve the angle $2\phi'_0+\psi'_0+(\beta+2n'')t$, because the coefficient is small and does not rise into importance in the subsequent integration,

$$\text{L}' = \frac{3n''^2(\text{C}-\text{A})}{2(1+e)}\left\{-\sin\text{I}\cos\text{I}\{1-\cos 2(\mu+\nu t)\}+i\cos 2\text{I}\cos(\psi'_0+\beta t)\right\}, \quad (251)$$

$$\text{M}' = \frac{3n''^2(\text{C}-\text{A})}{2(1+e)}\left\{\sin\text{I}\sin 2(\mu+\nu t)+i\cos\text{I}\sin(\psi'_0+\beta t)\right\}; \quad (252)$$

the moments of these couples in the time dt are severally $\text{L}\,dt$, $\text{M}\,dt$, $\text{L}'dt$, and $\text{M}'dt$; and besides them we have also the couple G. Their effects are to be considered.

The axis of $\text{L}\,dt$ is the line of equinoxes, and as the axis of G is perpendicular (approximately) to the plane of the equator, it is perpendicular to this line. Consequently the axis of the resultant of $\text{L}\,dt$ and G is the diagonal of the rectangle, of which the line-representatives meeting at the earth's centre are the adjacent sides: let OG be this diagonal; and let $d\lambda$ be the angle at which it is inclined to the axis of G, $d\lambda$ being necessarily infinitesimal because $\text{L}\,dt$ is infinitesimal and G is finite. And thus

$$\begin{aligned}d\lambda &= \frac{\text{L}\,dt}{\text{G}}\\ &= -\frac{3n'^2}{2n}\,\frac{\text{C}-\text{A}}{\text{C}}\sin\text{I}\cos\text{I}(1-\cos 2n't)\,dt. \quad (253)\end{aligned}$$

As $d\lambda$ lies in the plane which contains the axes of L and G, the axis about which the body revolves through $d\lambda$ is the line in the plane of the equator perpendicular to the line of equinoxes; this infinitesimal rotation therefore will not produce an appreciable change of obliquity, but only a change of position of the line of equinoxes; and if $d\psi$ is that angle,

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$$d\psi = \frac{d\lambda}{\sin \mathrm{I}}$$

$$= -\frac{3n'^2}{2n}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\cos \mathrm{I}\,\{1-\cos 2n't\}\,dt; \qquad (254)$$

$$\therefore \quad \psi = -\frac{3n'^2}{2n}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\cos \mathrm{I}\left\{t-\frac{\sin 2n't}{2n'}\right\}; \qquad (255)$$

which result is the same as (224).

As the axis of L' is the same as that of L, it may be treated in the same way relatively to G; thus, if ψ' is the angle of precession due to the effect of $\mathrm{L}'dt$, from (251) we have

$$\psi' = \frac{3n''^2}{2n(1+e)}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\left\{-\cos \mathrm{I}\,t+\frac{\cos \mathrm{I}}{2\nu}\sin 2(\mu+\nu t)+\frac{i\cos 2\mathrm{I}}{\beta \sin \mathrm{I}}\sin(\psi'_0+\beta t)\right\}; \qquad (256)$$

and replacing μ and ν by their values, and omitting small quantities, this becomes

$$\psi' = -\frac{3n''^2\cos \mathrm{I}}{2n(1+e)}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}t+\frac{3n''\cos \mathrm{I}}{4n(1+e)}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\left\{\sin 2\{\phi'_0+\psi'_0+(\beta+n'')t\}\right.$$
$$\left.+\frac{4n''\cos 2\mathrm{I}}{\sin 2\mathrm{I}}\frac{i}{\beta}\sin(\psi'_0+\beta t)\right\}; \qquad (257)$$

which is precisely the same result as (237). The sum of (255) and (257) is the total luni-solar precession, and nutation of the equinoxes.

Next let us consider the effects of M and M'. Since the axis of M is in the plane of the equator, and perpendicular to the line of equinoxes, the rotation-axis of the resultant of $\mathrm{M}dt$ and G is in the plane perpendicular to the line of equinoxes; and if $d\mu$ is the angle at which the axis of this new couple is inclined to that of G,

$$d\mu = \frac{\mathrm{M}dt}{\mathrm{G}}; \qquad (258)$$

as $d\mu$ lies in the plane of the axes of G and $\mathrm{M}dt$, this shifting of the rotation-axis is equivalent to a rotation of the body through a small angle $d\mu$ about the line of equinoxes; but hereby θ will be diminished by $d\theta$; so that $d\mu = -d\theta$;

$$\therefore \quad d\theta = -\frac{\mathrm{M}dt}{\mathrm{G}}$$

$$= -\frac{3n'^2}{2n}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\sin \mathrm{I}\sin 2n't\,dt;$$

$$\therefore \quad \theta = \mathrm{I}+\frac{3n'\sin \mathrm{I}}{4n}\frac{\mathrm{C}-\mathrm{A}}{\mathrm{C}}\cos 2n't, \qquad (259)$$

which is the same result as (226).

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As the axis of M' is the same as that of M, it may be combined in the same way with G; and if θ' is the obliquity due to the action of $\text{M}'dt$,

$$\theta' = \text{I} + \frac{3n''^2}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\left\{\sin\text{I}\,\frac{\cos 2(\mu+\nu t)}{2\nu} + i\cos\text{I}\,\frac{\cos(\psi'_0+\beta t)}{\beta}\right\}$$

$$= \text{I} + \frac{3n''}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\left\{\frac{\sin\text{I}}{2}\cos 2\{\phi'_0+\psi'_0+(\beta+n'')t\}\right.$$

$$\left. + n''\cos\text{I}\,\frac{i}{\beta}\cos(\psi'_0+\beta t)\right\}; \quad (260)$$

which is the same result as (238). And thus the addition of (255) and (257) will give (240); and that of (259) and (260) will give (241). But it is of course unnecessary to repeat them.

An account, with great exactness, of the effects of all the terms in the lunar and solar precession and nutation, will be found in the Memoir of M. Poinsot; but it would be out of place to insert it here.

181.] It remains for us still to examine the pressure borne by the fixed point of the body through which the rotation-axis always passes.

The pressure P, as well as the direction-cosines of its line of action, are to be determined by means of equations (24) or (26), Art. 148.

Let us refer the line of pressure to the principal axes fixed in the moving body; let $\bar{\text{M}}$ be the mass of the body, and $(\bar{x}, \bar{y}, \bar{z})$ the place of its centre of gravity at the time t; then from (26), Art. 148,

$$\left.\begin{aligned}
\text{P}\cos\lambda &= \Sigma.m\text{X} - \bar{\text{M}}\left\{\bar{z}\frac{d\omega_2}{dt} - \bar{y}\frac{d\omega_3}{dt}\right\} - \bar{\text{M}}\omega_1(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + \bar{\text{M}}\omega^2\bar{x},\\
\text{P}\cos\mu &= \Sigma.m\text{Y} - \bar{\text{M}}\left\{\bar{x}\frac{d\omega_3}{dt} - \bar{z}\frac{d\omega_1}{dt}\right\} - \bar{\text{M}}\omega_2(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + \bar{\text{M}}\omega^2\bar{y},\\
\text{P}\cos\nu &= \Sigma.m\text{Z} - \bar{\text{M}}\left\{\bar{y}\frac{d\omega_1}{dt} - \bar{x}\frac{d\omega_2}{dt}\right\} - \bar{\text{M}}\omega_3(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + \bar{\text{M}}\omega^2\bar{z};
\end{aligned}\right\}(261)$$

of these equations we have the following particular results.

If the centre of gravity is the origin, $\bar{x} = \bar{y} = \bar{z} = 0$; therefore

$$\text{P}\cos\lambda = \Sigma.m\text{X}, \qquad \text{P}\cos\mu = \Sigma.m\text{Y}, \qquad \text{P}\cos\nu = \Sigma.m\text{Z}; \quad (262)$$

that is, the pressure at the origin is due to the impressed forces only.

And if the body is not subject to the action of any force, then $\text{P} = 0$, and there is no pressure at the fixed point.

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If no forces act, so that $\Sigma . m\text{X} = \Sigma . m\text{Y} = \Sigma . m\text{Z} = 0$; and consequently $\text{L} = \text{M} = \text{N} = 0$; then, replacing $\frac{d\omega_1}{dt}, \frac{d\omega_2}{dt}, \frac{d\omega_3}{dt}$ by their values in (29), Art. 149, we have

$$\left.\begin{aligned}
\text{P}\cos\lambda &= \bar{\text{M}}\,\bar{x}(\omega_2^2+\omega_3^2) + \bar{\text{M}}(\text{A}-\text{B}-\text{C})\,\omega_1\left\{\frac{\omega_2\bar{y}}{\text{C}} + \frac{\omega_3\bar{z}}{\text{B}}\right\},\\
\text{P}\cos\mu &= \bar{\text{M}}\,\bar{y}(\omega_3^2+\omega_1^2) + \bar{\text{M}}(\text{B}-\text{C}-\text{A})\,\omega_2\left\{\frac{\omega_3\bar{z}}{\text{A}} + \frac{\omega_1\bar{x}}{\text{C}}\right\},\\
\text{P}\cos\nu &= \bar{\text{M}}\,\bar{z}(\omega_1^2+\omega_2^2) + \bar{\text{M}}(\text{C}-\text{A}-\text{B})\,\omega_3\left\{\frac{\omega_1\bar{x}}{\text{B}} + \frac{\omega_2\bar{y}}{\text{A}}\right\}.
\end{aligned}\right\} \quad (263)$$

Let us consider what properties are involved in these equations when $\text{P} = 0$. If

$$\frac{\text{B}-\text{C}}{\text{A}\,\omega_1} = a_1, \qquad \frac{\text{C}-\text{A}}{\text{B}\,\omega_2} = a_2, \qquad \frac{\text{A}-\text{B}}{\text{C}\,\omega_3} = a_3, \quad (264)$$

these equations take the symmetrical forms

$$\left.\begin{aligned}
\omega^2\bar{x} - \omega_1(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + (a_3\bar{y}-a_2\bar{z})\,\omega_1\omega_2\omega_3 &= 0,\\
\omega^2\bar{y} - \omega_2(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + (a_1\bar{z}-a_3\bar{x})\,\omega_1\omega_2\omega_3 &= 0,\\
\omega^2\bar{z} - \omega_3(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + (a_2\bar{x}-a_1\bar{y})\,\omega_1\omega_2\omega_3 &= 0;
\end{aligned}\right\} \quad (265)$$

multiplying these severally by $\bar{x}, \bar{y}, \bar{z}$, and adding, we have

$$(\omega_3\bar{y}-\omega_2\bar{z})^2 + (\omega_1\bar{z}-\omega_3\bar{x})^2 + (\omega_2\bar{x}-\omega_1\bar{y})^2 = 0;$$

$$\therefore \quad \frac{\bar{x}}{\omega_1} = \frac{\bar{y}}{\omega_2} = \frac{\bar{z}}{\omega_3}; \quad (266)$$

and replacing $\omega_1, \omega_2, \omega_3$ in (265) by the proportionals given in these last equations, we have

$$\frac{\text{A}\bar{x}^2}{\text{B}-\text{C}} = \frac{\text{B}\bar{y}^2}{\text{C}-\text{A}} = \frac{\text{C}\bar{z}^2}{\text{A}-\text{B}}; \quad (267)$$

which can only be satisfied, if $\bar{x} = \bar{y} = \bar{z} = 0$; so that whenever the equations (265) are true, the centre of gravity is the origin.

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CHAPTER VII.

THE MOTION OF A RIGID BODY, OR OF AN INVARIABLE MATERIAL SYSTEM, FREE FROM ALL CONSTRAINT.

SECTION 1.—*Motion of a free invariable system under the action of instantaneous forces.*

182.] As our inquiry proceeds, our problem becomes more general; and the conditions of constraint become fewer. The subject of motion is still a rigid body, or a system of particles of invariable form; and thus all the internal forces, which enter into the equations of motion in the most general problem, in this disappear, because they are introduced in pairs neutralizing each other. We return therefore to the equations which are given in Chap. III, Art. 48; viz., (34) and (35) which are applicable to instantaneous forces; and (37) and (38) which express the action of finite accelerating forces.

In the solution of the problem we shall find it most convenient to employ the principle of the independence of the motion of translation of the centre of gravity, and of the rotation about an axis passing through it, which has been proved in Section 2 of Chapter III. For we shall thereby resolve into two distinct parts complicated motion which arises from the action of given forces; we shall consider the forces as they produce either simple translation or rotation, and shall investigate their effects; and the whole motion will be the result of these two separate motions. And the process too is most convenient for the course taken in our treatise; for the motion of a free invariable system is thus resolved into that of simple translation of a particle at its centre of gravity, and that of rotation about an axis passing through the centre of gravity considered as a fixed point; motion of the former kind has been completely discussed in Vol. III; and that of the latter, as far as is possible, in the Chapter preceding the present; and we have investigated these as the effects of forces similar to those which we have now to consider.

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This mode of resolution is most convenient for a dynamical reason also: because all the forces which act on the several particles of the system may be transferred, each in its own line of action, direction, and intensity, to the centre of gravity; and may there act on a particle of mass equal to that of the whole system; and the motion of the centre of gravity will be that of the particle under the action of the forces thus transferred; and because the rotation relative to the axis passing through the centre of gravity, which we may suppose to be a fixed point, is the effect of the forces, as they act at their several points of application. At no point, except the centre of gravity, or centre of masses, are these dynamical propositions true. And an examination of what has preceded shews the reason of this. The centrifugal forces generated in the motion neutralize themselves at the centre of gravity; they produce thereon no pressure; and thus cause no acceleration or retardation of it: whatever is the pressure at, or the motion of, the centre of gravity, this is due to the impressed forces, and to them alone. This fact has been presented to us again and again in the course of our work.

Cinematically indeed other modes of estimating motion might have been taken. In Chapter II it has been proved that whatever is the motion of a body, it always consists of a motion of translation of *any* particle of it along a definite path, and of a motion of rotation about an axis passing through that particle; and the choice of the particle whose motion of translation is considered is arbitrary. And when force acts on the body, the effect of it, in combination with the centrifugal forces developed in the motion, will be to change the line of motion and velocity of the particle, the rotation-axis of the body passing through that particle, and its angular velocity about that axis; and the equations of motion will be formed in a manner which indicates these several changes: these we shall hereafter exhibit. Or, again, the motion which takes place in an infinitesimal time-element always consists of a rotation about the central axis, and of a sliding or of a motion of translation along that axis; and the effects of the impressed forces and of the centrifugal forces developed in the motion will be a shifting of the central axis, a change of velocity along it, and a change of angular velocity about it; and as the shifting of the central axis may take place in the most general way possible, so will it consist of both a displacement of translation and of a subsequent rotation about

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one of its points, so that the central axes in the new and the old positions do not intersect each other; and these four separate effects will be produced by the acting forces; they will be exhibited therefore in the equations of motion, which will evidently be greatly complicated; and the acting forces will have to be resolved along lines, the position of which is continually changing.

It is right to say thus much as to other modes of considering the motion of a free invariable system; although we shall for the most part confine ourselves to that motion of the centre of gravity, and of rotation about it, to which our equations of motion most conveniently adapt themselves.

183.] In the present Section I shall consider the motion of a free invariable system under the action of instantaneous forces. In the notation of Art. 48, the general equations of motion applicable to this problem are

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{v}_x-\frac{dx}{dt}\right) &= 0,\\ \Sigma.m\left(\mathrm{v}_y-\frac{dy}{dt}\right) &= 0,\\ \Sigma.m\left(\mathrm{v}_z-\frac{dz}{dt}\right) &= 0;\end{aligned}\right\} \tag{1}$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{v}_z-\frac{dz}{dt}\right)-z\left(\mathrm{v}_y-\frac{dy}{dt}\right)\right\} &= 0,\\ \Sigma.m\left\{z\left(\mathrm{v}_x-\frac{dx}{dt}\right)-x\left(\mathrm{v}_z-\frac{dz}{dt}\right)\right\} &= 0,\\ \Sigma.m\left\{x\left(\mathrm{v}_y-\frac{dy}{dt}\right)-y\left(\mathrm{v}_x-\frac{dx}{dt}\right)\right\} &= 0.\end{aligned}\right\} \tag{2}$$

Now if $\bar{\mathrm{M}}$ is the mass of the moving system, and if $(\bar{x}, \bar{y}, \bar{z})$ is the place of the centre of gravity at the time t, and (x', y', z') is the place of m relative to a system of axes originating at the centre of gravity, and parallel to the original axes, then, by means of Section 2, Chap. III, the equivalents of these are

$$\left.\begin{aligned} \bar{\mathrm{M}}\frac{d\bar{x}}{dt} &= \Sigma.m\mathrm{v}_x,\\ \bar{\mathrm{M}}\frac{d\bar{y}}{dt} &= \Sigma.m\mathrm{v}_y,\\ \bar{\mathrm{M}}\frac{d\bar{z}}{dt} &= \Sigma.m\mathrm{v}_z;\end{aligned}\right\} \tag{3}$$

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and
$$\left.\begin{aligned} \Sigma.m\left\{y'\left(\mathrm{v}_z-\frac{dz'}{dt}\right)-z'\left(\mathrm{v}_y-\frac{dy'}{dt}\right)\right\} &= 0,\\ \Sigma.m\left\{z'\left(\mathrm{v}_x-\frac{dx'}{dt}\right)-x'\left(\mathrm{v}_z-\frac{dz'}{dt}\right)\right\} &= 0,\\ \Sigma.m\left\{x'\left(\mathrm{v}_y-\frac{dy'}{dt}\right)-y'\left(\mathrm{v}_x-\frac{dx'}{dt}\right)\right\} &= 0. \end{aligned}\right\} \qquad (4)$$

Now (3) express the motion of the centre of gravity relative to a system of axes fixed in space, and do not generally admit of farther reduction. Whereas (4) express the motion of rotation of the body, or material system of invariable form, relative to a system of axes originating at the centre of gravity, and in other respects undetermined. If Ω_x, Ω_y, Ω_z are the axial components of the instantaneous angular velocity Ω which is due to the impulsive forces, these equations become, see Art. 76,

$$\left.\begin{aligned} \mathrm{A}\,\Omega_x - \mathrm{F}\,\Omega_y - \mathrm{E}\,\Omega_z &= \mathrm{L},\\ -\mathrm{F}\,\Omega_x + \mathrm{B}\,\Omega_y - \mathrm{D}\,\Omega_z &= \mathrm{M},\\ -\mathrm{E}\,\Omega_x - \mathrm{D}\,\Omega_y + \mathrm{C}\,\Omega_z &= \mathrm{N}; \end{aligned}\right\} \qquad (5)$$

where L, M, N are the axial components of the moment of the couple or couples due to the forces of impulsion.

If, as in the last Chapter, we refer the rotation to two systems of axes, one of which is the central principal system fixed in the body and moving with it, and the other is a system fixed in space, then, in reference to the former, $\mathrm{D} = \mathrm{E} = \mathrm{F} = 0$, and (5) assume the simple form

$$\mathrm{A}\,\Omega_1 = \mathrm{L}, \qquad \mathrm{B}\,\Omega_2 = \mathrm{M}, \qquad \mathrm{C}\,\Omega_3 = \mathrm{N}; \qquad (6)$$

where L, M, and N are the axial components of the moments of the couples of impulsion relative to the central principal axes.

The solution of every problem depends on these equations; they will hereafter be applied to particular cases; but some corollaries from them require investigation.

184.] When a free material system of variable or invariable form, which was at rest, has been acted on by one or more impulsive forces, the preceding equations express the momentum which has been communicated to it; and if no other momentum has been imparted, and if no momentum is lost, they also express the whole momentum of the system, and consequently that which is capable of abstraction from it. This fact, which is a consequence of the law of inertia, is commonly termed the principle of conservation of momentum or of force. Now when a material system has been put into motion by one or more forces of translation, these will generally be reducible to a single

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force; and therefore a single force equal to this, and acting in the opposite direction, will bring the system to rest. If the system has been put into motion by a couple, no one force generally can bring it to rest, because no one definite force acting at a definite distance can neutralize a couple. Generally, I say, in both cases; because in the course of our inquiry it will be seen that under certain circumstances, which will be expressed in an equation of condition, a single force of impulsion, or moving forces which will compound into a single resultant, will cause a system to rotate about an instantaneous axis, either with or without a motion of translation along that axis.

As the position of the axes fixed in space is arbitrary, let us take them to be parallel to the central principal axes of the body at the instant when the impulsive forces act. Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the axial components of the impressed momenta; and let $\bar{\mathrm{M}}$ be the mass of the system; then the axial components of the resulting velocity of the centre of gravity are

$$\frac{\mathrm{X}}{\bar{\mathrm{M}}}, \qquad \frac{\mathrm{Y}}{\bar{\mathrm{M}}}, \qquad \frac{\mathrm{Z}}{\bar{\mathrm{M}}}; \tag{7}$$

let v be the velocity of the particle at the place (x, y, z) relative to the central principal axes; and let v_x, v_y, v_z be the axial components of v; then, as $\Omega_1, \Omega_2, \Omega_3$ are the axial components of Ω relative to these central axes,

$$\left.\begin{aligned} v_x &= \frac{\mathrm{X}}{\bar{\mathrm{M}}} + z\,\Omega_2 - y\,\Omega_3, \\ v_y &= \frac{\mathrm{Y}}{\bar{\mathrm{M}}} + x\,\Omega_3 - z\,\Omega_1, \\ v_z &= \frac{\mathrm{Z}}{\bar{\mathrm{M}}} + y\,\Omega_1 - x\,\Omega_2; \end{aligned}\right\} \tag{8}$$

and if $\Omega_1, \Omega_2, \Omega_3$ are replaced by their values given in (6),

$$\left.\begin{aligned} v_x &= \frac{\mathrm{X}}{\bar{\mathrm{M}}} + \frac{z\,\mathrm{M}}{\mathrm{B}} - \frac{y\,\mathrm{N}}{\mathrm{C}}, \\ v_y &= \frac{\mathrm{Y}}{\bar{\mathrm{M}}} + \frac{x\,\mathrm{N}}{\mathrm{C}} - \frac{z\,\mathrm{L}}{\mathrm{A}}, \\ v_z &= \frac{\mathrm{Z}}{\bar{\mathrm{M}}} + \frac{y\,\mathrm{L}}{\mathrm{A}} - \frac{x\,\mathrm{M}}{\mathrm{B}}; \end{aligned}\right\} \tag{9}$$

which give the components of the velocity of any particle of the system, and consequently the velocity thereof.

185.] For the first application of these equations, I propose to determine the locus of all points of the system which move at the first instant with the same velocity v.

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Let v_x, v_y, v_z be the axial components of v; then, if (x, y, z) is the place of a particle which moves with this given velocity,

$$\left.\begin{aligned} v_x &= \frac{X}{\bar{M}} + z\,\Omega_2 - y\,\Omega_3, \\ v_y &= \frac{Y}{\bar{M}} + x\,\Omega_3 - z\,\Omega_1, \\ v_z &= \frac{Z}{\bar{M}} + y\,\Omega_1 - x\,\Omega_2; \end{aligned}\right\} \tag{10}$$

$$\therefore \quad v^2 = \left(\frac{X}{\bar{M}} + z\Omega_2 - y\,\Omega_3\right)^2 + \left(\frac{Y}{\bar{M}} + x\,\Omega_3 - z\,\Omega_1\right)^2 + \left(\frac{Z}{\bar{M}} + y\,\Omega_1 - x\,\Omega_2\right)^2; \tag{11}$$

which is evidently the equation to a cylinder of the second order, since it is of the form given in (33), Art. 356, Vol. I, 2nd. Ed.; all the generating lines of which are parallel to that whose equations are

$$\frac{x}{\Omega_1} = \frac{y}{\Omega_2} = \frac{z}{\Omega_3}. \tag{12}$$

If not only v is constant for all points of the required locus, but v_x, v_y, v_z, which are the axial components of v, are also constant; then each equation of (10) separately holds good; and the locus is evidently a straight line whose equations are

$$\frac{x + \dfrac{Y\,\Omega_3 - Z\,\Omega_2}{\bar{M}\,\Omega^2} - \dfrac{v_y\,\Omega_3 - v_z\,\Omega_3}{\Omega^2}}{\Omega_1} = \ldots = \ldots; \tag{13}$$

Equations (10) however are not independent; but are evidently subject to the condition

$$\bar{M}\,(v_x\,\Omega_1 + v_y\,\Omega_2 + v_z\,\Omega_3) = X\,\Omega_1 + Y\,\Omega_2 + Z\,\Omega_3; \tag{14}$$

and consequently instantaneous forces can produce the preceding effect only when this condition is satisfied.

Again, let us suppose the motion which results from the instantaneous forces to consist of a rotation about an instantaneous axis, and of a translation along that axis. In this case, if x, y, z are the current coordinates of the rotation-axis, v_x, v_y, v_z are constant for all points along that axis, and consequently (13) are the equations to the axis, and its direction-cosines are proportional to $\Omega_1, \Omega_2, \Omega_3$; and as v is the velocity of a particle in and along that axis,

$$\frac{v_x}{\Omega_1} = \frac{v_y}{\Omega_2} = \frac{v_z}{\Omega_3} = \frac{v}{\Omega}; \tag{15}$$

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in which case the condition (14) becomes

$$\bar{\mathrm{M}}\,\Omega\,\mathrm{V} = \mathrm{X}\,\Omega_1 + \mathrm{Y}\,\Omega_2 + \mathrm{Z}\,\Omega_3\,; \tag{16}$$

so that

$$\mathrm{V} = \frac{\mathrm{X}\,\Omega_1 + \mathrm{Y}\,\Omega_2 + \mathrm{Z}\,\Omega_3}{\Omega\,\bar{\mathrm{M}}}\,; \tag{17}$$

which shews that the velocity of displacement along the rotation-axis is the sum of the resolved parts of the components of the velocity of the centre of gravity along that axis.

The equations to the axis are

$$\frac{x + \dfrac{\mathrm{Y}\,\Omega_3 - \mathrm{Z}\,\Omega_2}{\bar{\mathrm{M}}\,\Omega^2}}{\Omega_1} = \ldots = \ldots \tag{18}$$

186.] Lastly, let us suppose the system to rotate about an axis without any motion of translation; so that for all points along that axis $v_x = v_y = v_z = 0$; then from (8), if (x, y, z) is a point on that axis, we have

$$\left.\begin{aligned} \frac{\mathrm{X}}{\bar{\mathrm{M}}} + z\,\Omega_2 - y\,\Omega_3 &= 0, \\ \frac{\mathrm{Y}}{\bar{\mathrm{M}}} + x\,\Omega_3 - z\,\Omega_1 &= 0, \\ \frac{\mathrm{Z}}{\bar{\mathrm{M}}} + y\,\Omega_1 - x\,\Omega_2 &= 0\,; \end{aligned}\right\} \tag{19}$$

which are the equations to the axis; and may be expressed in the form given in (18). This axis is called the spontaneous axis of rotation. Its direction-cosines are evidently proportional to $\Omega_1, \Omega_2, \Omega_3$, so that it is parallel to what would be the instantaneous axis, if the centre of gravity were a fixed point. It passes through the centre of gravity, if $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$; which will be the case when the impressed forces of impulsion compound into a couple.

Equations (19) are not independent, but are subject to the equation of condition,

$$\mathrm{X}\,\Omega_1 + \mathrm{Y}\,\Omega_2 + \mathrm{Z}\,\Omega_3 = 0\,; \tag{20}$$

which shews that the line of action of the resultant of the impressed momenta is perpendicular to the instantaneous rotation-axis, which is the spontaneous axis. This condition is the same as that determined in Art. 34, and numbered (52).

If in (20) $\Omega_1, \Omega_2, \Omega_3$ are replaced by their values given in (6), the condition for the existence of a spontaneous axis becomes,

$$\frac{\mathrm{L}\,\mathrm{X}}{\mathrm{A}} + \frac{\mathrm{M}\,\mathrm{Y}}{\mathrm{B}} + \frac{\mathrm{N}\,\mathrm{Z}}{\mathrm{C}} = 0. \tag{21}$$

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If the motion is due to a single blow applied at the point (ξ, η, ζ) whose momentum is Q, of which X, Y, Z are the axial components; then

$$L = \eta Z - \zeta Y, \qquad M = \zeta X - \xi Z, \qquad N = \xi Y - \eta X; \tag{22}$$

and (21) becomes after division by X, Y, Z,

$$\frac{\xi}{X}\left(\frac{1}{B} - \frac{1}{C}\right) + \frac{\eta}{Y}\left(\frac{1}{C} - \frac{1}{A}\right) + \frac{\zeta}{Z}\left(\frac{1}{A} - \frac{1}{B}\right) = 0; \tag{23}$$

so that (20), (21) or (23) is the condition to be satisfied, when a body being struck by a blow rotates at the first instant of its motion about an axis, without any other motion of translation.

I may in passing observe, that the necessary condition is satisfied when the line of action of the blow is parallel to a central principal axis, say that of z; because in that case $X = Y = 0$; and $\Omega_3 = 0$. These circumstances however will be hereafter considered at length.

187.] When the impulsive forces which act on a given system satisfy the condition (21), so that the system rotates about a spontaneous axis, that axis has the following property: the sum of the vires vivæ of all the particles of the system due to the impulsive forces is greater for that axis than it would be for any other rotation-axis. So that the spontaneous axis may be defined as that axis for which the sum of the vires vivæ of all the particles due to the impulsive forces is greater than for any other axis.

Although the theorem is true for any system, I will, for the sake of simplicity, confine the proof in this Article to that of an invariable system, the components of the velocity of any particle of which are given in equations (8); at least as to form: for herein we must assume the position of the rotation-axis to be undetermined, and we shall shew that the vis viva is a maximum when the rotation-axis coincides with the spontaneous axis.

Let ω be the angular velocity about the assumed axis which passes through the centre of gravity; and let $\omega_1, \omega_2, \omega_3$ be the axial components of ω; then

$$\left.\begin{aligned} v_x &= \frac{X}{M} + z\,\omega_2 - y\,\omega_3, \\ v_y &= \frac{Y}{M} + x\,\omega_3 - z\,\omega_1, \\ v_z &= \frac{Z}{M} + y\,\omega_1 - x\,\omega_2. \end{aligned}\right\} \tag{24}$$

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As the change of the velocity is supposed to be due to a change only in the position of the rotation-axis,

$$\left.\begin{aligned} dv_x &= z\,d\omega_2 - y\,d\omega_3, \\ dv_y &= x\,d\omega_3 - z\,d\omega_1, \\ dv_z &= y\,d\omega_1 - x\,d\omega_2. \end{aligned}\right\} \qquad (25)$$

Let X, Y, Z be now the components of the velocity impressed on m; then the equation of motion given by D'Alembert's Principle, in combination with the principle of virtual velocities, is

$$\Sigma.m\{(\mathrm{X}-v_x)\,\delta x + (\mathrm{Y}-v_y)\,\delta y + (\mathrm{Z}-v_z)\,\delta z\} = 0. \qquad (26)$$

In this problem δx, δy, δz may be replaced by the actual velocities v_x, v_y, v_z, whereby we have,

$$\Sigma.m(v_x^2 + v_y^2 + v_z^2) = \Sigma.m(\mathrm{X}v_x + \mathrm{Y}v_y + \mathrm{Z}v_z). \qquad (27)$$

Let the vis viva thus generated $= \mathrm{U}$; then

$$\begin{aligned} \mathrm{U} &= \Sigma.mv^2, \\ &= \Sigma.m(v_x^2 + v_y^2 + v_z^2); \\ \therefore\quad \mathrm{D}\mathrm{U} &= 2\Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z); \end{aligned} \qquad (28)$$

and we have to shew that by reason of (27) this quantity vanishes.

From (27) we have

$$\begin{aligned} &2\Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z) \\ &\quad = \Sigma.m\{\mathrm{X}\,dv_x + \mathrm{Y}\,dv_y + \mathrm{Z}\,dv_z\} \\ &\quad = \Sigma.m\{(y\mathrm{Z} - z\mathrm{Y})\,d\omega_1 + (z\mathrm{X} - x\mathrm{Z})\,d\omega_2 + (x\mathrm{Y} - y\mathrm{X})\,d\omega_3\}; \end{aligned}$$

and as $d\omega_1$, $d\omega_2$, $d\omega_3$ are independent of x, y, z, X, Y, and Z in the right hand member, we may substitute from (35), Article 48, observing that $\mathrm{V}_x, \mathrm{V}_y, \mathrm{V}_z$, $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are severally X, Y, Z, v_x, v_y, v_z in our present notation; so that we have

$$\begin{aligned} &2\Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z) \\ &\quad = \Sigma.m\{(yv_z - zv_y)\,d\omega_1 + (zv_x - xv_z)\,d\omega_2 + (xv_y - yv_x)\,d\omega_3\}, \\ &\quad = \Sigma.m\{(z\,d\omega_2 - y\,d\omega_3)\,v_x + (x\,d\omega_3 - z\,d\omega_1)\,v_y + (y\,d\omega_1 - x\,d\omega_2)\,v_z\}, \\ &\quad = \Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z); \\ &\qquad \therefore\quad \Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z) = 0; \end{aligned} \qquad (29)$$

and therefore from (28), $\mathrm{D}\mathrm{U} = 0$; and consequently U is a maximum or a minimum.

And it is evidently a maximum; for if we give to v_x, v_y, v_z increments dv_x, dv_y, dv_z of such a finite magnitude that their squares are not to be neglected, then the right-hand member of (28) becomes

$$2\Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z) + \Sigma.m\{(dv_x)^2 + (dv_y)^2 + (dv_z)^2\};$$

and we shall eventually have

$$\Sigma.m(v_x dv_x + v_y dv_y + v_z dv_z) + \Sigma.m\{(dv_x)^2 + (dv_y)^2 + (dv_z)^2\} = 0; \qquad (30)$$

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that is, the increment of the vis viva for the finite variation is less than it is for the infinitesimal variation by

$$\Sigma . m \{(dv_x)^2 + (dv_y)^2 + (dv_z)^2\};$$

that is, by the sum of the vires vivæ due to the velocities lost by the different particles of the system; and consequently the vis viva determined as above is a maximum.

The preceding proof of this theorem is due to Lagrange*; and the proof that the vis viva corresponding to the spontaneous axis is a maximum is due to his editor, M. Bertrand. The theorem was originally discovered by Euler†, and restated by Lagrange; and although the proof given by the latter holds true for a material system of invariable form, yet his mode of expression is so obscure, that it is almost impossible to understand his meaning when it is applied to a system of variable form. Another proof is given by M. Delaunay‡, and this is sufficient for all material systems.

Let U be the vis viva of the system arising from the angular velocities due to the impulsive forces, then

$$U = A\omega_1^2 + B\omega_2^2 + C\omega_3^2.$$

And if l, m, n are the direction-cosines of the undetermined axis, this becomes

$$U = (Al^2 + Bm^2 + Cn^2)\omega^2;$$

$$= \frac{(Ll + Mm + Nn)^2}{Al^2 + Bm^2 + Cn^2}; \tag{31}$$

and equating to zero the total differential of this, we have

$$(BLm^2 + CLn^2 - AMlm - ANln)dl + \ldots + \ldots = 0;$$

also,

$$l\,dl + m\,dm + n\,dn = 0;$$

whence we have

$$\frac{L}{Al} = \frac{M}{Bm} = \frac{N}{Cn}; \tag{32}$$

and therefore by means of (6)

$$\frac{l}{\Omega_1} = \frac{m}{\Omega_2} = \frac{n}{\Omega_3}; \tag{33}$$

so that the vis viva is a maximum or a minimum when the rotation-axis through the centre of gravity is parallel to the spontaneous axis.

188.] In the concluding paragraph of Art. 186 I have ob-

* See Mécanique Analytique, tome I, p. 271, ed. 3, par M. J. Bertrand, Paris, 1853.

† Theoria Motus corporum solidorum, cap. IX, Theorema 8 (Art. 637.) Gryphiswaldiæ, 1790.

‡ Liouville's Journal, tome V, p. 255.

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served, that if a body is put into motion by a single blow, the condition necessary for the existence of a spontaneous axis is satisfied whenever the line of action of the blow is parallel to a central principal axis. I propose to consider this circumstance more at length. The investigation is for the most part due to Poinsot*; and although fuller explanation will be found in his Memoirs than I am able to insert in this place, still all his important theorems are here given.

In the first place, let us assume the line of action of the blow to be not only parallel to a central principal axis, but also to lie in a central principal plane. Let Q = the momentum of the blow; let the central principal plane in which its line of action lies be that of (x, y), and let the line of action be parallel to the axis of y; these several lines are represented in Fig. 39; where G is the centre of gravity, and is the origin; $\mathrm{G}x$, $\mathrm{G}y$, $\mathrm{G}z$ are the three central principal axes; CQ is the line of action of the blow Q, and C is the point at which it intersects $\mathrm{G}x$; C is called the centre of percussion†. Let $\mathrm{GC} = h$; the mass of the body $= m$; $mk^2 = \mathrm{C}$ = its central principal moment of inertia about the axis of z; Ω = the angular velocity which is due to Q; then, in this case,

$$\mathrm{X} = \mathrm{Z} = 0; \qquad \mathrm{Y} = \mathrm{Q};$$

$$\Omega_1 = \Omega_2 = 0; \qquad \Omega_3 = \frac{\mathrm{Q}h}{\mathrm{C}} = \Omega; \tag{34}$$

and thus the equations (19) to the spontaneous axis are

$$y = 0; \qquad z = \frac{0}{0}; \qquad x = -\frac{\mathrm{C}}{mh} = -\frac{k^2}{h}; \tag{35}$$

which represent a line parallel to the axis of z, and intersecting the axis of x at a distance $= \frac{k^2}{h}$ on the negative side from the origin: this is the line OR in Fig. 39, which is therefore the spontaneous axis; the point O, in which the spontaneous axis intersects the plane of (x, y), is called the centre of spontaneous rotation. Thus the effect of the blow Q is to cause the body to rotate about the axis OR with the angular velocity Ω, which is given in (34). The centre of gravity at the first instant moves along the axis of y with a velocity

$$= \frac{\mathrm{Q}}{m}. \tag{36}$$

* Liouville's Journal, Deuxième Série, tome II, 1857, p. 281.

† The reader will observe the difference between the term 'centre of percussion' as here used, and as used in Art. 127.

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Let $\text{OG} = h'$; so that omitting the negative sign which enters into (35),

$$hh' = k^2; \tag{37}$$

and hence it appears that if C is the centre of percussion, O is the centre of spontaneous rotation; and if O is the centre of percussion, C is the centre of spontaneous rotation. Thus the centres of spontaneous rotation and of percussion are reciprocal.

Also, since the product hh' is constant, it follows that the smaller h is, the greater is h', and vice versâ. If $h = 0$, $h' = \infty$; that is, if the blow is given at the centre of gravity, the axis of spontaneous rotation is at an infinite distance, so that the body has only a motion of translation. If the axis of spontaneous rotation passes through the centre of gravity, $h' = 0$, and consequently $h = \infty$; which indicates that the blow must be given at an infinite distance from G, or that the impressed force must be a couple.

Hence also

$$\Omega = \frac{Qh}{C} = \frac{Qh}{mk^2},$$

$$= \frac{Q}{mh'}. \tag{38}$$

Let $\text{OC} = l = h + h'$; therefore

$$l = h + \frac{k^2}{h};$$

and consequently, corresponding to a variation of h, l is a minimum, when $h = k$; in which case $\text{OG} = k = \text{GC}$, and $\text{OC} = 2k$; and this is the shortest possible distance between the centres of percussion and of spontaneous rotation.

In all these expressions occurs k, which is a central principal radius of gyration; of this there are generally three different values, corresponding to the three central moments of inertia; of which the greatest and least are those corresponding to the greatest and least moments of inertia, and the mean corresponds to the mean moment of inertia. Thus l is the minimum minimorum when k is the least; and is the maximum minimorum when k is the greatest.

The velocity of the centre of percussion after the blow

$$= (h + h')\Omega;$$

$$= \frac{Q}{m}\left(1 + \frac{h}{h'}\right).$$

189.] The preceding investigation leads to this result: when a body is rotating freely about an axis parallel to one of its central principal axes, and lying in one of its central principal planes,

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the whole momentum of the body may be considered to be due to a single blow impressed on it in a line parallel to the central principal axis which is perpendicular to the former principal plane, and lying in the principal plane which is perpendicular to the former principal axis. And consequently, if the body at that instant met with a fixed obstacle at the point where the blow acted, the whole momentum would be taken from the body, and the body would be brought to rest, the fixed obstacle being struck with a momentum equal to that which was originally imparted to the body. Now, in reference to the given rotation-axis considered as a spontaneous axis, the centre of percussion would be the position of the fixed obstacle, and the momentum of the blow which it would receive would evidently be Q. Is, however, the point C thus determined the position of the fixed obstacle against which the body m would impinge with the greatest momentum? Let us consider this question.

Suppose the body to impinge on an obstacle fixed at C′, whose distance from G $= x$, see Fig. 40, and suppose the momentum which the obstacle receives to be P; let O′ be the point reciprocal to C′; that is, O′ is the spontaneous centre of rotation, when C′ is the centre of percussion; and thus

$$\text{GO}' = \frac{k^2}{x}. \tag{39}$$

At the instant of impact of the body on C′, Q is the whole momentum of the body, and its line of action is CQ; let us suppose it to be resolved into two parts P and P′, acting at C′ and O′ with lines of action parallel to CQ; then, by the laws of composition of parallel forces,

$$\text{Q} = \text{P} + \text{P}';$$

$$\left.\begin{aligned} \text{Q} \times \text{O}'\text{C} &= \text{P} \times \text{O}'\text{P}', \\ \text{Q} \times \text{CC}' &= \text{P}' \times \text{O}'\text{C}'; \end{aligned}\right\} \tag{40}$$

so that

$$\text{P} = \text{Q}\,\frac{k^2 + hx}{k^2 + x^2}; \tag{41}$$

$$\text{P}' = \text{Q}\,\frac{x^2 - hx}{k^2 + x^2}. \tag{42}$$

As O′ is reciprocal to C′, P′ produces no momentum at C′; so that P is the only part of Q which affects the obstacle at C′.

If $x = -\dfrac{k^2}{h} = -h' = -\text{GO}$, $\text{P} = 0$; that is, an obstacle placed at the spontaneous centre receives no blow.

If $x = 0$, $\text{P} = \text{Q}$; that is, an obstacle placed at the centre of gravity receives a blow equal to the whole momentum.

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If $x = h$, $P = Q$; that is, an obstacle placed at the centre of percussion receives a blow equal to the whole momentum.

Thus, a body strikes with the same momentum at its centre of gravity and at its centre of percussion; but with this difference; when it strikes an obstacle placed at its centre of percussion, it is brought to rest; when it strikes an obstacle placed at its centre of gravity, its angular velocity continues what it was before impact.

If $x = \infty$, $P = 0$, $P' = Q$; and $GO' = 0$, so that the body strikes at its centre of gravity with a momentum $P' = Q$.

If x is negative, and less than $\frac{k^2}{h}$ or h', P is still positive; but if x is negatively greater than h', P is negative. In this case, C' falls on the negative side of O, and the body strikes an obstacle in a direction contrary to that for all points on the right hand side of O; and thus the obstacle must be placed on the opposite side of the line OGC.

190.] To determine the position of the obstacle, when the momentum of the blow with which the body strikes it is a maximum or minimum, we must take the x-differential of (41) and equate it to zero. Thus,

$$\frac{dP}{dx} = Q\,\frac{-hx^2 - 2k^2x + hk^2}{(k^2+x^2)^2} = 0, \text{ if}$$

$$x = -h' \pm (h'^2 + k^2)^{\frac{1}{2}}; \tag{43}$$

and changes sign from $+$ to $-$ for the upper sign, and from $-$ to $+$ for the lower sign. Hence we have two critical values of P, which are respectively a maximum and a minimum: let these be T and T', where T is the maximum corresponding to $x = -h' + (h'^2 + k^2)^{\frac{1}{2}}$; in which case

$$T = Q\,\frac{(k^2+h'^2)^{\frac{1}{2}} + h'}{2h'} = \frac{Q}{2}\left\{1 + \left(1 + \frac{h}{h'}\right)^{\frac{1}{2}}\right\}, \tag{44}$$

which is manifestly greater than Q.

And corresponding to $x = -h' - (h'^2 + k^2)^{\frac{1}{2}}$,

$$T' = -Q\,\frac{(k^2+h'^2)^{\frac{1}{2}} - h'}{2h'} = -\frac{Q}{2}\left\{\left(1 + \frac{h}{h'}\right)^{\frac{1}{2}} - 1\right\}, \tag{45}$$

which is evidently negative, and acts in a direction opposite to that of T; and thus satisfying the criteria of a minimum, it is indeed the greatest negative value.

The former result is apparently paradoxical; for as T is greater

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than Q, the momentum of the blow with which the obstacle is struck is greater than that of the whole moving body; a momentum therefore is extracted from the body greater than that which it has. The explanation of the seeming paradox is, that an opposite momentum, viz. T′, has been generated; and $T + T' = Q$; so that the sum of the two resulting momenta is equal to the whole momentum of the body; and the principle of the conservation of momentum still rules this case: more however will be said on this subject hereafter.

Let the points of application of T and T′ be R and R′; see Fig. 41; these are called the centres of greatest percussion; they are evidently reciprocal to each other, as centres of percussion and of spontaneous rotation. Also, since

$$x + h' = \pm (h'^2 + k^2)^{\frac{1}{2}};$$

$$OR = OR' = (h'^2 + hh')^{\frac{1}{2}},$$

$$= (GO \times OC)^{\frac{1}{2}}; \qquad (46)$$

so that the two centres of maximum percussion are equally distant from the spontaneous centre; and the distance is a mean proportional between the distances of the centre of gravity and of the centre of percussion from that same centre. This property gives an easy geometrical construction for the determination of the centres. Also this distance is equal to the radius of gyration of the body about the spontaneous axis; because k is the radius of gyration about the axis through the centre of gravity parallel to the spontaneous axis.

If $h = 0$, that is, if the original blow Q is given at the centre of gravity, so that the spontaneous centre is at an infinite distance, and the body has only a motion of translation, then

$$P = Q \frac{k^2}{k^2 + x^2}; \qquad (47)$$

$$\therefore \quad \frac{dP}{dx} = -\frac{2Qk^2x}{(k^2 + x^2)} = 0, \qquad (48)$$

if $x = 0$, and changes sign from + to −; and the greatest value of P is Q; that is, the greatest blow which the body is capable of giving is at its centre of gravity.

If the body is originally put into motion by a couple whose moment is N, so that the body has only a motion of rotation about an axis passing through its centre of gravity, then in (41), $Q = 0$, $h = \infty$, and $Qh = N$; so that

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$$\text{P} = \frac{\text{N}x}{k^2+x^2}; \tag{49}$$

$$\therefore \quad \frac{d\text{P}}{dx} = \frac{\text{N}(k^2-x^2)}{(k^2+x^2)^2} = 0,$$

if $x = \pm k$; and P has two corresponding values, which are respectively positive and negative; each of which $= \frac{\text{N}}{2k}$; and their lines of action are equidistant from the centre of gravity, the distance being equal to the central radius of gyration.

Thus, if a sphere of radius a rotates about a vertical diameter, the greatest blow will be given on an obstacle at a distance $= a(.4)^{\frac{1}{2}}$ from the centre of the sphere.

If a circular plate of radius a revolves about an axis through its centre perpendicular to its plane, it will strike an obstacle in its plane with the greatest effect when that obstacle is at a distance $= a(.5)^{\frac{1}{2}}$ from the centre.

If Ω is the angular velocity of the body, $\text{N} = mk^2\Omega$; and therefore from (49),

$$\text{P} = \Omega x \frac{mk^2}{k^2+x^2}.$$

Now the velocity of a point in the body at a distance $= x$ from the rotation-axis through the centre of gravity is Ωx; and since momentum is equal to the product of the mass and the velocity, a mass $= \frac{mk^2}{k^2+x^2}$ moving with the velocity with which the body impinges on the obstacle at its point of impact would produce a blow of equal momentum. And since, when $x = k$, this mass $= \frac{m}{2}$, it follows, that when the body impinges on the obstacle with the greatest effect, the momentum of the blow is the same as that of a particle of half the mass of the body, moving directly with a velocity equal to that of the corresponding centre of maximum percussion.

A similar result is true for the centre of maximum percussion corresponding to $x = -k$.

191.] The subject from this point of view requires more consideration. For suppose the body to impinge against, not a fixed obstacle, but a finite moveable mass m', then the velocities after impact, both of the body and of m', depend on the mass of m', and on the mass which, moving with the velocity of impact,

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would have the same momentum as the blow P due to the moving body.

In the general expressions for P and P′ given in (41) and (42), let Q be replaced by its value $mh'\Omega$ given in (38); and let us inquire what masses moving with the velocities at the points of impact of P and P′ respectively, will produce the momenta P and P′; let M and M′ be the masses required; then, since the velocities of the points of impact are respectively

$$(h'+x)\,\Omega \text{ and } \left(h'-\frac{k^2}{x}\right)\Omega,$$

$$\therefore \quad \mathrm{M}(h'+x)\,\Omega = \mathrm{Q}\,\frac{k^2+hx}{k^2+x^2};$$

$$= m\,h'\Omega\,\frac{hh'+hx}{k^2+x^2};$$

$$\therefore \quad \mathrm{M} = \frac{m\,k^2}{k^2+x^2}; \tag{50}$$

similarly

$$\mathrm{M}' = \frac{m\,x^2}{k^2+x^2}: \tag{51}$$

these equations assign the fractions of m, which, moving with the velocities of the body at C′ and O′, would produce momenta equal to P and P′ respectively.

In reference to these values, let it be observed, that

(1) $\mathrm{M}+\mathrm{M}'=m$; so that the sum of the two masses is equal to that of the whole moving body.

(2) $\mathrm{M}x = \mathrm{M}'\dfrac{k^2}{x}$; so that the two masses statically equilibrate about G, the centre of gravity of the body; and thus M and M′ have the same centre of gravity as m.

And thus the masses, which, placed at two reciprocal centres, may equivalently replace a body so far as impact at these centres is concerned, are equal to the whole body, and are to each other inversely as the distances from the centre of gravity.

(3) $\mathrm{M}x^2+\mathrm{M}'\dfrac{k^4}{x^2}=mk^2$; so that the moment of inertia of the two masses relative to the central principal axis, which is perpendicular to the line of blow, is equal to that of the body; and consequently the moment of the two relative to the spontaneous axis is equal to that of the body relative to the same axis.

In all these respects, then, a rigid inflexible straight bar, whose mass must be neglected, of any length $= x+\dfrac{k^2}{x}$, with masses

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equal to M and M′ at its two ends, will equivalently replace a body; it will have the same mass, the same centre of gravity, and the same moment of inertia; and when it is charged with the same momentum of impulsion, it will have the same spontaneous axis, and the same percussion at a corresponding point.

From (50) it appears, that $M = m$ only when $x = 0$; the centre of gravity therefore is the only point at which the momentum of the blow is the same as that of the whole body.

If $x = h$, $M = \frac{mh'}{l}$, which is only a fraction of m. Now we have already shewn that $P = Q$, both when the point of impact is at the centre of percussion and at the centre of gravity; in the latter case the momentum is due to the mass m, moving with the velocity $\frac{Q}{m}$; in the former it is due to the mass $\frac{mh'}{l}$, moving with a velocity $\frac{Ql}{mh'}$; see (39); that is, in the former case we have a smaller mass and a greater velocity. Although the effects will be the same when the impact takes place against a fixed obstacle, yet they will not be the same when the object impinged upon is a moveable finite mass, say m'. Thus, if m' is at rest, when the body strikes it, and its elasticity is e, then, if v' is the velocity of m' after impact, we have, from Vol. III, Art. 215, (9), the following values:

If the centre of gravity is the point of impact,

$$v' = \frac{Q(1+e)}{m+m'}. \tag{52}$$

And if the centre of percussion is the point of impact,

$$v' = \frac{Q(1+e)l}{mh'+m'l}; \tag{53}$$

and therefore is greater in the latter case than in the former.

192.] Let us however investigate the position of m', when the velocity communicated to it at rest by the impact of the body is a maximum.

Let the distance of m' from the centre of gravity $= x$; then P, the momentum of impact, is given by (41), and the mass corresponding to the momentum of the blow is given by (50); so that by (9), Art. 215, Vol. III,

$$v' = \frac{Q(1+e)h(h'+x)}{(m+m')k^2+m'x^2}. \tag{54}$$

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Of this quantity let the x-differential be taken and equated to zero; thence we have

$$x^2+2h'x-\left(1+\frac{m}{m'}\right)k^2=0;$$

$$\therefore\quad x+h'=\pm\left\{h'^2+\left(1+\frac{m}{m'}\right)k^2\right\}^{\frac{1}{2}}; \tag{55}$$

so that, as in the case of greatest percussion against a fixed obstacle, two points give critical effects; to one of which corresponds a positive, and to the other a negative maximum: these points are equidistant from the spontaneous centre; and the distance of each from that centre

$$=\left\{h'^2+\left(1+\frac{m}{m'}\right)k^2\right\}^{\frac{1}{2}};$$

this distance depends on m', the mass of the particle impinged upon, and is less the greater m' is. The points determined will coincide with the centres of greatest percussion only when $m'=\infty$; which is a result in accordance with the fact, that a fixed obstacle is nothing else than a particle or body of infinite mass. Thus we have arrived at two new points; which, however, are not reciprocal to each other, as the centres of greatest percussion are.

Also, corresponding to the values of x, given in (55),

$$\mathrm{v}'=\frac{\mathrm{Q}(1+e)}{2(m+m')}\left[\pm\left\{1+\left(1+\frac{m}{m'}\right)\frac{h}{h'}\right\}^{\frac{1}{2}}+1\right]; \tag{56}$$

which are the greatest values of the velocity with which the particle m' can be projected after impact by the body.

When the body impinges on m' at rest at a distance x from the centre of gravity, the velocity of m' is given in (54); if v is the velocity of the impinging point of the body after collision, then, by (8), Art. 215, Vol. III,

$$\mathrm{v}=\frac{(m-em')k^2-em'x^2}{(m+m')k^2+m'x^2}(h'+x)\frac{\mathrm{Q}}{mh'}; \tag{57}$$

and the momentum at that point after collision

$$=\frac{(m-em')k^2-em'x^2}{(m+m')k^2+m'x^2}\;\frac{(h'+x)h\mathrm{Q}}{k^2+x^2}. \tag{58}$$

If we take the x-differential of (57) and equate it to zero, the point will be determined at which the body must impinge on m', and continue to proceed with the greatest velocity.

Thus, if a body, which has been put into motion by a blow whose momentum is Q, impinges on a particle m' at a distance $=x$ from its centre of gravity, under the circumstances of the pre-

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ceding Articles so that the momentum of a blow given against a fixed obstacle at the point $= \text{Q}\dfrac{k^2+hx}{k^2+x^2}$; then after impact on m' the momentum is given by (58). Now as the point at which P', see equation (42), acts is the point reciprocal to that at which P acts, so P' will not affect the momentum given by (58), and consequently the spontaneous axis is not altered by the collision. But, if Ω' is the angular velocity about the spontaneous axis after the collision,

$$\Omega' = \frac{(m-em')k^2-em'x^2}{(m+m')k^2+m'x^2}\,\frac{(h'x+x^2)\,\Omega}{k^2+x^2}. \tag{59}$$

If $m' = \infty$, then the momentum of the blow which the body is capable of at a distance $= x$ from the centre of gravity after impact

$$= -e\frac{k^2+hx}{k^2+x^2}\text{Q},$$

$$= -e\text{P}; \tag{60}$$

that is, is e times the momentum before impact, and acts in an opposite direction. And

$$\Omega' = -e\frac{h'x+x^2}{k^2+x^2}\,\Omega. \tag{61}$$

In this case, if $x = h$, $\Omega' = -e\Omega$; and the effect of the impact is to change the direction of the angular velocity about the spontaneous axis, and to diminish it in the ratio of $e : 1$.

193.] Suppose however that, when the body impinges on the fixed obstacle at a distance $= x$ from the centre of gravity, the point of impact is brought to rest, and has no further motion; that is, suppose that in the preceding Article $m' = \infty$ and $e = 0$; then the momentum of the body is reduced to a quantity P', whose value is given in (42), and which acts at a distance $= -\dfrac{k^2}{x}$ from the centre of gravity. Let us consider the effect of this.

Since
$$\text{P}' = \text{Q}\frac{x^2-hx}{k^2+x^2},$$

the velocity of the centre of gravity

$$= \frac{\text{Q}}{m}\,\frac{x^2-hx}{k^2+x^2} = u' \text{ (say)}; \tag{62}$$

and the angular velocity about an axis passing through the centre of gravity, which is also the angular velocity about the spontaneous axis which passes through the fixed point,

$$= \frac{\text{Q}}{m}\,\frac{h-x}{k^2+x^2} = \Omega' \text{ (say)}. \tag{63}$$

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If $x = h$, $u' = 0$, and $\Omega' = 0$: in this case the fixed obstacle is at the centre of percussion, and the body is brought entirely to rest.

From these values however of u' and Ω', some interesting questions arise: we can determine the values of x, which will render u' a maximum; or will give it a given value; say, will make it equal to the original velocity of the centre of gravity, and in an opposite direction. Similar values too may be found for Ω'.

Let us first take u'; if u' is a maximum, then

$$x + h' = \pm (h'^2 + k^2)^{\frac{1}{2}};$$

which give the two centres of greatest percussion. In reference to this property M. Poinsot has called these points the centres of greatest reflexion. One will be a centre of reflexion in a direction opposite to that in which the centre of gravity was going previously to the impact; and the other will be the centre of reflexion in the same direction.

For if we take the upper sign,

$$u' = \frac{Q}{2mh'} \{h' - (h'^2 + k^2)^{\frac{1}{2}}\},$$

$$= -\frac{\Omega}{2} \{(h'^2 + k^2)^{\frac{1}{2}} - h'\}; \tag{64}$$

which is evidently negative; and therefore the centre of gravity of the body moves after the impact in a direction contrary to that of its former motion; and thus has undergone a true reflexion.

If we take the lower sign,

$$u' = \frac{Q}{2mh'} \{h' + (h'^2 + k^2)^{\frac{1}{2}}\},$$

$$= \frac{\Omega}{2} \{h' + (h'^2 + k^2)^{\frac{1}{2}}\}; \tag{65}$$

in which case the centre of gravity of the body proceeds in the same direction as before the impact, and with an increased velocity.

If the velocity of the centre of gravity after the impact is the same as it was before the impact, but in an opposite direction; then, from (62),

$$\frac{Q}{m} \frac{x^2 - hx}{k^2 + x^2} = -\frac{Q}{m};$$

$$\therefore \quad x = \frac{h}{4} \pm \frac{1}{4} (h^2 - 8k^2)^{\frac{1}{2}}; \tag{66}$$

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these two points have been called by Poinsot centres of perfect reflexion. They are however only possible when h^2 is not less than $8k^2$; that is, only when the original blow Q has been given at a greater distance from the centre of gravity than the limits thus assigned.

Next let us consider the value of Ω', given in (63). Ω' has a maximum value when the x-differential of it $=0$; in which case

$$x = h \pm (h^2+k^2)^{\frac{1}{2}}; \tag{67}$$

which are two values always real, one being positive and the other negative. These points are situated at equal distances from the original centre of percussion; and the distance is equal to the radius of gyration of the body about an axis passing through the centre of percussion, and parallel to the spontaneous axis. These points have been named by Poinsot centres of greatest conversion. On comparing the values of x which assign these centres with those which assign the centres of greatest reflexion, it is evident that these bear the same relation to the centre of percussion as those do to the spontaneous centre. So that the centres of greatest conversion in a body become the centres of greatest reflexion, and vice versâ, if the centre of percussion and the spontaneous centre are interchanged.

If in the value for Ω' given in (63) we substitute for x the value given in (67) with the upper sign,

$$\Omega' = -\frac{\Omega}{2h}\{(h^2+k^2)^{\frac{1}{2}}-h\}; \tag{68}$$

which is negative, and thus indicates that for this centre of conversion the angular velocity of the body is in a direction the contrary of what it was before the impact.

If we take the lower sign in (67),

$$\Omega' = \frac{\Omega}{2h}\{(h^2+k^2)^{\frac{1}{2}}+h\}, \tag{69}$$

which is positive; and this shews that the direction of the angular velocity for this centre of greatest conversion is the same as that of the body before impact.

If the angular velocity of the body after impact is the same as before impact, but in an opposite direction, then $\Omega' = -\Omega$; and

$$\frac{Q}{m}\,\frac{h-x}{k^2-x^2} = -\frac{Qh}{mk^2};$$

$$\therefore\quad x = \frac{h'}{2} \pm \frac{1}{2}(h'^2-8k^2)^{\frac{1}{2}}; \tag{70}$$

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which gives possible values only, provided that h'^2 is not less than $8k^2$. These two points have been called by Poinsot centres of perfect conversion.

If the angular velocity after impact is to have a given value; say, if the angular velocity after impact $= n$ times the angular velocity before impact, it is only necessary to equate the value of Ω' given in (63) to $n\Omega$, and the resulting quadratic equation will give the positions of the corresponding points of impact, and will, by the nature of its roots, also assign the limits of possibility of the problem.

I may in conclusion observe, that in the Memoir by M. Poinsot a geometrical construction is given whereby the several centres may be determined.

194.] Let us now consider a problem of the same kind, though somewhat less special, in which the condition (21), necessary for the existence of a spontaneous axis, is also satisfied; that, namely, in which the line of action of the impulsive blow is parallel to a central principal axis, although it does not, as in the problem just discussed, lie in a central principal plane.

Let the line of the blow be parallel to the central principal axis of z; and let the point of impact be (x_0, y_0) in the principal plane of (x, y); let Q = the momentum of the blow; and let all these circumstances be delineated in Fig. 42; wherein G, the centre of gravity, is the origin, $\text{G}x$, $\text{G}y$, $\text{G}z$ are the three central principal axes, relative to which the moments of inertia of the body are A, B, C respectively, and the corresponding radii of gyration are severally a, b, c. Let C be the point (x_0, y_0) in the plane (x, y), whereat the blow, whose moment is Q, strikes the body; the point C will be called the centre of impulsion. Now, in this case,

$$\text{X} = \text{Y} = 0; \qquad \text{Z} = \text{Q}; \tag{71}$$

$$\Omega_1 = \frac{\text{Q}\,y_0}{\text{A}}, \qquad \Omega_2 = -\frac{\text{Q}\,x_0}{\text{B}}, \qquad \Omega_3 = 0. \tag{72}$$

Let the mass of the body $= m$, and let x, y, z be the current coordinates of the spontaneous axis; then its equations, which are given in (19), Art. 186, become

$$z = 0,$$

$$\frac{\text{Q}}{m} + \frac{\text{Q}}{\text{A}}\,y y_0 + \frac{\text{Q}}{\text{B}}\,x x_0 = 0; \tag{73}$$

which are the equations to a line in the plane of (x, y). Let A and B be replaced respectively by their equivalents ma^2 and mb^2,

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then the equation to the spontaneous axis in the plane of (x, y) becomes

$$\frac{x x_0}{b^2} + \frac{y y_0}{a^2} + 1 = 0; \tag{74}$$

which is the line EF in Fig. 42: thus the body by reason of the blow Q begins to revolve about a line which is in the central principal plane perpendicular to the line of blow.

Let CG be produced, and cut the spontaneous axis in the point O; of which let the coordinates be x'_0, y'_0; then

$$x'_0 = -\frac{a^2 b^2 x_0}{a^2 x_0^2 + b^2 y_0^2}, \qquad y'_0 = -\frac{a^2 b^2 y_0}{a^2 x_0^2 + b^2 y_0^2}; \tag{75}$$

which give the coordinates of O in terms of those of C.

Since x, y and x_0, y_0 are symmetrically involved in (74), it follows that the points to which they correspond are thus far at least reciprocal. And as C is called the centre of impulsion, O is called the spontaneous centre. Thus, if O is the centre of impulsion, C is the spontaneous centre through which the spontaneous axis passes; and vice versâ.

Let h and h' be the distances of C and of O respectively from G; then

$$h^2 = x_0^2 + y_0^2,$$

$$h'^2 = x'^2_0 + y'^2_0,$$

$$= \frac{a^4 b^4 (x_0^2 + y_0^2)}{(a^2 x_0^2 + b^2 y_0^2)^2};$$

$$\therefore \quad h h' = \frac{a^2 b^2 (x_0^2 + y_0^2)}{a^2 x_0^2 + b^2 y_0^2}; \tag{76}$$

and

$$a^2 x_0^2 + b^2 y_0^2 = \frac{a^2 b^2 h}{h'}. \tag{77}$$

Let us interpret this result by means of the central momental ellipsoid. The equation to that ellipsoid which is given in Art. 102 is

$$\mathrm{A}\xi^2 + \mathrm{B}\eta^2 + \mathrm{C}\zeta^2 = \mu,$$

where μ is undetermined. Let A, B, C be replaced by ma^2, mb^2, mc^2 respectively; and, for the sake of simplification, let $\mu = ma^2 b^2$; then the equation becomes

$$a^2 \xi^2 + b^2 \eta^2 + c^2 \zeta^2 = a^2 b^2;$$

and thus the trace of this on the plane of (x, y) is the ellipse

$$a^2 \xi^2 + b^2 \eta^2 = a^2 b^2; \tag{78}$$

which we will call the central ellipse. The ξ- and the η-principal semi-axes of this ellipse are evidently b and a respectively; and the moments of inertia about these axes are A and B; which are respectively equal to ma^2 and mb^2; so that

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$$\text{A} = m a^2 = \frac{\mu}{b^2}; \qquad \text{B} = m b^2 = \frac{\mu}{a^2};$$

and consequently the moments of inertia are inversely proportional to the square of the corresponding radii vectores of the ellipse. If K is the moment of inertia about a radius vector δ of this ellipse,

$$\text{K} = \frac{\mu}{\delta^2} = \frac{m a^2 b^2}{\delta^2}. \tag{79}$$

Let (ξ, η) be the point P where this ellipse is intersected by the line GC; see Fig. 43; and let the radius vector GP $= \delta$; then

$$\frac{\xi}{x_0} = \frac{\eta}{y_0} = \frac{\delta}{h};$$

and consequently from (76),

$$hh' = \xi^2 + \eta^2 = \delta^2; \tag{80}$$

$$\therefore \quad \text{GC} \times \text{GO} = \text{GP}^2; \tag{81}$$

and thus, if the central ellipse is described, the spontaneous centre which is relative to a given centre of impulsion can be determined immediately. Of this theorem, (37), Art. 188, is evidently a particular case.

Hence, if C is the focus of the ellipse, the spontaneous axis is the farther directrix.

Again, draw the diameter GD which is conjugate to GP; its equation is

$$\frac{x\xi}{b^2} + \frac{y\eta}{a^2} = 0; \tag{82}$$

and the spontaneous axis is evidently parallel to this line. Thus, if through O we draw OS parallel to GD, or to the tangent of the central ellipse at P, OS is the spontaneous axis.

Similarly, if through C the line CT is drawn parallel to GD and OS, CT is the spontaneous axis relative to O as a centre of impulsion.

If C is at P, O is at P′; and CT and OS are tangents to the central ellipse at P and P′ respectively. In this case PP′ is the shortest possible distance along the line CGO, between the centre of impulsion and the spontaneous centre. But of all minima distances between these centres, AA′ is the least and BB′ is the greatest.

195.] The spontaneous axis and spontaneous centre, which are relative to a given centre of impulsion, give rise to many interesting theorems.

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(1) The equation to the spontaneous axis, in reference to a given centre of impulsion, being

$$\frac{x_0 x}{b^2} + \frac{y_0 y}{a^2} = 1;$$

it is evident that, if a series of spontaneous axes pass through the same point (x'_0, y'_0), all the corresponding centres of impulsion lie along the straight line

$$\frac{x x'_0}{b^2} + \frac{y y'_0}{a^2} = 1. \tag{83}$$

This line is parallel to GD, which is conjugate to the diameter GOP′ of the central ellipse: and thus all the centres lie along the line CT, which is the spontaneous axis relative to O as a centre of impulsion.

Similarly, of centres of impulsion lying in the line OS, the corresponding spontaneous axes pass through the point C.

Thus, wherever in OS the centre of impulsion is, C does not move; and wherever in CT the centre of impulsion is, O does not move.

(2) Let us suppose the centre of impulsion to move on a given curve; then the spontaneous axis will envelope another curve, of which the equation may be found.

Thus suppose the centre of impulsion to move on the circle

$$x_0^2 + y_0^2 = r^2; \tag{84}$$

then the equation to the spontaneous axis is

$$\frac{x x_0}{b^2} + \frac{y y_0}{a^2} = 1; \tag{85}$$

let these be differentiated, on the supposition that x_0 and y_0 vary; then the envelope of the last is

$$\frac{x^2}{b^4} + \frac{y^2}{a^4} = \frac{1}{r^2}; \tag{86}$$

which is the equation to an ellipse, coaxal and concentric with the central ellipse.

(3) Or, again, the curve may be given, all the tangents to which are to be spontaneous axes; and it may be required to determine the locus of the corresponding centres of impulsion.

Thus, if the spontaneous axes all touch the circle $x^2 + y^2 = r^2$, the locus of the corresponding centres of impulsion is the ellipse

$$\frac{x^2}{b^4} + \frac{y^2}{a^4} = \frac{1}{r^2}.$$

Indeed these reciprocal properties give rise to a complete system

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of duality, to a great extent similar to that of polar lines and their reciprocals. This however is not the occasion for a farther development of it.

(4) Or, again, since the coordinates to the centre of impulsion and to the spontaneous centre are related by the equations (75), if the locus of one centre is given, that of the other can be found.

Thus, suppose the centre of impulsion to move along the ellipse $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, then the locus of the spontaneous centre is also the same ellipse.

If the centre of impulsion moves along a straight line, then the locus of the spontaneous centre is an ellipse. Let the equation to the straight line be put into the form

$$\frac{x x_0}{b^2} + \frac{y y_0}{a^2} = 1; \tag{87}$$

where x_0 and y_0 are any constants. Then the locus of the spontaneous centres is

$$\frac{\xi^2}{b^2} + \frac{\eta^2}{a^2} + \frac{\xi x_0}{b^2} + \frac{\eta y_0}{a^2} = 0; \tag{88}$$

which is evidently an ellipse similar, and similarly situated, to the central ellipse; whose centre is at $\left(-\frac{x_0}{2}, -\frac{y_0}{2}\right)$, and which passes through the origin. The form of the equation to the straight line which I have taken shews that the line is the spontaneous axis to a centre of impulsion situated at $(-x_0, -y_0)$.

196.] Let thus much suffice for the circumstanees of the spontaneous axis of the body in its relation to the centre of impulsion; and let us investigate other incidents of its motion at the first instant.

Let Ω be the angular velocity; then, from (72),

$$\begin{aligned} \Omega^2 &= \Omega_1^2 + \Omega_2^2, \\ &= \frac{Q^2}{m^2}\left\{\frac{x_0^2}{b^4} + \frac{y_0^2}{a^4}\right\}; \end{aligned} \tag{89}$$

$$\therefore \quad \Omega = \frac{Q}{mp},$$

if p is the length of the perpendicular from the centre of gravity on the spontaneous axis.

The velocity of the centre of gravity is evidently $\frac{Q}{m}$.

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Also, since the length of the perpendicular from C (x_0, y_0) on the spontaneous axis

$$= \frac{a^2 x_0^2 + b^2 y_0^2 + a^2 b^2}{(a^4 x_0^2 + b^4 y_0^2)^{\frac{1}{2}}}; \tag{90}$$

therefore the velocity at the centre of impulsion at the first instant after the blow

$$= \frac{Q}{m} \left\{ \frac{x_0^2}{b^2} + \frac{y_0^2}{a^2} + 1 \right\} \tag{91}$$

$$= \frac{Q}{m} \frac{h + h'}{h'}.$$

If the body impinges against a fixed obstacle, or indeed against any mass at its centre of impulsion C in the plane of the central principal axes of x and y, the momentum of the blow will be Q: let us inquire what will be the momentum at any other point of impact in the same plane; say at R, which is (x, y); with a view to a farther inquiry of the position of the point when the momentum is a maximum.

Let C be as heretofore (x_0, y_0), and let R be (x, y), Fig. 44; and let the momentum of the blow given at R be P; let O′ be the spontaneous centre relative to R, and let O′U be its spontaneous axis. Join RC, and produce it to meet O′U in U. Resolve Q into two parallel forces, P and P′, acting at U and R, with lines of action parallel to that of Q; so that by the laws of composition of parallel forces,

$$P + P' = Q, \tag{92}$$

$$P \times UR = Q \times UC, \tag{93}$$

$$P' \times UR = Q \times CR. \tag{94}$$

As O′U is the spontaneous axis relative to R as centre of impulsion, whatever in O′U be the point at which a blow is given, R remains at rest; so that P′ impressed at U produces no effect on R; and consequently P is the whole effect at R; and P is determined by (93): we have therefore to find UR and UC.

The equations to UO′ and CR are respectively

$$\frac{\xi x}{b^2} + \frac{\eta y}{a^2} + 1 = 0,$$

$$\xi (y - y_0) + \eta (x_0 - x) + y_0 x - x_0 y = 0;$$

and if U is (ξ, η),

$$\frac{UC}{UR} = \frac{x_0 - \xi}{x - \xi} = \frac{a^2 x x_0 + b^2 y y_0 + a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}; \tag{95}$$

$$\therefore \quad P = Q \frac{a^2 x x_0 + b^2 y y_0 + a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}, \tag{96}$$

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which is the momentum of the blow with which the body would strike any obstacle at the point (x, y) in the central principal plane of (x, y).

Certain particular values of P deserve mention. P = 0, if

$$a^2 x x_0 + b^2 y y_0 + a^2 b^2 = 0; \tag{97}$$

that is, when the obstacle is at any point on the spontaneous axis.

And P = Q, when

$$a^2 x^2 + b^2 y^2 - a^2 x_0 x - b^2 y_0 y = 0; \tag{98}$$

that is, when the obstacle is at any point on the ellipse, similar and similarly placed to the central ellipse, of which the line GC is a diameter, and of which, of course, the centre is at the middle point of GC. The points G and C are on this ellipse; and consequently at both these points P = Q.

The case also in which P has a given value, say nQ, deserves consideration; of course it gives a locus of centres of percussion, which is generally an ellipse; and in certain cases becomes a point; and in certain other cases is imaginary. The subject however does not offer any particular difficulty: and the student can easily work it out for himself.

Also, if the body were originally put into motion by the blow Q at G, so that it has a motion of translation only, then $x_0 = y_0 = 0$, and

$$\text{P} = \text{Q}\,\frac{a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}. \tag{99}$$

Let us also investigate the value of P′, and consider certain particular values of it: from (92) or (94),

$$\text{P}' = \text{Q}\,\frac{a^2 x^2 + b^2 y^2 - a^2 x x_0 - b^2 y y_0}{a^2 x^2 + b^2 y^2 + a^2 b^2}. \tag{100}$$

Hence P′ = 0, when

$$a^2 x^2 + b^2 y^2 - a^2 x x_0 - b^2 y y_0 = 0;$$

that is, when the point of impact is on the ellipse given in (98), in which case P = Q. Hence P′ = 0, if the obstacle is at the centre of gravity, or at the centre of impulsion.

Also P′ = Q, when

$$a^2 x x_0 + b^2 y y_0 + a^2 b^2 = 0;$$

that is, when the obstacle is at a point on the spontaneous axis, in which case P = 0.

We might also investigate the locus of the place of the obstacle when P′ = nQ, say; but as the problem presents no particular difficulty, the reader may work it out for himself.

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If the centre of impulsion is at the centre of gravity, $x_0 = y_0 = 0$;

and
$$\text{P}' = \text{Q}\,\frac{a^2x^2 + b^2y^2}{a^2x^2 + b^2y^2 + a^2b^2}. \tag{101}$$

197.] I propose now to investigate the points for which P is a maximum. In this case $\left(\frac{d\text{P}}{dx}\right) = \left(\frac{d\text{P}}{dy}\right) = 0$. And hence we have

$$\left.\begin{aligned} x_0(a^2x^2 + b^2y^2 + a^2b^2) - 2x(a^2x\,x_0 + b^2y\,y_0 + a^2b^2) &= 0, \\ y_0(a^2x^2 + b^2y^2 + a^2b^2) - 2y(a^2x\,x_0 + b^2y\,y_0 + a^2b^2) &= 0; \end{aligned}\right\} \tag{102}$$

which give
$$\frac{x}{x_0} = \frac{y}{y_0}; \tag{103}$$

whence it follows that the place of the obstacle, when P has a critical value, is on the line GC. If we substitute in either of (102) from (103), we have

$$(a^2x_0^2 + b^2y_0^2)x^2 + 2a^2b^2x_0x - a^2b^2x_0^2 = 0, \tag{104}$$
$$(a^2x_0^2 + b^2y_0^2)y^2 + 2a^2b^2y_0y - a^2b^2y_0^2 = 0; \tag{105}$$

which are quadratic equations in terms of x and y respectively; and thus give two positions of a point of impact for which P has a critical value. Let these be called centres of greatest percussion; and let r be their distance from the centre of gravity; then $r^2 = x^2 + y^2$; also $h^2 = x_0^2 + y_0^2$; and

$$\frac{r}{h} = \frac{x}{x_0} = \frac{y}{y_0}; \tag{106}$$

so that (102) give

$$(a^2x_0^2 + b^2y_0^2)r^2 + 2a^2b^2hr - a^2b^2h^2 = 0; \tag{107}$$

therefore, substituting from (77),

$$r^2 + 2h'r - hh' = 0; \tag{108}$$

$$\therefore\quad r = -h' \pm (h'^2 + hh')^{\frac{1}{2}}; \tag{109}$$

thus the two centres of greatest percussion are equally distant from the spontaneous centre O. Let V and V′ be these centres; Fig. 45; then

$$\text{OV} = \text{OV}' = (\text{OG} \times \text{OC})^{\frac{1}{2}}; \tag{110}$$

and the distance is a mean proportional between the distances of the centre of impulsion and of the centre of gravity from the spontaneous centre.

Let T and T′ be the corresponding momenta: then making the above substitutions in (96),

$$\text{P} = \text{Q}\,\frac{rh + hh'}{r^2 + hh'}; \tag{111}$$

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therefore
$$\text{T} = \frac{\text{Q}}{2}\,\frac{h' + (h'^2 + hh')^{\frac{1}{2}}}{h'}; \tag{112}$$
$$\text{T}' = \frac{\text{Q}}{2}\,\frac{h' - (h'^2 + hh')^{\frac{1}{2}}}{h'}. \tag{113}$$

of which T, which acts at the centre V, is positive, and is greater than Q; T′, which acts at the centre V′, is negative, and thus m gives a blow against an obstacle at V′ in a direction opposite to that in which it strikes the obstacle at V; and thus, as the obstacle at V must be on the upper or positive side of the plane of (x, y), that at V′ must be on the lower or negative side. On applying the criteria for a maximum or minimum to these values of P, viz., T and T′, it will be found that T is a maximum, and that T′ is a minimum; but as T′ is negative, it is a negative maximum, so that both centres may be called centres of greatest percussion.

If the point of application of the blow Q, by which the body is originally put into motion, is the centre of gravity, so that $x_0 = y_0 = 0$, and the body has only motion of translation; then, from (99),
$$\text{P} = \text{Q}\,\frac{a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}; \tag{114}$$
and the maximum value corresponds to $x = 0$, $y = 0$; in which case P = Q; and thus the centre of gravity is the centre of greatest percussion.

If $\text{P} = \frac{\text{Q}}{n}$, then
$$a^2 x^2 + b^2 y^2 = a^2 b^2 (n - 1); \tag{115}$$
which represents an ellipse concentric, coaxal, and similar to the central ellipse; and therefore the intensity of a blow against an obstacle is the same for all points on this ellipse. If $n = 2$, the ellipse is the central ellipse.

198.] Again, if the body is put into motion by a couple whose axis is perpendicular to the axis of z, so that the spontaneous axis passes through the centre of gravity, and the body has only rotation about that axis which is in the plane of (x, y), the momentum of a blow P at the point (x, y) in the plane of (x, y), may be determined in the following manner, which is independent of the preceding process:

Let L and M be the components of the moment of the impressed couple about the axes of x and y respectively; then we have
$$\left.\begin{array}{c} \text{X} = \text{Y} = \text{Z} = 0; \\ \Omega_1 = \dfrac{\text{L}}{ma^2}, \qquad \Omega_2 = \dfrac{\text{M}}{mb^2}, \qquad \Omega_3 = 0; \end{array}\right\} \tag{116}$$

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and the equation to the spontaneous axis is, see (19),

$$\frac{\text{L}y}{a^2} - \frac{\text{M}x}{b^2} = 0. \tag{117}$$

Since the point of application of P is (x, y), the equation to the corresponding spontaneous axis is

$$a^2 x\xi + b^2 y\eta + a^2 b^2 = 0; \tag{118}$$

the perpendicular distance on which from the point (x, y)

$$= \frac{a^2 x^2 + b^2 y^2 + a^2 b^2}{(a^4 x^2 + b^4 y^2)^{\frac{1}{2}}}. \tag{119}$$

Let us suppose the couple of impulsion, of which the axial components are L and M, to be replaced by a couple whose forces are P at (x, y), and $-$P at the point of intersection of the spontaneous axis with the perpendicular on it from (x, y); then the moment of that couple

$$= \text{P}\frac{a^2 x^2 + b^2 y^2 + a^2 b^2}{(a^4 x^2 + b^4 y^2)^{\frac{1}{2}}}; \tag{120}$$

and the x- and the y-direction-cosines of its axis are

$$\frac{b^2 y}{(a^4 x^2 + b^4 y^2)^{\frac{1}{2}}}, \quad \text{and} \quad \frac{-a^2 x}{(a^4 x^2 + b^4 y^2)^{\frac{1}{2}}}. \tag{121}$$

And (120) is to be equal to the couple of which L and M are the axial components; hence

$$\text{P}(a^2 x^2 + b^2 y^2 + a^2 b^2) = \text{L}\, b^2 y - \text{M}\, a^2 x;$$

$$\therefore \quad \text{P} = \frac{\text{L}\, b^2 y - \text{M}\, a^2 x}{a^2 x^2 + b^2 y^2 + a^2 b^2}. \tag{122}$$

This quantity might also be deduced from the general expression of P, given in (96). For when the body is put into motion by a couple, that couple is equivalent to a force $= 0$ acting at an infinite distance; so that in the numerator $x_0 = y_0 = \infty$, and consequently $a^2 b^2$ must be omitted; and thus

$$\text{P} = \frac{\text{Q}\, a^2 x x_0 + \text{Q}\, b^2 y y_0}{a^2 x^2 + b^2 y^2 + a^2 b^2};$$

but $\text{Q}y_0 =$ the moment of the couple about the axis of $x =$ L,

and $-\text{Q}x_0 =$ - - - - - - - - - - - - - - - - - $y =$ M;

$$\therefore \quad \text{P} = \frac{\text{L}\, b^2 y - \text{M}\, a^2 x}{a^2 x^2 + b^2 y^2 + a^2 b^2}.$$

Now since $-$P acts at a point on the spontaneous axis which corresponds to the centre (x, y), $-$P produces no effect at (x, y); so that P, which is given in (122), is the momentum of the whole blow given by the body on the obstacle.

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When P thus determined is a maximum,

$$\left(\frac{dP}{dx}\right) = \left(\frac{dP}{dy}\right) = 0;$$

whence we have

$$\left.\begin{aligned} -M(a^2x^2 + b^2y^2 + a^2b^2) &= 2x(Lb^2y - Ma^2x), \\ L(a^2x^2 + b^2y^2 + a^2b^2) &= 2y(Lb^2y - Ma^2x); \end{aligned}\right\} \quad (123)$$

$$\therefore \quad Lx + My = 0; \quad (124)$$

$$a^2x^2 + b^2y^2 = a^2b^2. \quad (125)$$

Thus, the point (x, y) which gives the greatest percussion, is in the central ellipse, at the points in which the plane of the couple of impulsion passing through the centre of gravity intersects it. And the greatest value of P

$$= \pm \frac{(a^2L^2 + b^2M^2)^{\frac{1}{2}}}{2ab}; \quad (126)$$

the two signs corresponding to the two extremities of the diameter of the central ellipse, which coincides with the plane of the couple passing through the centre of gravity; at which points the two values of P are equal, but as they have opposite signs they act in opposite directions.

199.] And let us farther investigate the nature of the blow, when the body strikes against a moveable mass at the point (x, y).

Let p be the perpendicular distance from (x, y) to the spontaneous axis; so that

$$p = \frac{a^2x_0x + b^2y_0y + a^2b^2}{(a^4x_0^2 + b^4y_0^2)^{\frac{1}{2}}}; \quad (127)$$

and let Ω be the angular velocity of the body about the spontaneous axis due to the force of impulsion; then, from (89),

$$\Omega = \frac{Q(a^4x_0^2 + b^4y_0^2)^{\frac{1}{2}}}{ma^2b^2}; \quad (128)$$

and consequently if v is the velocity of the point (x, y) which is due to the force of impulsion,

$$v = \Omega p = \frac{Q}{m}\,\frac{a^2x_0x + b^2y_0y + a^2b^2}{a^2b^2}. \quad (129)$$

Hence, if P is the momentum of the blow which the body is capable of at the point (x, y),

$$\begin{aligned} P &= Q\frac{a^2xx_0 + b^2yy_0 + a^2b^2}{a^2x^2 + b^2y^2 + a^2b^2}, \\ &= v\frac{ma^2b^2}{a^2x^2 + b^2y^2 + a^2b^2}. \end{aligned} \quad (130)$$

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Let M be the mass which, moving with the velocity v, would produce on the obstacle at (x, y) a blow of this momentum; then

$$\text{M} = \frac{m a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}. \tag{131}$$

Also, let M′ be the mass of a particle which, moving with the velocity of the spontaneous centre (x', y') corresponding to (x, y), would produce against an obstacle placed there a blow whose momentum is equal to that of the body. Then

$$\begin{aligned} \text{M}' &= \frac{m a^2 b^2}{a^2 x'^2 + b^2 y'^2 + a^2 b^2}, \\ &= \frac{m (a^2 x^2 + b^2 y^2)}{a^2 x^2 + b^2 y^2 + a^2 b^2}, \end{aligned} \tag{132}$$

by reason of (75). And thus masses are assigned, which are fractions of m, and which, moving with the velocities of any point and of its corresponding spontaneous centre, would have momenta equal to those of the blows which the body would give to obstacles placed at those points.

The values of these masses thus determined may be conveniently put into another form: let r and r' be the distances of M and M′ from the centre of gravity; and let δ be the radius vector of the central ellipse which coincides with the line joining M and M′: then, as the places of M and M′ are reciprocal as a centre of impulsion and a spontaneous centre, $r r' = \delta^2$; and from (77),

$$a^2 x^2 + b^2 y^2 = \frac{a^2 b^2 r}{r'}; \quad \text{so that}$$

$$\left.\begin{aligned} \text{M} &= \frac{m r'}{r + r'}, & \text{M}' &= \frac{m r}{r + r'}, \\ &= \frac{m \delta^2}{r^2 + \delta^2}; & &= \frac{m r^2}{r^2 + \delta^2}. \end{aligned}\right\} \tag{133}$$

In reference to these masses let it be observed, that

(1) $\text{M} + \text{M}' = m$; so that the sum of the two is equal to the whole moving body.

(2) $\text{M} r = \text{M}' r'$; so that the centre of gravity of M and M′ coincides with that of m.

(3) $\text{M}(x^2 + y^2) + \text{M}'(x'^2 + y'^2) = \dfrac{m a^2 b^2 (x^2 + y^2)}{a^2 x^2 + b^2 y^2} = m \delta^2$;

so that the radius of gyration of the masses about any axis passing through their centre of gravity and perpendicular to the line joining them is equal to the radius vector of the central ellipse which coincides with that line.

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In these respects therefore the body may be equivalently replaced by a straight, inflexible, and immaterial bar, having masses M and M′ at its two ends, which are determined by equations (133): this bar will not only at its two ends, but at any point in its length, strike an obstacle with a blow of the same momentum as the body.

From (131) it appears, that $M = m$, only when $x = y = 0$; that is, the centre of gravity is the only point at which the body will strike an obstacle as if it were a mass equal to its own mass; and in this case $P = Q$.

If the centre of percussion is the point of impact, $P = Q$; but

$$M = \frac{m a^2 b^2}{a^2 x_0^2 + b^2 y_0^2 + a^2 b^2} = m \frac{h'}{h + h'}; \tag{134}$$

and from (91), $$v = \frac{Q}{m} \frac{h + h'}{h'};$$

so that the momentum is produced by a mass smaller than m, moving with a greater velocity. Although therefore against a fixed obstacle the momentum of the blow P is the same, whether the obstacle be at the centre of gravity or at the centre of percussion; yet against a particle of finite mass, say m', the effects will be different. These we proceed to investigate; and we shall determine both the velocity of m' after impact from the body, as well as the velocity of the impinging point of the body after impact on m'.

200.] Let v and v′ be the velocities of the body and of m' after collision at the point (x, y); let e = the elasticity; and let us suppose m' to be at rest when the impact takes place; then, from (9), Art. 215, Vol. III,

$$v = Q \frac{(a^2 x x_0 + b^2 y y_0 + a^2 b^2)}{m a^2 b^2} \frac{(m - e m') a^2 b^2 - e m' (a^2 x^2 + b^2 y^2)}{(m + m') a^2 b^2 + m' (a^2 x^2 + b^2 y^2)}; \tag{135}$$

$$v' = \frac{(1 + e) Q (a^2 x x_0 + b^2 y y_0 + a^2 b^2)}{(m + m') a^2 b^2 + m' (a^2 x^2 + b^2 y^2)}. \tag{136}$$

If we equate to zero the x- and y-differentials of v′, the point will be determined at which m' must be struck so that it may move after collision with the greatest velocity: this process gives

$$\frac{x}{x_0} = \frac{y}{y_0}; \tag{137}$$

which shews that the point of greatest percussion is in the line joining the centre of gravity and the centre of impulsion. If r

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is the distance of the required point from the centre of gravity, then, as in Article 197,

$$r = -h' \pm \left\{h'^2 + hh'\left(1 + \frac{m}{m'}\right)\right\}^{\frac{1}{2}}. \qquad (138)$$

Thus there are two points at which a body impinging on a particle m' will cause it to move after collision with a maximum velocity; these points are equidistant from the spontaneous centre which corresponds to the centre of impulsion, and that on the positive side of the spontaneous centre lies farther from it than the centre of gravity. These two points are the centres of greatest percussion when $m' = \infty$; that is, when the mass of the particle against which the body impinges is infinitely great, and is thus equivalent to a fixed obstacle.

And corresponding to these distances,

$$v' = \frac{Q(1+e)}{2(m+m')}\left[1 \pm \left\{1 + \left(1 + \frac{m}{m'}\right)\frac{h}{h'}\right\}^{\frac{1}{2}}\right]; \qquad (139)$$

of which values one is positive and the other is negative: the former shews that the particle m' will move with a velocity whose direction is the same as that of Q; the latter, which corresponds to the point of percussion on the side of the spontaneous axis away from the centre of gravity, gives a velocity of m' in the opposite direction.

In a similar way may the point be determined, at which, if the body impinges on m', the velocity of the point of impact after collision will be a maximum; for if we take the x- and y-partial differentials of (135), and equate them to zero, the points will be determined by means of these two equations.

201.] Now at the instant when the body has impinged against a fixed obstacle at the point (x, y), that point of the body is at rest; yet there remains the momentum P′, which is given in (100), whose point of application is U, see Fig. 44: as U however is a point in O′S, which is the spontaneous axis relative to R as a centre of impulsion at which P acts, P′ produces no effect on R; and thus the motion of the body at that instant is due to the force P′ only applied at U; and consequently the spontaneous axis passes through R. U, which must now be considered as a centre of impulsion, is, from Art. 196,

$$\left(\frac{b^2y(x_0y - y_0x) - a^2b^2(x - x_0)}{a^2x^2 + b^2y^2 - a^2xx_0 - b^2yy_0},\ \frac{a^2x(y_0x - x_0y) - a^2b^2(y - y_0)}{a^2x^2 + b^2y^2 - a^2xx_0 - b^2yy_0}\right). \qquad (140)$$

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The spontaneous axis corresponding to which is

$$\xi\{y(x_0y-y_0x)-a^2(x-x_0)\}+\eta\{x(y_0x-x_0y)-b^2(y-y_0)\}$$
$$+a^2x^2+b^2y^2-a^2xx_0-b^2yy_0=0. \quad (141)$$

Now the momentum of the blow of impulsion, namely P', is given in (100); and consequently if u' = the velocity of the centre of gravity,

$$u'=\frac{\text{P}'}{m}, \quad (142)$$

$$=\frac{\text{Q}}{m}\,\frac{a^2x^2+b^2y^2-a^2xx_0-b^2yy_0}{a^2x^2+b^2y^2+a^2b^2}. \quad (143)$$

And if Ω' = the angular velocity about the spontaneous axis through R due to P', and if p = the perpendicular distance from the centre of gravity on that line,

$$\Omega'=\frac{\text{P}'}{mp}; \quad (144)$$

where p =

$$\frac{a^2x^2+b^2y^2-a^2xx_0-b^2yy_0}{\{(x^2+y^2)(x_0y-y_0x)^2-2\{a^2(x-x_0)+b^2(y-y_0)\}(x_0y-y_0x)+a^4(x-x_0)^2+b^4(y-y_0)^2\}^{\frac{1}{2}}}$$

If the point (x,y) lies on the line joining the centre of gravity and the original centre of impulsion, these expressions become much simplified; because, in that case,

$$x_0y-y_0x=0;$$

thus the equation to the spontaneous axis through R becomes

$$a^2(x-x_0)\xi+b^2(y-y_0)\eta=a^2x^2+b^2y^2-a^2xx_0-b^2yy_0; \quad (145)$$

$$p=\frac{a^2x^2+b^2y^2-a^2xx_0-b^2yy_0}{\{a^4(x-x_0)^2+b^4(y-y_0)^2\}^{\frac{1}{2}}};$$

$$\Omega'=\frac{\text{Q}}{m}\,\frac{\{a^4(x-x_0)^2+b^4(y-y_0)^2\}^{\frac{1}{2}}}{a^2x^2+b^2y^2+a^2b^2}; \quad (146)$$

a particular case of this last simplification is that in which the obstacle is placed at a centre of greatest percussion; see Art. 197.

Questions exactly analogous to those which I have alluded to in Art. 197 arise out of the preceding values of u' and Ω', and give points which may be called points of greatest reflexion and of greatest conversion.

202.] Thus as to u'; the problem may be to determine the place of a fixed obstacle, or of a particle of given mass, so that it may be a maximum; or the place of a fixed obstacle, so that it may have a given value; say, be equal to the original value of the velocity of the centre of gravity but in an opposite direc-

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tion; or to find the place of the obstacle, so that u' may be equal to 0.

As to critical values of u' I would observe, that

$$u' = \frac{\text{P}'}{m}$$

$$= \frac{\text{Q} - \text{P}}{m}, \text{ by reason of (92)};$$

so that whatever values of x and y give critical values for P, also give critical values for P′ and for u': these values have been already investigated in Art. 197, and give what are therein called centres of greatest percussion; these centres then are also centres of greatest reflexion. Also, since there are two such centres, we have also two critical values of u',

$$\left.\begin{aligned} u_1' &= \frac{\text{Q}}{m}\,\frac{h' - (h'^2 + hh')^{\frac{1}{2}}}{2h'}, \\ u_2' &= \frac{\text{Q}}{m}\,\frac{h' + (h'^2 + hh')^{\frac{1}{2}}}{2h'}. \end{aligned}\right\} \qquad (147)$$

The latter of these is positive, and is evidently greater than $\frac{\text{Q}}{m}$, which is the original value of the velocity of the centre of gravity: this is paradoxical: it seems contrary to the first principles of mechanics that a body should strike against a fixed obstacle, and after impact rebound with the velocity of the centre of gravity greater than that velocity before impact. Now consider this in reference to fig. 45; V and V′ in it are the centres of greatest percussion, and consequently of greatest reflexion; and u'_2 corresponds to the point V′, so that when the obstacle is placed at that point, and the body impinges against it, the velocity of G after the impact is greater than before. The body moves by the blow Q, which is given at C, from below to above the paper; and rotates about the axis OS; if however it impinges against the obstacle at V′, that angular velocity becomes modified, and OS, which was at rest before the impact, moves in the direction CQ, and the velocity of G is increased. We must not however hence infer that the momentum of the body is increased; for that would be contrary to the principles of mechanics; but some of the momentum, which is due to the angular velocity, by means of the obstacle becomes momentum of translation; and hence it is that the velocity of the centre of gravity is after impact greater than it was before. Thus, a ball from a rifled

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gun, having velocity both of translation of its centre of gravity and of rotation about an axis through that centre, may have its velocity of translation increased by meeting with an obstacle, and thus may be carried farther than if it never met with such an obstacle. This is one of the peculiar and surprising facts of *ricochet* practice.

If $u' = -\frac{Q}{m}$, the velocity of the centre of gravity will be after impact the same as it was before but in an opposite direction; then

$$2a^2x^2 + 2b^2y^2 - a^2xx_0 - b^2yy_0 + a^2b^2 = 0; \qquad (148)$$

which is the equation to an ellipse similar to the central ellipse, and similarly situated; the points which give this value of u' are called points of perfect reflexion; and the ellipse (148) is called the ellipse of perfect reflexion.

Again, as to Ω'; it is a function of x and y, and the values of those quantities may be found which will give to Ω' a critical value. Also, those may be found which will assign a point on which, when the body impinges, the angular velocity after impact will have a given value; say, be equal to that before impact, and in an opposite direction. Points which give these values to Ω' are called respectively, points of maximum conversion, points of given conversion, and points of perfect conversion.

Thus, if $\Omega' = 0$, $P' = 0$; and

$$a^2x^2 + b^2y^2 - a^2xx_0 - b^2yy_0 = 0;$$

so that for all points on the ellipse given by (98), $P = Q$, $P' = 0$, $u' = 0$, $\Omega' = 0$; that is, if the obstacle is on that ellipse, the body impinges on it with a momentum equal to that of original impulsion; the centre of gravity of the body is brought to rest, and there is also no angular velocity; in fact the body is brought to rest.

If $\Omega' = -\frac{Q}{m}\frac{(a^4x_0^2 + b^4y_0^2)^{\frac{1}{2}}}{a^2b^2}$, see (89); in this case the angular velocity after the impact is equal to, but of contrary direction to, that before impact; then, if we take the particular case given in (146), we have

$$\frac{\{a^4(x-x_0)^2 + b^4(y-y_0)^2\}^{\frac{1}{2}}}{a^2x^2 + b^2y^2 + a^2b^2} = -\frac{(a^4x_0^2 + b^4y_0^2)^{\frac{1}{2}}}{a^2b^2}; \qquad (149)$$

which gives an equation of the fourth degree in terms of x and y; all points on the curve expressed by which are points of perfect conversion.

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To determine the points for which Ω' has a critical value, the x- and y-partial differentials of (144) or of (146) must be equated to zero: in the general case however they lead to results so much complicated that it is useless to insert them.

203.] In the preceding Articles, see Art. 191, (50) and (51), and Art. 199, (133), it has been shewn, that a body may be equivalently replaced by two particles of definite and determinate masses at the ends of an immaterial rigid and straight bar, so far as the effects of momentum communicated to the body by a blow, and the effects of impact of the body on a fixed obstacle are concerned. This property is of considerable use in the solution of another problem: A body of given mass moving with a given velocity impinges on a given body at a given point, it is required to determine the motion of the bodies at the instant after impact.

I will assume the line of motion of the moving mass to be in a central principal plane and to be parallel to a principal axis of the body on which it impinges. Let m be the mass of the latter body, $m' =$ the impinging mass, of which let the velocity at the point of impact be v'; let the line of motion of m' be in the central principal plane of (x, y), and be parallel to the axis of y; let C, see Fig. 46, the point of impact, be in the axis of x at a distance $= x$ from the centre of gravity G; and let k be the radius of gyration of the body about the axis Gz, which is perpendicular to the line of action of the blow. Let O be the spontaneous centre reciprocal to C; so that $\text{OG} = \dfrac{k^2}{x}$. Now in Art. 191 we have shewn, that so far as concerns blows given by it, the body m may be replaced equivalently by two masses M and M′, which are therein assigned, of which the former is placed at C, and the latter at O; and that as O is a centre reciprocal to C, the mass M′ placed at O neither affects nor is affected by the blow given at C; so that as far as the momentum of a blow at C is concerned, the effect of the body will be the same as that of the mass M placed there: all this is explained in Art. 191. The problem then which was proposed for solution is this; m' moving with a velocity v' impinges on M at rest: it is required to determine the motion of M and m' after collision. The principles of Art. 215, Vol. III, are sufficient for the purpose.

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Let $\text{V} =$ the velocity of M after impact; $\text{V}' =$ the velocity of m' after impact; and let $e =$ the elasticity. Then, since

$$\text{M} = \frac{m k^2}{k^2 + x^2}, \tag{150}$$

$$\text{V} = \frac{(1+e)\, m' (k^2 + x^2)\, v'}{(m+m') k^2 + m' x^2}; \tag{151}$$

$$\text{V}' = \frac{m' (k^2 + x^2) - e m k^2}{(m+m') k^2 + m' x^2}; \tag{152}$$

thus the momentum of M after impact $= \text{M}\text{V}$

$$= \frac{m m' (1+e) k^2 v'}{(m+m') k^2 + m' x^2}; \tag{153}$$

and this is the momentum of the body at the point C, and is that which has hitherto been symbolised by Q.

The momentum imparted to the body decreases as x increases; and vanishes when $x = \infty$; and the greatest value is that which corresponds to $x = 0$.

If the bodies are perfectly inelastic, $e = 0$; in which case the momentum imparted to the body

$$= \frac{m m' k^2 v'}{(m+m') k^2 + m' x^2}. \tag{154}$$

Suppose now that m' and v' are variables, with the condition of their product being constant; that is, the momentum of the impinging ball is constant, although its mass and velocity vary; say,

$$m' v' = \mu_0 v_0. \tag{155}$$

And suppose moreover that, whatever is the distance from G at which m' impinges, the momentum imparted to the body is the same; say, $= m \text{V}_0$; then,

$$\text{V}_0 = \frac{k^2 \mu_0 v_0}{m k^2 + m' (k^2 + x^2)};$$

and consequently $m' (k^2 + x^2)$ is constant, $= m_0 b^2$, say;

$$\therefore \quad m' = \frac{m_0 b^2}{k^2 + x^2}; \tag{156}$$

$$v' = \frac{\mu_0 v_0 (k^2 + x^2)}{m_0 b^2}; \tag{157}$$

in which equations m' and v' are expressed as functions of x, and are thus determinable for any distance from the centre of gravity of the point of impact.

Thus, if a hammer is to be constructed and used, so that the same quantity of momentum is to be imparted to a body whose

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mass is m, whatever is the distance of the point of impact from the centre of gravity, the momentum of the blow of the hammer being always the same; then the mass of the hammer and the velocity of the blow are given by (156) and (157).

204.] One other problem of a practical kind deserves insertion in this branch of the subject which treats of spontaneous axes and their properties.

A body m rests on a prop and is struck by a blow whose momentum is Q, the line of motion of the blow being in a central principal plane, and parallel to one central principal axis; and the line of reaction of the prop being in the same principal plane, and parallel to the line of the blow.

Let OGR be a central principal axis of the body whose mass is m, and centre of gravity is G. Let C be the prop and R the point of application of the blow whose momentum is Q, and of which QR is the line of action; this line of motion being in the central principal plane xGz: see Fig. 47. It is evident that if R coincides with C, that is, if the blow is given at the prop, the momentum of the pressure borne by the prop = Q; suppose however that the point of impact of the blow is R, where GR $= x$; let P = the momentum borne by C, and let P′ be that applied at O, which is the spontaneous centre reciprocal to C, both these being due to Q; then the pressure P′ does not affect the pressure at C, which is a point reciprocal to O, so that P is the whole pressure on the prop.

Let GC $= h$; and consequently GO $= \dfrac{k^2}{h}$; then as P and P′ are the components of Q, Q = P − P′;

and
$$\text{P}\left(h + \frac{k^2}{h}\right) = \text{Q}\left(\frac{k^2}{h} + x\right);$$

$$\therefore \quad \text{P} = \text{Q}\,\frac{k^2 + hx}{k^2 + h^2}; \tag{158}$$

which assigns the pressure borne by the prop. If $x = h$, P = Q; that is, the blow is applied at the prop, and the pressure borne by the prop is equal to the momentum of the blow. If $x = -\dfrac{k^2}{h}$, that is, if the blow is struck at O, the spontaneous centre relative to C, P = 0.

P increases as x increases, and P is greater than Q when x is greater than h: it follows therefore that by means of an inter-

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vening body m, a blow of given momentum can produce a pressure of any intensity on a given prop. If $x = \infty$, $P = \infty$.

If however the blow is caused by a hammer of mass m', and impinging with a velocity v' on a point whose distance from the centre of gravity is x; then, from (154),

$$Q = \frac{m m' k^2 v'}{(m + m') k^2 + m' x^2};$$

and therefore $$P = \frac{m m' k^2 v' (k^2 + h x)}{(k^2 + h^2) \{(m + m') k^2 + m' x^2\}}. \qquad (159)$$

If in this expression m' and v' have, for a distance x, the values found for them in (156) and (157), then the momentum of the blow of the hammer is always the same, and the momentum borne by the prop is given by (158), and may consequently be of any magnitude whatever.

P is a maximum in (159) when

$$x = -h' \pm \left(h'^2 + \frac{m + m'}{m} k^2\right)^{\frac{1}{2}}, \qquad (160)$$

where $GO = h' = \frac{k^2}{h}$; which gives two points equidistant from O.

Thus if $m = m'$; and $h = h' = k$,

$$x = (-1 \pm 3^{\frac{1}{2}}) k.$$

Similarly may the points of impact be determined, so that the momentum of the pressure borne by the obstacle may be of a given value.

I cannot conclude this subject, in which I have borrowed largely from the Memoirs of Poinsot, contained in Vols. II and IV of the second series of Liouville's Journal, without alluding to a remark which he makes of the process by which the circumstances of motion of a rigid system having a fixed axis or a fixed point may be deduced from those of a similar free system. He considers a fixed point to be a particle of a certain definite mass, introduces this mass and its incidents into all the equations of motion, and in the final results makes this mass infinite; and this particle of infinite mass he considers to be a fixed point; on which, of course, as to translation, a finite force has no effect; but for an axis passing through that point the moment of inertia of the body is finite, and consequently the impressed couples will produce their own rotatory effects.

205.] Thus far, the bodies on which the impulsive forces have acted have been assumed to be entirely free from all constraint; and we have considered only those whose circumstances under

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the action of the given forces satisfy the condition necessary for the existence of a spontaneous axis. But the principles from which we started were general, and are applicable to all kinds of bodies and instantaneous forces. The general cases will be most conveniently discussed as they arise in the course of particular problems in the following sections; but it is also expedient to exhibit the form which the general equations take when they are thus applied to such problems. For this purpose I will consider the effects of a blow on an ordinary billiard ball.

A heavy spherical billiard ball on a rough horizontal table is struck by a cue at a given point with a blow of given intensity in a given direction; it is required to determine the resulting motion of the ball.

Let a = the radius, M = the mass of the ball; Q = the momentum of the blow; α = the angle at which its line of action is inclined to the plane of the table.

Let the horizontal plane which passes through the centre of the ball and is parallel to that of the table be the plane of (x, y); and let the line in it parallel to the vertical plane which contains the line of Q be the axis of x; let h = the horizontal distance from the centre of the ball to the vertical plane which contains the line of blow; and let k be the perpendicular distance on the line of blow from the point where h meets the vertical plane containing that line.

Let $\text{F}\cos\beta$ and $\text{F}\sin\beta$ be the components parallel to the axes of x and y respectively of the friction against the table which is brought into action by the blow; let $\Omega_1, \Omega_2, \Omega_3$ be the resulting angular velocities about axes through the centre of the ball which are parallel to the coordinate axes; and let u_0, v_0 be the resulting expressed velocities parallel to the coordinate axes. Then the equations of translation parallel to the axes of x and y are

$$\left.\begin{aligned} \text{M}u_0 &= \text{Q}\cos\alpha - \text{F}\cos\beta, \\ \text{M}v_0 &= \qquad\quad - \text{F}\sin\beta; \end{aligned}\right\} \tag{161}$$

and if A is the moment of inertia of the ball about an axis through its centre,

$$\left.\begin{aligned} \text{A}\Omega_1 &= -\text{Q}h\sin\alpha - a\text{F}\sin\beta, \\ \text{A}\Omega_2 &= \quad \text{Q}k + a\text{F}\cos\beta, \\ \text{A}\Omega_3 &= -\text{Q}h\cos\alpha. \end{aligned}\right\} \tag{162}$$

If R = the pressure of the ball on the table,

$$\text{R} = \text{Q}\sin\alpha + \text{M}g; \tag{163}$$

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but since the line of action of R passes through the centre of the ball, it produces no effect on Ω_1, Ω_2, or Ω_3.

Thus the axial components of the velocity of the point of contact of the ball with the plane are $u_0 - a\,\Omega_2$, $v_0 + a\,\Omega_1$; consequently, if

$$s^2 = (u_0 - a\,\Omega_2)^2 + (v_0 + a\,\Omega_1)^2, \tag{164}$$

and as β is the angle at which the initial path of the point of contact is inclined to the axis of x,

$$\frac{\cos\beta}{u_0 - a\,\Omega_2} = \frac{\sin\beta}{v_0 + a\,\Omega_1} = \frac{1}{s}; \tag{165}$$

and since the friction acts as a retarding force along the line of motion of the point of contact, its line of action is thus determined; and the friction is known in terms of the pressure R, so that the four unknown quantities u_0, v_0, Ω_1, Ω_2 are involved in four independent equations, and may be determined without difficulty; and thus the initial motion of the ball will be determined. Applications of these results will be made hereafter; see Art. 222.

SECTION 2.—*Motion of a free invariable system under the action of finite accelerating forces.*

206.] WE now come to the most general case of absolute motion of a body, or of any system or systems of material particles under the action of finite forces. Many processes have been devised for the purpose, and several of them are especially adapted to particular classes of problems. All however are founded upon the principle of D'Alembert; and their equations of motion are derived from, or are identical with, those six equations in which we have expressed that theorem. I propose to apply these to the solution of problems of motion in preference to other and derived processes; because we shall hereby maintain an uniformity of process and of principle, and because the circumstances of the problems will be resolved into their most simple elements. We shall indeed take the forms which these equations admit of, in virtue of the theorems proved in Section 2 of Chap. III; we shall consider the motion of translation of the centre of gravity, by assuming all the forces to act on a particle, whose mass is equal to the whole mass of the moving system, placed therein; and in our inquiry into the rotation of the system, we shall assume the centre of gravity to be a fixed point, and the body or material system to rotate

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about an axis passing through that point. Thus the motion of the system in the first place depends on the two following groups of equations:

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right) &= 0, \\ \Sigma.m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right) &= 0, \\ \Sigma.m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right) &= 0; \end{aligned}\right\} \tag{166}$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)-z\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{z\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)-x\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{x\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)-y\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\right\} &= 0. \end{aligned}\right\} \tag{167}$$

If $\overline{\mathrm{M}}$ is the mass of the whole moving system, $(\bar{x}, \bar{y}, \bar{z})$ is the place of the centre of gravity at the time t, and if (x', y', z') is the place of m relatively to a system of coordinate axes originating at the centre of gravity, and parallel to the original system of axes; then, by the theorems of Section 2, Chap. III, these take the forms

$$\left.\begin{aligned} \overline{\mathrm{M}}\frac{d^2\bar{x}}{dt^2} &= \Sigma.m\,\mathrm{X}, \\ \overline{\mathrm{M}}\frac{d^2\bar{y}}{dt^2} &= \Sigma.m\,\mathrm{Y}, \\ \overline{\mathrm{M}}\frac{d^2\bar{z}}{dt^2} &= \Sigma.m\,\mathrm{Z}; \end{aligned}\right\} \tag{168}$$

$$\left.\begin{aligned} \Sigma.m\left\{y'\left(\mathrm{Z}-\frac{d^2z'}{dt^2}\right)-z'\left(\mathrm{Y}-\frac{d^2y'}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{z'\left(\mathrm{X}-\frac{d^2x'}{dt^2}\right)-x'\left(\mathrm{Z}-\frac{d^2z'}{dt^2}\right)\right\} &= 0, \\ \Sigma.m\left\{x'\left(\mathrm{Y}-\frac{d^2y'}{dt^2}\right)-y'\left(\mathrm{X}-\frac{d^2x'}{dt^2}\right)\right\} &= 0. \end{aligned}\right\} \tag{169}$$

By reason of the former of these last two groups, the motion of translation of the body is reduced to that of a single material particle whose mass is $\overline{\mathrm{M}}$; and to this motion all that has been said in Vol. III is applicable. The second group reduces the motion of rotation to that of a body rotating about an axis passing through a fixed point of it; and consequently to this motion all that has been said in the preceding Chapter is ap-

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plicable. The problem therefore requires two processes in combination, each of which has been separately discussed; and little else remains than to illustrate the combination by means of particular examples. Indeed I have already anticipated the process in the investigation of the phænomena of terrestrial precession and nutation in the preceding Chapter; because we have assumed the centre of the earth to be fixed, whereas it has a motion of translation in space.

In investigating the motion of rotation of the body about the point which is assumed to be fixed, we may use the simplifications and substitutions of the last and preceding Chapters. Thus, if $\omega_x, \omega_y, \omega_z$ are the angular velocities at the time t about any three coordinate axes originating at the fixed point, (169) become (48), (49), and (50), of Art. 79; which however it is unnecessary to repeat in this place as we shall employ simplified forms of them. We shall investigate the angular velocities of the body at the time t relatively to the three principal axes of the body; and the equations for determining these are

$$\left.\begin{aligned} \mathrm{A}\frac{d\omega_1}{dt} + (\mathrm{C}-\mathrm{B})\,\omega_2\,\omega_3 &= \mathrm{L}, \\ \mathrm{B}\frac{d\omega_2}{dt} + (\mathrm{A}-\mathrm{C})\,\omega_3\,\omega_1 &= \mathrm{M}, \\ \mathrm{C}\frac{d\omega_3}{dt} + (\mathrm{B}-\mathrm{A})\,\omega_1\,\omega_2 &= \mathrm{N}; \end{aligned}\right\} \qquad (170)$$

because hereby (theoretically at least) the angular velocity of the body, and the position of the instantaneous rotation-axis relatively to the principal axes, may be determined at the time t; and thence we may determine, as in the preceding Chapter, the motion of the body in reference to fixed axes, by means of the three connecting angles θ, ϕ, ψ.

And if the position of the rotation-axis which passes through the centre of gravity of the body is invariable relatively to the body, then the rotation is determined by the simple equation,

$$\frac{d\omega}{dt} = \frac{\text{moment of impressed forces}}{\text{moment of inertia}}. \qquad (171)$$

In the solution of mechanical problems, the theorems of vis viva, and of conservation of areas, which have been derived in Chapter III from the general equations of motion, may frequently be applied, to the saving of considerable trouble; not indeed because they contain any truth besides those involved

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explicitly or implicitly in the equations of motion, but because they are first integrals of these equations. In a didactic treatise, like the present, however, I consider clearness of conception and accuracy of expression to be of paramount importance; and I am convinced that it is most likely that these will be obtained when the circumstances of a problem are resolved into their simplest elements. In all the following problems, the equations of motion are given in their original forms, and for the complete solution of a problem they require two successive integrations. In many cases the intelligent student will readily recognise the equations of areas and the equation of vis viva; and the latter will frequently present itself to him in the derived form which has been proved in Art. 66; viz., the vis viva of the system is equal to the sum of the vis viva of the whole system condensed into its centre of gravity, and of the vis viva of the several particles relative to the centre of gravity.

207.] The following are mechanical problems, on the motion of rigid bodies, in which the rotation-axis moves parallel to itself.

Ex. 1. A heavy homogeneous sphere rolls down a rough inclined plane; it is required to determine the motion.

We suppose the sphere to be placed at rest on the plane, and to roll down it so that the point of contact describes a straight line perpendicular to the line of intersection of the inclined and horizontal planes. Let fig. 48 represent a section of the sphere and plane at the time t, made by a vertical plane passing through C the centre of the sphere. Let A be the point of the sphere which was originally in contact with the plane at the point O; let a be the radius of the sphere; $\text{OP} = s$, $\text{ACP} = \theta$, M = the mass of the sphere, F = the friction of rolling, R = the pressure of the sphere on the plane, α = the angle of elevation of the plane.

Now C evidently moves along a straight line parallel to the plane; so that for its motion of translation we have

$$\text{M}\frac{d^2s}{dt^2} = \text{M}g\sin\alpha - \text{F};$$

and if C is considered fixed, the sphere evidently rotates about a horizontal axis parallel to the plane; and if k is the radius of gyration of the sphere relative to this axis,

$$\text{M}k^2\frac{d^2\theta}{dt^2} = \text{F}a;$$

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and since the plane is perfectly rough, so that the sphere does not slide, $ds = a d\theta$; also $k^2 = \frac{2a^2}{5}$;

$$\therefore \quad \frac{d^2 s}{dt^2} = a \frac{d^2\theta}{dt^2} = \frac{5}{7} g \sin \alpha;$$

which assigns the motion; also $\text{R} = mg \cos \alpha$. If the plane were perfectly smooth, the impressed velocity-increment along the plane would be $g \sin \alpha$; so that the roughness of the plane which causes the rolling diminishes the action of gravity along the plane by two-sevenths of its full value.

If the rolling body were a circular cylinder with its axis horizontal, then $k^2 = \frac{a^2}{2}$; and

$$\frac{d^2 s}{dt^2} = \frac{2}{3} g \sin \alpha;$$

so that the roughness of the plane would diminish the action of gravity along the plane by one-third of its full value.

Ex. 2. A hollow spherical shell is filled with fluid and rolls down a rough inclined plane, determine its motion.

Let M and M' be the masses of the shell and fluid respectively; and let k and k' be the radii of gyration of them respectively about a diameter; let a and a' be the radii of the exterior and interior surfaces of the shell; then, employing the same notation as in Ex. 1, we have

$$(\text{M} + \text{M}') \frac{d^2 s}{dt^2} = (\text{M} + \text{M}') g \sin \alpha - \text{F}.$$

As the spherical shell rotates in its descent down the plane, the fluid has only motion of translation; so that the equation of rotation is

$$\text{M} k^2 \frac{d^2\theta}{dt^2} = \text{F} a;$$

$$\therefore \quad \{(\text{M} + \text{M}') a^2 + \text{M} k^2\} \frac{d^2 s}{dt^2} = (\text{M} + \text{M}') a^2 g \sin \alpha.$$

If the interior were solid, and rigidly joined to the shell, the equation of motion would be

$$\{(\text{M} + \text{M}') a^2 + \text{M} k^2 + \text{M}' k'^2\} \frac{d^2 s}{dt^2} = (\text{M} + \text{M}') a^2 g \sin \alpha.$$

Thus if s and s' are the spaces through which the centre moves during the time t in these two cases respectively, then

$$\frac{s}{s'} = \frac{(\text{M} + \text{M}') a^2 + \text{M} k^2 + \text{M}' k'^2}{(\text{M} + \text{M}') a^2 + \text{M} k^2};$$

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so that a greater space is described by the sphere which has the fluid than by that which has the solid in its interior.

If the densities of the solid and the fluid are the same, replacing k and k' by their values,

$$\frac{s}{s'} = \frac{7a^5}{7a^5 - 2a'^5}.$$

Ex. 3. A heavy solid wheel, in the form of a right circular cylinder, is composed of two substances, whose volumes are equal, and whose densities are ρ and ρ'; these substances are arranged in two different forms; in one case, that whose density is ρ occupies the central part of the wheel, and the other is placed as a ring around it; in the second case, the places of the substances are interchanged; t and t' are the times in which the wheels roll down a given rough inclined plane from rest; shew that $\quad t^2 : t'^2 :: 5\rho + 7\rho' : 5\rho' + 7\rho.$

Ex. 4. A homogeneous heavy sphere rolls down within a rough spherical bowl; it is required to determine the motion.

Let the circumstances be represented in Fig. 49, where a is the radius of the rolling sphere, and b is the radius of the spherical bowl; let us suppose the sphere to be placed in the bowl at rest. Let $\text{OCQ} = \phi$, $\text{QPA} = \theta$, $\text{BCO} = \alpha$; $\text{OM} = x$, $\text{MP} = y$; m = the mass of the ball. Then

$$m\frac{d^2x}{dt^2} = -\text{R}\sin\phi + \text{F}\cos\phi,$$

$$m\frac{d^2y}{dt^2} = \text{R}\cos\phi + \text{F}\sin\phi - mg;$$

and if ω = the angular velocity of the ball about an axis through its centre P, and k is the corresponding radius of gyration,

$$mk^2\frac{d\omega}{dt} = a\text{F};$$

where $k^2 = \dfrac{2a^2}{5}$. Now $x = (b-a)\sin\phi$, $\quad y = b - (b-a)\cos\phi$;

$$\therefore \quad \frac{d^2x}{dt^2} = (b-a)\cos\phi\frac{d^2\phi}{dt^2} - (b-a)\sin\phi\left(\frac{d\phi}{dt}\right)^2,$$

$$\frac{d^2y}{dt^2} = (b-a)\sin\phi\frac{d^2\phi}{dt^2} + (b-a)\cos\phi\left(\frac{d\phi}{dt}\right)^2;$$

$$\frac{d^2x}{dt^2}\cos\phi + \frac{d^2y}{dt^2}\sin\phi = (b-a)\frac{d^2\phi}{dt^2};$$

$$\therefore \quad m(b-a)\frac{d^2\phi}{dt^2} = \text{F} - mg\sin\phi.$$

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Now to determine the angular velocity of the ball, we must estimate the angle described by a fixed line in it from a line fixed in direction, and the ratio of the infinitesimal increase of this angle to that of the time will be the angular velocity of the ball. Let us take MPA to be the angle whose increment we will consider,

$$\therefore \quad \omega = \frac{d.\text{MPA}}{dt} = \frac{d(\phi+\theta)}{dt},$$

$$= \frac{d\phi}{dt} + \frac{d\theta}{dt}.$$

Since however the sphere rolls, and does not slide, $a\theta = b(a-\phi)$;

$$\therefore \quad \omega = \frac{a-b}{a}\frac{d\phi}{dt};$$

$$\therefore \quad \frac{d\omega}{dt} = \frac{a-b}{a}\frac{d^2\phi}{dt^2};$$

whence, eliminating F, and reducing, we have finally,

$$(b-a)\frac{d^2\phi}{dt^2} = -\frac{5}{7}g\sin\phi;$$

$$\therefore \quad (b-a)\left(\frac{d\phi}{dt}\right)^2 = \frac{10g}{7}(\cos\phi - \cos a).$$

Substituting in the preceding equations,

$$\text{R} = \frac{mg}{7}\{17\cos\phi - 10\cos a\};$$

therefore the pressure at the lowest point $= \frac{mg}{7}\{17 - 10\cos a\}$.

And the pressure of the ball on the bowl vanishes when

$$\cos\phi = \frac{10}{17}\cos a.$$

If the ball rolls over a small arc at the lowest part of the bowl, so that a and ϕ are always small, then replacing $\cos\phi$ and $\cos a$ by $1 - \frac{\phi^2}{2}$ and $1 - \frac{a^2}{2}$ respectively, we have

$$\frac{-d\phi}{(a^2-\phi^2)^{\frac{1}{2}}} = \left\{\frac{5g}{7(b-a)}\right\}^{\frac{1}{2}} dt;$$

$$\therefore \quad \phi = a\cos\left\{\frac{5g}{7(b-a)}\right\}^{\frac{1}{2}} t:$$

thus the ball comes to rest at points whose angular distance is a on both sides of O, the lowest point of the bowl; and the periodic time $= \pi\left\{\frac{7(b-a)}{5g}\right\}^{\frac{1}{2}}$; consequently the oscillations are

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performed isochronously with those of a simple pendulum whose length is $\frac{7}{5}(b-a)$.

Ex. 5. A heavy homogeneous sphere rolls down a rough inclined plane; the inclined plane rests on a smooth horizontal plane, along which it slides by reason of the pressure of the sphere; it is required to determine the motions of the sphere and of the inclined plane.

The circumstances of motion at the time t are delineated in Fig. 50. m = the mass of the ball, M = the mass of the plane or wedge; a = the radius of the ball, α = the angle of the inclined plane; Q the apex of the wedge, O the place of Q when $t=0$; O′ the point on the plane which was in contact with the point A of the sphere, when $t=0$; at which time let us suppose all to be at rest; ACP $=\theta$, the angle through which the sphere has revolved in the time t.

Let O be the origin, and let the horizontal and vertical lines through it be the axes of x and y; OQ $=x'$; and let (x, y) and (h, k) be the places of the centre of gravity of the sphere at the times $t=t$, and $t=0$ respectively. Then from the geometry, we have

$$x = h + x' - a\theta\cos\alpha,$$
$$y = k - a\theta\sin\alpha;$$

$$\therefore \quad \left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{d^2x'}{dt^2} - a\cos\alpha\frac{d^2\theta}{dt^2}, \\ \frac{d^2y}{dt^2} &= \qquad\quad - a\sin\alpha\frac{d^2\theta}{dt^2}. \end{aligned}\right\}$$

The equations of motion of the sphere are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \text{F}\cos\alpha - \text{R}\sin\alpha, \\ m\frac{d^2y}{dt^2} &= \text{F}\sin\alpha + \text{R}\cos\alpha - mg, \\ m\frac{2a^2}{5}\frac{d^2\theta}{dt^2} &= a\text{F}; \end{aligned}\right\}$$

and the equation of motion of the plane is

$$\text{M}\frac{d^2x'}{dt^2} = -\text{F}\cos\alpha + \text{R}\sin\alpha;$$

from which we obtain

$$a\frac{d^2\theta}{dt^2} = \frac{5g\sin\alpha}{\{7-5(\cos\alpha)^2\}m + 7\text{M}};$$

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$$\therefore \quad a\theta = \frac{5 \sin a}{\{7-5(\cos a)^2\} m + 7\,\text{M}} \frac{g t^2}{2};$$

and
$$x' = \frac{m \cos a}{m + \text{M}} a\theta,$$
$$= \frac{m}{m + \text{M}} \frac{5 \sin a \cos a}{\{7-5(\cos a)^2\} m + 7\,\text{M}} \frac{g t^2}{2};$$

from which values x and y may be determined in terms of t.

Also
$$m \frac{d^2 x}{dt^2} + \text{M} \frac{d^2 x'}{dt^2} = 0;$$
$$\therefore \quad m(x-h) + \text{M} x' = 0;$$

and consequently,
$$(m + \text{M})(x-h) \sin a - \text{M}(y-k) \cos a = 0;$$

which shews that the path described by the centre of the sphere is a straight line.

Ex. 6. A heavy beam OP, see Fig. 51, turns about a hinge at O, and its end P rests on a smooth inclined plane or wedge, which slides along a smooth horizontal plane which passes through O: it is required to determine the motion of the beam and of the inclined plane.

Let m and M be the masses of the beam and plane respectively; $2a$ = the length of the beam, a = the angle of inclination of the plane to the horizon; POQ $= \theta$, OQ $= x$. Then, from the geometry, we have,
$$x \sin a = 2a \sin(a-\theta).$$

For the motion of the beam we have
$$m \frac{4a^2}{3} \frac{d^2\theta}{dt^2} = -mga\cos\theta + 2a\,\text{R}\cos(a-\theta);$$

and for the motion of the plane
$$\text{M} \frac{d^2 x}{dt^2} = \text{R} \sin a;$$

whence eliminating R, we have
$$\frac{4a^2 m}{3} \frac{d^2\theta}{dt^2} = -mag\cos\theta + 2a\,\text{M} \frac{\cos(a-\theta)}{\sin a} \frac{d^2 x}{dt^2}.$$

But from the preceding geometrical condition we have
$$\sin a \frac{dx}{dt} = -2a\cos(a-\theta) \frac{d\theta}{dt};$$
$$\therefore \quad \frac{4a^2 m}{3} \frac{d^2\theta}{dt^2} \frac{d\theta}{dt} = -mag\cos\theta \frac{d\theta}{dt} - \text{M} \frac{dx}{dt} \frac{d^2 x}{dt^2};$$

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and if θ_0 is the value of θ, when the system is at rest,

$$\frac{4a^2m}{3}\left(\frac{d\theta}{dt}\right)^2 = 2mag(\sin\theta_0 - \sin\theta) - \text{M}\left(\frac{dx}{dt}\right)^2;$$

which is indeed the equation of vis viva. And, substituting for $\frac{dx}{dt}$, we have

$$2a\left\{\frac{m}{3} + \text{M}\left(\frac{\cos(a-\theta)}{\sin a}\right)^2\right\}\left(\frac{d\theta}{dt}\right)^2 = mg(\sin\theta_0 - \sin\theta).$$

This equation determines the angular velocity of the beam, and consequently the velocity of the plane; but as it does not admit of further integration, we cannot find θ or x in terms of t.

Ex. 7. A heavy body whose bounding surface is a circular cylinder, but whose centre of gravity is not in the axis of the surface, rolls on a rough horizontal plane: it is required to determine its motion.

Let M = the mass of the body; and let Fig. 52 represent the circumstances at the time t; in which G is the centre of gravity, C is the point of intersection of the axis of the cylinder by a vertical plane passing through the centre of gravity G; Ox is the horizontal plane; O, the origin, is the point where the sphere touches the plane when it is in equilibrium, A being the corresponding point of the sphere. Let CA $= a$, CG $=$ C, ACP $= \theta$, so that OP $= a\theta$; and let k be the radius of gyration of the body relative to its rotation-axis through G; let G be (x, y); then from the geometry we have,

$$x = a\theta - c\sin\theta,$$
$$y = a - c\cos\theta.$$

Now if F is the friction of rolling, and R is the pressure on the plane, the equations of motion are

$$\text{M}\frac{d^2x}{dt^2} = -\text{F};$$

$$\text{M}\frac{d^2y}{dt^2} = \text{R} - \text{M}g;$$

$$\text{M}k^2\frac{d^2\theta}{dt^2} = \text{F}(a - c\cos\theta) - \text{R}c\sin\theta.$$

Whence we have

$$\frac{d^2\theta}{dt^2}\{k^2 + a^2 + c^2 - 2ac\cos\theta\} + ac\sin\theta\left(\frac{d\theta}{dt}\right)^2 = -cg\sin\theta;$$

multiplying through by $2d\theta$, and assuming $\frac{d\theta}{dt} = 0$, when $\theta = a$, we have

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$$\{k^2+a^2+c^2-2ac\cos\theta\}\left(\frac{d\theta}{dt}\right)^2 = 2cg(\cos\theta-\cos\alpha); \quad (172)$$

which equation gives the angular velocity of the body about a horizontal axis through its centre of gravity. From this may be found $\frac{dy}{dt}$ and $\frac{dx}{dt}$, which will give the linear velocity of the centre of gravity. As these equations in their general forms do not admit of further integration, θ, x, and y cannot be found in terms of t.

If however the angle through which the body rotates is always small, so that α and θ are always so small that all powers of them above the second may be neglected, and that the second powers may be neglected when they are added to finite quantities; then we may replace (172) by its approximate equation,

$$\{k^2+(a-c)^2\}\left(\frac{d\theta}{dt}\right)^2 = cg(\alpha^2-\theta^2);$$

$$\therefore \quad \theta = \alpha\cos\left\{\frac{cg}{k^2+(a-c)^2}\right\}^{\frac{1}{2}}t; \quad (173)$$

so that the body oscillates or rocks through an angle 2α; and the time of an oscillation

$$= \pi\left\{\frac{k^2+(a-c)^2}{cg}\right\}^{\frac{1}{2}}. \quad (174)$$

This result applies to a problem which is physically of considerable importance. In making observations with the pendulum, the mode of suspension which is found most convenient for the determination of the distance between the centres of suspension and oscillation is that of knife edges, resting on horizontal plates of agate or of some other hard material. Although the knife edges are made of steel, and brought to as fine an edge as possible, yet they are not mathematical straight lines, but approximate to cylinders, which we may, without sensible error, suppose to be circular, and of a very small radius; in which case the pendulum is suspended by a horizontal cylinder which rests and rolls on two parallel horizonal bars which are perpendicular to the axis, and of which a diagram is given in Fig. 53. Here the centre of gravity of the rolling body is below the horizontal plane, so that c is greater than a; then, if τ is the time of an oscillation,

$$\tau = \pi\left\{\frac{k^2+(c-a)^2}{cg}\right\}^{\frac{1}{2}};$$

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but if the pendulum is suspended by an exact edge, the time of oscillation

$$= \pi \left\{ \frac{k^2 + c^2}{cg} \right\}^{\frac{1}{2}}:$$

thus the effect of the want of accuracy in the edge diminishes the time of vibration in the ratio of

$$\{k^2 + (c-a)^2\}^{\frac{1}{2}} \quad \text{to} \quad \{k^2 + c^2\}^{\frac{1}{2}}.$$

Ex. 8. Let us suppose the rocking body to be homogeneous, and to be bounded by a cylindrical surface whose section perpendicular to the generating lines is semicircular, as in Fig. 54; then, if $\text{CA} = a$, $\text{CG} = c = \frac{4a}{3\pi}$; $k^2 = \frac{a^2}{2} - \left(\frac{4a}{3\pi}\right)^2$; and the time of oscillation

$$= \pi \left\{ \frac{9\pi - 16}{8g} a \right\}^{\frac{1}{2}}.$$

If the rocking body is a homogeneous hemisphere, then the time of an oscillation

$$= \pi \left(\frac{26a}{15g}\right)^{\frac{1}{2}}.$$

Ex. 9. A heavy homogeneous beam is suspended by two vertical strings of equal length, so that the position of the beam is horizontal; the beam is slightly twisted through a small angle about a vertical axis passing through its centre of gravity: it is required to determine the motion.

We shall suppose the length of the strings, in reference to that of the beam and the angle through which the beam is turned, to be so great that the vertical displacement of the beam may be neglected.

Let M = the mass of the beam; $2a$ = the length of the beam; l = the length of each of the suspending strings; θ = the angle between the line of the beam at the time t, and its line in the position of rest; so that each end of the beam is displaced through a distance $= a\theta$.

The tension of each string when the beam is in equilibrium $= \frac{\text{M}g}{2}$: as the displacement of the beam is small, I shall assume the tension to be unchanged in the displaced state; so that the horizontal component of this force which acts at the end of the beam, in a line perpendicular to it, $= \frac{\text{M}g}{2} \frac{a\theta}{l}$; and if k is the radius of gyration of the beam relative to an axis passing

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through its middle point and perpendicular to it, the equation of motion is

$$\text{M}\, k^2 \frac{d^2\theta}{dt^2} = -\text{M}\, g \frac{a^2\theta}{l};$$

but $k^2 = \frac{a^2}{3}$; so that

$$\frac{d^2\theta}{dt^2} = -\frac{3g}{l}\theta.$$

Let a = the angle through which the beam is turned; then, by integration, we have

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{3g}{l}(a^2-\theta^2);$$

whence it is manifest that the beam oscillates; and that the time of an oscillation

$$= \pi\left(\frac{l}{3g}\right)^{\frac{1}{2}},$$

which is independent of the length of the beam, and depends only on the length of the string by which it is suspended.

Ex. 10. A fine string is coiled round a heavy cylindrical wheel; one end of the string is fixed, and the wheel descends, unwinding the string: it is required to determine the motion of the wheel.

Let M = the mass of the wheel; T = the tension of the string at the time t; a = the radius; OP = x, see Fig. 55; θ = the angle through which the wheel has revolved from its position of rest; k = the radius of gyration of the wheel. Then the equations of motion are

$$\text{M}\frac{d^2x}{dt^2} = \text{M}g - \text{T}; \qquad \text{M}\, k^2\frac{d^2\theta}{dt^2} = a\,\text{T};$$

also $dx = a\,d\theta$; so that

$$(a^2+k^2)\frac{d^2\theta}{dt^2} = ag; \text{ and since } k^2 = \frac{a^2}{2}, \text{ we have}$$

$$a\frac{d^2\theta}{dt^2} = \frac{2g}{3}; \qquad \therefore\ \frac{d^2x}{dt^2} = \frac{2g}{3}; \qquad \therefore\ x = x_0 + \frac{gt^2}{3};$$

so that the space described in a given time is two-thirds of that which would be described by the wheel falling freely.

Ex. 11. To determine the motion of a system of pulleys and weights, each of which hangs by a separate string, as in Fig. 56.

The system consists of a fixed pulley whose centre is C, and of a series of pulleys whose centres at the time t are $C_1, C_2, C_3 \ldots$; we will assume all these pulleys to be equal, a to be the radius, k to be the radius of gyration, and m to be the mass of each; let M be the mass of the weight which acts round the fixed

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pulley, and M′ the mass of that which is attached to the last moving pulley. Let T be the tension of the string by which M acts on the fixed pulley, and $T_1, T_2, T_3, \ldots$ be the tensions of the strings which severally pass from the fixed pulley to the first moving pulley, from the first moving pulley to the second moving pulley, and so on; and let $T_1', T_2', T_3', \ldots$ be the tensions of the strings which are fastened severally to $A_1, A_2, A_3, \ldots$; let x be the distance of M, and let $x_1, x_2, x_3, \ldots$ be the distances of $C_1, C_2, C_3, \ldots$ from the horizontal line $A_1 A_2, \ldots$ at the time t; let $d\theta, d\theta_1, d\theta_2, \ldots$ be the angles through which the fixed and the several moving pulleys respectively rotate in the time dt; and, to fix our thoughts, let us suppose M′ to descend, so that $x_1, x_2, x_3, \ldots$ increase as t increases; and let us suppose $\theta, \theta_1, \theta_2, \ldots$ to increase as t increases. Then we have the following series of equations of motion;

$$\left.\begin{aligned}
M\left(g-\frac{d^2x}{dt^2}\right) &= T,\\
m\left(g-\frac{d^2x_1}{dt^2}\right) &= T_1+T_1'-T_2,\\
m\left(g-\frac{d^2x_2}{dt^2}\right) &= T_2+T_2'-T_3,\\
&\cdots\cdots\cdots\\
m\left(g-\frac{d^2x_{n-1}}{dt^2}\right) &= T_{n-1}+T'_{n-1}-T_n,\\
(m+M')\left(g-\frac{d^2x_n}{dt^2}\right) &= T_n+T'_n;
\end{aligned}\right\} \quad (175)$$

$$\left.\begin{aligned}
\frac{mk^2}{a}\,\frac{d^2\theta}{dt^2} &= T_1-T,\\
\frac{mk^2}{a}\,\frac{d^2\theta_1}{dt^2} &= T'_1-T_1,\\
\frac{mk^2}{a}\,\frac{d^2\theta_2}{dt^2} &= T'_2-T_2,\\
&\cdots\cdots\cdots\\
\frac{mk^2}{a}\,\frac{d^2\theta_{n-1}}{dt^2} &= T'_{n-1}-T_{n-1},\\
\frac{mk^2}{a}\,\frac{d^2\theta_n}{dt^2} &= T'_n-T_n.
\end{aligned}\right\} \quad (176)$$

We have also the two following series of conditions; (177), because the pulleys are rough and the cords roll round them and do not slip; and (178), by reason of the geometrical relations of the system. (179) follow from (178);

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$$\left.\begin{aligned} dx &= -a\,d\theta, \\ dx_1 &= a\,d\theta_1, \\ dx_2 &= a\,d\theta_2, \\ &\cdots\cdots \\ dx_n &= a\,d\theta_n; \end{aligned}\right\} (177) \qquad \left.\begin{aligned} dx &= -2\,dx_1, \\ dx_1 &= 2\,dx_2, \\ dx_2 &= 2\,dx_3, \\ &\cdots\cdots \\ dx_{n-1} &= 2\,dx_n; \end{aligned}\right\} (178)$$

$$\left.\begin{aligned} dx &= -2^n\,dx_n, \\ dx_1 &= 2^{n-1}\,dx_n, \\ dx_2 &= 2^{n-2}\,dx_n, \\ &\cdots\cdots \\ dx_{n-1} &= 2\,dx_n; \end{aligned}\right\} (179)$$

Now $k^2 = \frac{a^2}{2}$; and consequently, taking the horizontal pairs of (175) and (176), and introducing the conditions (177), we have

$$\left.\begin{aligned} \text{M}\left(g - \frac{d^2x}{dt^2}\right) - \frac{m}{2}\,\frac{d^2x}{dt^2} &= \text{T}_1, \\ m\left(g - \frac{3}{2}\,\frac{d^2x_1}{dt^2}\right) &= 2\text{T}_1 - \text{T}_2, \\ m\left(g - \frac{3}{2}\,\frac{d^2x_2}{dt^2}\right) &= 2\text{T}_2 - \text{T}_3, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ m\left(g - \frac{3}{2}\,\frac{d^2x_{n-1}}{dt^2}\right) &= 2\text{T}_{n-1} - \text{T}_n, \\ \text{M}'\left(g - \frac{d^2x_n}{dt^2}\right) + m\left(g - \frac{3}{2}\,\frac{d^2x_n}{dt^2}\right) &= 2\text{T}_n; \end{aligned}\right\} (180)$$

whence, eliminating $\text{T}_1, \text{T}_2, \ldots \text{T}_n$, and replacing $x_1, x_2, \ldots$ in terms of x_n according to the values given in (179), we have

$$\left\{\text{M}' + \frac{m}{2}(2^{2n+1}-1) + \text{M}2^{2n}\right\}\frac{d^2x_n}{dt^2} = \{\text{M}' + m(2^n-1) - \text{M}\,2^n\}g; \quad (181)$$

and this determines the place of M' at the time t. From this value may be deduced, by means of (179), the places of M and of the centre of every pulley at any time.

If $n = 1$, the system is that of a single fixed and of a single moving pulley; and we have

$$\left\{\text{M}' + \frac{7m}{2} + 4\text{M}\right\}\frac{d^2x'}{dt^2} = \{\text{M}' + m - 2\text{M}\}g.$$

If we equate to zero the right hand member of (181) we have the condition of statical equilibrium of the system of weights and pulleys, when the weights of the pulleys are taken account of.

Ex. 12. Determine the motion of a screw which descends in

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its nut by reason of a given weight acting on it, when a constant power applied at the end of the lever is not sufficient to maintain equilibrium.

Let a = the radius of the cylinder which carries the screw, α = the angle of inclination of the thread to the axis of the screw; l = the length of the lever, P = the force applied perpendicularly to l at its extremity; m = the mass of the screw, and M = the mass by the weight of which the screw descends; R and F the sums of the reactions and sliding frictions respectively of the parts of the screw on its nut; μ = the coefficient of sliding friction. We shall suppose that M does not rotate as the screw revolves; then the equations which determine the motion are

$$(m+\text{M})\left(g-\frac{d^2z}{dt^2}\right) = \text{R}\sin\alpha + \text{F}\cos\alpha;$$

$$m\frac{a^2}{2}\frac{d^2\theta}{dt^2} = (\text{R}\cos\alpha - \text{F}\sin\alpha)a - \text{P}l;$$

$$\text{F} = \mu\text{R};$$

$$dz = a\,d\theta\tan\alpha;$$

whence $\frac{d^2z}{dt^2}$ may easily be found, and all the circumstances of motion may be determined.

208.] In the course of the preceding problems some subjects have incidentally arisen in particular forms which require fuller and more general discussion.

The first is the general case of the rocking, or titubation as it has been called, of a heavy body bounded by a cylindrical surface, resting on another rough cylindrical surface, the axes of the two surfaces being parallel and horizontal, when the upper body which rests on the lower surface is slightly displaced from its position of equilibrium.

Fig. 57, which represents a section of the two surfaces by a vertical plane perpendicular to the axes of the cylinders, shews the circumstances at the time t.

G is the centre of gravity of the upper body, whose mass $= m$; and when the upper body is at rest on the lower, A is in contact with O, and the line GA, which is the normal of the upper surface at A, is vertical, and is in the same straight line with OC, which is the normal to the lower surface at O. Let the upper body be slightly displaced by rolling, not sliding, on the lower;

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so that the arcs AP and OP are equal. Let C′ and C be the centres of curvature of the upper and lower surfaces at A and O respectively. The normals to the two surfaces at P are evidently in the same straight line; and since AP and OP are infinitesimal, C′A = C′P, and CO = CP. Let R = the normal pressure of the two surfaces on each other, and let F = the friction of rolling; also let k = the radius of gyration of the moving body relative to an axis through G parallel to the axes of the cylinders.

Let O be the origin, (x, y) the place of G at the time t; CO = CP = ρ; C′A = C′P = ρ'; C′G = c, OCP = θ, AC′P = θ'; consequently $\rho\theta = \rho'\theta'$, and

$$\theta + \theta' = \frac{\rho+\rho'}{\rho'}\theta.$$

The equations of motion are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \mathrm{R}\sin\theta - \mathrm{F}\cos\theta, \\ m\frac{d^2y}{dt^2} &= \mathrm{R}\cos\theta + \mathrm{F}\sin\theta - mg, \\ mk^2\frac{d^2(\theta+\theta')}{dt^2} &= -\mathrm{R}c\sin\theta' + \mathrm{F}(\rho' - c\cos\theta'); \end{aligned}\right\} \quad (182)$$

and the geometrical equations of condition are

$$\left.\begin{aligned} x &= (\rho+\rho')\sin\theta - c\sin(\theta+\theta'), \\ y &= -\rho + (\rho+\rho')\cos\theta - c\cos(\theta+\theta'). \end{aligned}\right\} \quad (183)$$

As the displacement which we are considering is very small, I shall assume θ and θ' to be so small that all powers of them above the first may be neglected. I shall also assume $\frac{d\theta}{dt}$ to be so small that all powers of it above the first may be neglected: under these suppositions, the preceding give

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \left\{\rho + \rho' - c\frac{\rho+\rho'}{\rho'}\right\}\frac{d^2\theta}{dt^2}, \\ \frac{d^2y}{dt^2} &= \left\{-(\rho+\rho')\theta + c\left(\frac{\rho+\rho'}{\rho'}\right)^2\theta\right\}\frac{d^2\theta}{dt^2}; \end{aligned}\right\} \quad (184)$$

and substituting these values in the first two of (182), we have

$$\mathrm{R} = mg + m\frac{(\rho+\rho')\rho}{\rho'}c\theta\frac{d^2\theta}{dt^2}; \quad (185)$$

$$\mathrm{F} = mg\theta - m\frac{\rho+\rho'}{\rho'}(\rho' - c)\frac{d^2\theta}{dt^2}; \quad (187)$$

which determine R and F in terms of $\frac{d^2\theta}{dt^2}$. And if we substitute

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these values in the last of (182), and omit terms involving powers of θ higher than the first, we have

$$\{k^2+(\rho'-c)^2\}\frac{d^2\theta}{dt^2}-\frac{\rho'^2-c(\rho+\rho')}{\rho+\rho'}g\theta=0; \qquad (187)$$

which is the equation of rotation of the upper body about its rotation-axis through G.

Since the coefficient of $\frac{d^2\theta}{dt^2}$ is positive, the form of the integral of this equation will depend on the sign of the coefficient of θ.

(1) If ρ'^2 is greater than $c(\rho+\rho')$; then the integral of (187) takes the logarithmic form, and θ will continually increase as t increases; so that the body moves farther away from its original position of rest, that position being one of unstable equilibrium. The geometrical meaning of this criterion is

$$\frac{1}{AG} \text{ is less than } \frac{1}{\rho}+\frac{1}{\rho'}.$$

(2) If $\rho'^2=c(\rho+\rho')$, that is, if

$$\frac{1}{AG}=\frac{1}{\rho}+\frac{1}{\rho'}, \qquad \frac{d^2\theta}{dt^2}=0,$$

and the body either remains at rest in its new position, or rotates with a constant angular velocity. The original equilibrium in this case is neutral.

(3) If ρ'^2 is less than $c(\rho+\rho')$, the integral of (187) takes the form of a circular function, and indicates an oscillatory motion; in which case the body rocks or titubates; and the time of an oscillation

$$=\pi\frac{\{k^2+(\rho'-c)^2\}^{\frac{1}{2}}(\rho+\rho')^{\frac{1}{2}}}{g^{\frac{1}{2}}\{c(\rho+\rho')-\rho'^2\}^{\frac{1}{2}}}. \qquad (188)$$

In this case, the original equilibrium is stable; and we have

$$\frac{1}{AG} \text{ greater than } \frac{1}{\rho}+\frac{1}{\rho'}.$$

The geometrical criteria for the stability, neutrality, and instability of equilibrium, are the same as those found from statical considerations in Art. 124, Vol. III.

The process and the results of this Article are equally true whatever are the signs and values of ρ, ρ', and c. Thus, if ρ is negative, the lower surface has its concavity upwards, and the problem is that of a body with a convex surface rolling on a concave surface. If ρ' is negative, we have a body with a con-

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cave surface rolling on a convex surface. If $\rho = \infty$, the lower surface is plane, and a body with a convex surface rolls upon it. If $\rho' = \infty$, the upper surface is plane, and the body with a plane surface rolls on a convex surface.

209.] Another principle, which has arisen incidentally in the preceding Examples, and which admits of more general application, is that of small oscillations.

When a system of material particles, subject to mutual connections, is slightly disturbed from a position of stable equilibrium, certain forces are brought into action, which tend to restore the system to its original place of rest. We have had examples of such forces in the preceding Article; and in Examples 4, 7, 8, of Art. 207. Now the equations of connection of the several particles enable us to express the coordinates of the places of these particles at the time t in their disturbed state as functions of new variables $\theta, \phi, \psi, \ldots$, which are, all and each, as well as their t-differentials, infinitesimals when the displacement of the system is infinitesimal, and are equal to zero when the system is in its place of stable equilibrium; and the number of these variables is of course that which is sufficient to determine the places of the particles, subject as they are to their mutual connections. If then we substitute these variables for the old variables in the equations of motion, we thereby obtain new differential equations which correctly represent the circumstances of motion in terms of the new variables. These equations are of the second order, and contain no term independent of the variables $\theta, \phi, \psi, \ldots$; because they must be satisfied when the system is in its position of stable equilibrium; that is, when $\theta = \phi = \psi = \ldots = 0$: and since the displacement of the system is, by our assumption, small, these variables and their differentials are small; if therefore our object is to acquire a general idea of the motion, that is, to obtain the principal motion, we must in the first place neglect all powers of the variables and of their t-differentials higher than the first. With this object in view the functions of the variables which enter into the equations must be expanded in ascending powers of the variable; and the squares, and all higher powers of them, are to be omitted; thus the differential equations will be linear, of the second order, of constant coefficients, and devoid of the second members; and their integrals will be of the following forms:

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$$\left.\begin{aligned}\theta &= \text{A}_1 \cos(r_1 t - a_1) + \text{A}_2 \cos(r_2 t - a_2) + \dots,\\ \phi &= \text{B}_1 \cos(r_1 t - a_1) + \text{B}_2 \cos(r_2 t - a_2) + \dots,\\ \psi &= \text{C}_1 \cos(r_1 t - a_1) + \text{C}_2 \cos(r_2 t - a_2) + \dots,\\ &\dots\dots\dots\dots\dots\dots;\end{aligned}\right\} \quad (189)$$

where $r_1, r_2, \dots$ are the roots of a certain algebraical equation of an order equal to the number of the variables; $\text{A}_1, \text{A}_2, \dots, a_1, a_2, \dots$ are constants depending on the initial or other circumstances of the system; and $\text{B}_1, \text{B}_2, \dots, \text{C}_1, \text{C}_2, \dots$ are other constants, which are functions of $\text{A}_1, \text{A}_2, \dots r_1, r_2, \dots$, and are determinable by means of the given differential equations.

If n is the number of the variables $\theta, \phi, \psi, \dots$, each of the equations in (189) is the sum of n terms, which are circular functions of t; each term by itself representing a small oscillation of the same nature as that of the simple pendulum; the times of oscillation corresponding to each term being different, and being severally $\frac{\pi}{r_1}, \frac{\pi}{r_2}, \dots$; and each variable generally containing a term of each period. Thus the motion of the system, slightly disturbed from its position of stable equilibrium, consists of simple oscillations of its several component particles, both the amplitudes and the periodic times being in general different for the several oscillations. As these oscillations coexist, and as each variable is the sum of many, the principle of their combination is commonly called the law of the coexistence of small oscillations.

If the quantities $r_1, r_2, r_3, \dots$ are commensurable, the system of particles will periodically pass through the same state; for suppose μ to be the greatest common measure of $r_1, r_2, \dots$; so that

$$r_1 = k_1 \mu, \qquad r_2 = k_2 \mu, \qquad r_3 = k_3 \mu, \dots;$$

where $k_1, k_2, k_2, \dots$ are whole numbers which have no common measure; then if τ is the time in which the system passes from a given state to the same state again, $k_1 \mu \tau, k_2 \mu \tau, \dots$ must all be multiples of 2π; and as $k_1, k_2, \dots$ have no common factor, the least value of τ which will satisfy this condition is

$$\tau = \frac{2\pi}{\mu}; \quad (190)$$

this therefore is the time in which the system of particles passes through all its forms from one state to the same state again.

If $\mu = 0$, this time is infinite; that is, if the quantities $r_1, r_2, \dots$ have no common measure, the system of particles is

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not periodic; and the state in which the particles may be at a given time is never taken by the particles again.

210.] Since each of these small motions takes effect separately, and independently of other similar motions; and since the whole effect is the sum of these separate and partial effects; the law of co-existence of small oscillations is a particular case of the so called principle of superposition of small motions. This principle may be explained in the following way.

Suppose that for certain initial values of the variables and their t-differentials, say,

$$\left.\begin{array}{lll} \theta = \theta_1, & \phi = \phi_1, & \psi = \psi_1, \ldots, \\ \dfrac{d\theta}{dt} = \theta_1', & \dfrac{d\phi}{dt} = \phi_1', & \dfrac{d\psi}{dt} = \psi_1', \ldots, \end{array}\right\} \tag{191}$$

the motion is represented by the integrals,

$$\theta = \Theta_1, \quad \phi = \Phi_1, \quad \psi = \Psi_1, \ldots; \tag{192}$$

and suppose that for another system of values, say,

$$\left.\begin{array}{ll} \theta = \theta_2, & \phi = \phi_2, \ldots, \\ \dfrac{d\theta}{dt} = \theta_2', & \dfrac{d\phi}{dt} = \phi_2', \ldots, \end{array}\right\} \tag{193}$$

the motion is represented by the integrals,

$$\theta = \Theta_2, \quad \phi = \Phi_2, \quad \psi = \Psi_2, \ldots; \tag{194}$$

and so on for n systems: then for the systems of values which are the sums of all these values, viz.

$$\theta = \theta_1 + \theta_2 + \ldots, \quad \phi = \phi_1 + \phi_2 + \ldots, \quad \psi = \psi_1 + \psi_2 + \ldots,$$

$$\frac{d\theta}{dt} = \theta_1' + \theta_2' + \ldots, \quad \frac{d\phi}{dt} = \phi_1' + \phi_2' + \ldots, \quad \frac{d\psi}{dt} = \psi_1' + \psi_2' + \ldots,$$

the motion is represented by the sum of the partial integrals; viz.,

$$\left.\begin{array}{l} \theta = \Theta_1 + \Theta_2 + \ldots, \\ \phi = \Phi_1 + \Phi_2 + \ldots, \\ \psi = \Psi_1 + \Psi_2 + \ldots; \end{array}\right\} \tag{195}$$

for these values will satisfy the differential equations of motion by reason of their linearity; and they reduce themselves to the several initial values when $t = 0$; thus they satisfy all the conditions of the problem.

The preceding processes are only applicable when we confine ourselves to small motions and to first approximations. If a more exact determination is required we must return to the

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original equations of motion in their complete forms, and substitute in terms of the second degree relatively to the variables those values which we have found in terms of t to a first approximation; and then, neglecting all the terms of a degree higher than the second, we shall have new equations which will differ from the first only by the addition of a new member, which is a known function of t. Values of $\theta, \phi, \ldots$ will be determined from these to an approximation higher than the former. And if an approximation is required still more exact, we must introduce the second values of the variables in the original equations, and pursue a process similar to the former.

211.] The following examples are illustrative of the preceding method.

Ex. 1. I will first take the simple case of a conical pendulum, that is, of a heavy particle constrained to move on the inside of a smooth spherical surface; this problem is the same as that which has been considered in Art. 370, Vol. III.

Let us refer the position of the moving particle to the point of suspension of the pendulum as the origin, and to two vertical planes passing through that point, and perpendicular to each other, as the planes of (x, z) and (y, z), the axis of z being taken vertically downwards. Let us moreover suppose the rod of the pendulum, whose length $= l$, initially to be in the plane of (x, z), and to be inclined at an angle $= a$ to the z-axis; and the bob to be projected with a velocity $= u$ perpendicularly to the plane of (x, z); let the line of the pendulum at the time t be projected on the planes of (x, z) and (y, z); and let the angles between these projections and the z-axis be respectively θ and ψ: now I shall assume that the oscillations of the pendulum are always small, and I shall consequently consider θ and ψ, the variables which determine its position, to be so small that powers of them higher than the second are to be neglected. The initial values of θ and ψ are respectively a and 0; and of $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}$ are 0 and $\frac{u}{l}$; then the equations of motion are

$$l\frac{d^2\theta}{dt^2} = -g\theta, \qquad l\frac{d^2\psi}{dt^2} = -g\psi;$$

$$\therefore \quad l\frac{d\theta^2}{dt^2} = g(a^2-\theta^2), \qquad l\left\{\frac{d\psi^2}{dt^2} - \frac{u^2}{l^2}\right\} = -g\psi^2;$$

$$\theta = a\cos\left(\frac{g}{l}\right)^{\frac{1}{2}}t, \qquad \psi = \frac{u}{(gl)^{\frac{1}{2}}}\sin\left(\frac{g}{l}\right)^{\frac{1}{2}}t.$$

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Let (x, y) be the projection on the horizontal plane of (x, y) of the place of the bob of the pendulum at the time t; so that

$$\begin{aligned} x &= l\sin\theta, & y &= l\sin\psi, \\ &= l\theta, & &= l\psi \\ &= la\cos\left(\frac{g}{l}\right)^{\frac{1}{2}}t; & &= u\left(\frac{l}{g}\right)^{\frac{1}{2}}\sin\left(\frac{g}{l}\right)^{\frac{1}{2}}t; \end{aligned}$$

let $y = x\tan\phi$; then

$$\tan\phi = \frac{u}{a(lg)^{\frac{1}{2}}}\tan\left(\frac{g}{l}\right)^{\frac{1}{2}}t;$$

so that ϕ, which determines the plane of oscillation of the pendulum at the time t, does not vary directly with t, and consequently the pendulum does not revolve uniformly. Also

$$\frac{x^2}{l^2a^2} + \frac{gy^2}{lu^2} = 1, \tag{196}$$

which represents an ellipse; so that the bob of the pendulum describes a path whose projection on the plane of (x, y) is an ellipse. All these results are in accordance with those of Art. 370, Vol. III.

Ex. 2. A system of n heavy rods OA_1, $\text{A}_1\text{A}_2, \ldots$, of given lengths $2a_1, 2a_2, \ldots 2a_n$ is formed by means of smooth hinges at their extremities $\text{A}_1, \text{A}_2, \ldots$, as in Fig. 58, and is suspended by the extremity O from a fixed point. Determine the small oscillations of the system, when the motions of all are in the same vertical plane of (x, y).

Let the angles which the rods respectively make with the vertical $\text{O}y$ be $\theta_1, \theta_2, \ldots$; and let $(x_1, y_1), (x_2, y_2), \ldots$ be the places of their centres of gravity at the time t; let $m_1, m_2, m_3, \ldots$ be the masses of the rods, and $k_1, k_2, \ldots$ their radii of gyration relative to axes passing through their centres of gravity, and perpendicular to the plane of (x, y). Let $\text{X}_1, \text{Y}_1, \text{X}_2, \text{Y}_2, \ldots$ be respectively the horizontal and vertical components of the actions of the hinges at $\text{A}_1, \text{A}_2, \ldots$; then the complete equations of motion are

$$m_1(a_1^2 + k_1^2)\frac{d^2\theta_1}{dt^2} = -m_1ga_1\sin\theta_1 + 2a_1(\text{X}_1\cos\theta_1 - \text{Y}_1\sin\theta_1); \tag{197}$$

$$\left.\begin{aligned} m_2\frac{d^2x_2}{dt^2} &= -\text{X}_1 + \text{X}_2, \\ m_2\frac{d^2y_2}{dt^2} &= m_2g - \text{Y}_1 + \text{Y}_2, \\ m_2k_2^2\frac{d^2\theta_2}{dt^2} &= a_2\{(\text{X}_1 + \text{X}_2)\cos\theta_2 - (\text{Y}_1 + \text{Y}_2)\sin\theta_2\}; \end{aligned}\right\} \tag{198}$$

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$$\left.\begin{aligned} m_3 \frac{d^2 x_3}{dt^2} &= -\mathrm{X}_2 + \mathrm{X}_3, \\ m_3 \frac{d^2 y_3}{dt^2} &= m_3 g - \mathrm{Y}_2 + \mathrm{Y}_3, \\ m_3 k_3^2 \frac{d^2 \theta_3}{dt^2} &= a_3 \{(\mathrm{X}_2 + \mathrm{X}_3) \cos\theta_3 - (\mathrm{Y}_2 + \mathrm{Y}_3) \sin\theta_3\}; \end{aligned}\right\} \quad (199)$$

. . . . . .

$$\left.\begin{aligned} m_n \frac{d^2 x_n}{dt^2} &= -\mathrm{X}_{n-1}, \\ m_n \frac{d^2 y_n}{dt^2} &= m_n g - \mathrm{Y}_{n-1}, \\ m_n k_n^2 \frac{d^2 \theta_n}{dt^2} &= a_n \{\mathrm{X}_{n-1} \cos\theta_n - \mathrm{Y}_{n-1} \sin\theta_n\}; \end{aligned}\right\} \quad (200)$$

but the variable coordinates are subject to the following equations:

$$\left.\begin{aligned} x_2 &= 2a_1 \sin\theta_1 + a_2 \sin\theta_2, \\ y_2 &= 2a_1 \cos\theta_1 + a_2 \cos\theta_2; \end{aligned}\right\} \quad (201)$$

$$\left.\begin{aligned} x_3 &= 2a_1 \sin\theta_1 + 2a_2 \sin\theta_2 + a_3 \sin\theta_3, \\ y_3 &= 2a_1 \cos\theta_1 + 2a_2 \cos\theta_2 + a_3 \cos\theta_3; \end{aligned}\right\} \quad (202)$$

. . . . . .

$$\left.\begin{aligned} x_n &= 2a_1 \sin\theta_1 + \ldots + 2a_{n-1} \sin\theta_{n-1} + a_n \sin\theta_n, \\ y_n &= 2a_1 \cos\theta_1 + \ldots + 2a_{n-1} \cos\theta_{n-1} + a_n \cos\theta_n. \end{aligned}\right\} \quad (203)$$

From these we have, omitting squares and higher powers of $\theta_1, \theta_2, \ldots$,

$$\left.\begin{aligned} \frac{d^2 x_2}{dt^2} &= 2a_1 \frac{d^2 \theta_1}{dt^2} + a_2 \frac{d^2 \theta_2}{dt^2}, \\ \frac{d^2 y_2}{dt^2} &= -2a_1 \theta_1 \frac{d^2 \theta_1}{dt^2} - a^2 \theta^2 \frac{d^2 \theta_2}{dt^2}, \end{aligned}\right\} \quad (204)$$

. . . . . .

and substituting these several values in the first two of each of the preceding groups (198), (199), ..., we shall have equations involving the second t-differentials of the θ's only; from these we can eliminate the X's and the Y's, and thereby obtain a series of equations in terms of the θ's and their second t-differentials only.

Let us take a particular case, and suppose that there are only two beams; then, in addition to (197), we have what (200) becomes when $n = 2$; viz.,

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$$\left.\begin{aligned} m_2 \frac{d^2 x_2}{dt^2} &= -\mathrm{X}_1, \\ m_2 \frac{d^2 y_2}{dt^2} &= m_2 g - \mathrm{Y}_1, \\ m_2 k_2^2 \frac{d^2 \theta_2}{dt^2} &= a_2 \{\mathrm{X}_1 \cos\theta_2 - \mathrm{Y}_1 \sin\theta_2\}; \end{aligned}\right\} \tag{205}$$

$$\left.\begin{aligned} \therefore \quad \mathrm{X}_1 &= -m_2 \left\{2 a_1 \frac{d^2 \theta_1}{dt^2} + a_2 \frac{d^2 \theta_2}{dt^2}\right\}, \\ \mathrm{Y}_1 &= m_2 g + m_2 \left\{2 a_1 \theta_1 \frac{d^2 \theta_1}{dt^2} + a_2 \theta_2 \frac{d^2 \theta_2}{dt^2}\right\}; \end{aligned}\right\} \tag{206}$$

and consequently from (197), and the last of (205), we have,

$$\{m_1(a_1^2 + k_1^2) + 4 m_2 a_1^2\} \frac{d^2 \theta_1}{dt^2} + 2 a_1 a_2 m_2 \frac{d^2 \theta_2}{dt^2} = -a_1 (m_1 + 2 m_2) \theta_1 g,$$

$$2 a_1 a_2 m_2 \frac{d^2 \theta_1}{dt^2} + m_2 (a_2^2 + k_2^2) \frac{d^2 \theta_2}{dt^2} = -m_2 a_2 \theta_2 g;$$

which are two linear differential equations whereby θ_1 and θ_2 are to be found in terms of t. The form of them is that which has been explained in Art. 209; for both are satisfied for the position of equilibrium, when the beams are vertical, and $\theta_1 = \theta_2 = 0$.

Now these equations are of the form.

$$\left.\begin{aligned} \mathrm{A} \frac{d^2 \theta_1}{dt^2} + \mathrm{B} \frac{d^2 \theta_2}{dt^2} &= -\alpha \theta_1, \\ \mathrm{B} \frac{d^2 \theta_1}{dt^2} + \mathrm{C} \frac{d^2 \theta_2}{dt^2} &= -\beta \theta_2; \end{aligned}\right\} \tag{207}$$

which may be expressed in the form

$$\left.\begin{aligned} \left(\mathrm{A} \frac{d^2}{dt^2} + \alpha\right) \theta_1 + \mathrm{B} \frac{d^2}{dt^2} \theta_2 &= 0, \\ \mathrm{B} \frac{d^2}{dt^2} \theta_1 + \left(\mathrm{C} \frac{d^2}{dt^2} + \beta\right) \theta_2 &= 0; \end{aligned}\right\} \tag{208}$$

and eliminating θ_2, we have

$$\left\{(\mathrm{AC} - \mathrm{B}^2) \frac{d^4}{dt^4} + (\mathrm{A}\beta + \mathrm{C}\alpha) \frac{d^2}{dt^2} + \alpha\beta\right\} \theta_1 = 0. \tag{209}$$

Now the four values of $\frac{d}{dt}$, which make the first factor in the left hand member of this equation to vanish, are impossible; let them be $\pm (-)^{\frac{1}{2}} r_1$, $\pm (-)^{\frac{1}{2}} r_2$; then the solution of (209) is

$$\theta_1 = \mathrm{E}_1 \cos(r_1 t - \gamma_1) + \mathrm{F}_1 \cos(r_2 t - \gamma_2); \tag{210}$$

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where $E_1, F_1, \gamma_1, \gamma_2$, are four constants introduced in integration; which are to be determined from initial or other circumstances.

If we eliminate $\frac{d^2\theta_2}{dt^2}$ from the two equations of (207), we have

$$(AC-B^2)\frac{d^2\theta_1}{dt^2}+Ca\theta_1-B\beta\theta_2=0;$$

and replacing θ_1 in terms of t, by means of (210),

$$\theta_2 = E_2\cos(r_1 t-\gamma_1)+F_2\cos(r_2 t-\gamma_2); \tag{211}$$

where E_2 and F_2 are constants, which are functions of the former constants, of r_1, r_2, and of E_1 and F_1; we are consequently able to determine all these constants in terms of initial values of $\theta_1, \theta_2, \frac{d\theta_1}{dt}, \frac{d\theta_2}{dt}$; thus the variables which determine the position of the beams consist of two circular functions, whose complete periodic times are respectively $\frac{2\pi}{r_1}$ and $\frac{2\pi}{r_2}$. If r_1 and r_2 are prime to each other, a position of the beams, which exists either initially or at any other time, never recurs. If however they have a common measure $=\mu$, the state of the beams is the same after every interval of time $=\frac{2\pi}{\mu}$.

Ex. 3. An uniform heavy rod of length $2a$ is suspended from a fixed point by means of a string of length l, whose weight may be neglected. The rod is slightly displaced from its position of equilibrium; it is required to determine its small oscillations.

Let the point of suspension be taken for the origin; and let the horizontal plane through it be the plane of (x, y), and let the z-axis be taken positively downwards. Let m = the mass of the rod; and at the time t let T = the tension of the string, (x, y, z) the place of the extremity of the string, (x', y', z') the place of the centre of gravity of the rod; (ξ, η, ζ) the place of any element dm of the rod, the distance of which from the upper extremity $= s$. Then the equations of motion of the centre of gravity of the rod are

$$\left.\begin{aligned} m\frac{d^2x'}{dt^2} &= -T\frac{x}{l}, \\ m\frac{d^2y'}{dt^2} &= -T\frac{y}{l}, \\ m\frac{d^2z'}{dt^2} &= mg-T\frac{z}{l}; \end{aligned}\right\} \tag{212}$$

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but since the displacement of the string and beam is always small, x, y, x', y' are always small, and approximately, $z = l$, $z' = l + a$; consequently for the preceding equations we have their approximate values

$$\left.\begin{aligned} \frac{d^2 x'}{dt^2} &= -g\frac{x}{l}, \\ \frac{d^2 y'}{dt^2} &= -g\frac{y}{l}, \\ \text{T} &= mg. \end{aligned}\right\} \qquad (213)$$

The equation of moments relative to the x-axis is

$$\text{Z}.m\left\{\eta\frac{d^2\zeta}{dt^2} - \zeta\frac{d^2\eta}{dt^2}\right\} = \text{Z}.m(\eta\text{Z} - \zeta\text{Y});$$

but approximately,

$$\frac{d^2\zeta}{dt^2} = 0; \qquad \eta = y + (y' - y)\frac{s}{a}; \qquad \zeta = l + s;$$

so that we have

$$-\int_0^{2a} m\frac{l+s}{2a}\left\{\frac{d^2y}{dt^2} + \frac{d^2(y'-y)}{dt^2}\frac{s}{a}\right\}ds = mgy' - y\text{T} + l\text{T}\frac{y}{l};$$

$$\therefore \quad a\frac{d^2y}{dt^2} - (4a + 3l)\frac{d^2y'}{dt^2} = 3gy'. \qquad (214)$$

Similarly the equation of moments relative to the y-axis is

$$a\frac{d^2x}{dt^2} - (4a + 3l)\frac{d^2x'}{dt^2} = 3gx'. \qquad (215)$$

These two equations together with the first two are sufficient to determine the motion.

Eliminating y' between the second of (213) and (214) we have

$$\frac{d^4y}{dt^4} + \frac{4a+3l}{al}g\frac{d^2y}{dt^2} + \frac{3g^2}{al}y = 0; \qquad (216)$$

and taking the symbols of operation only, we have

$$\frac{d^4}{dt^4} + \frac{4a+3l}{al}g\frac{d^2}{dt^2} + \frac{3g^2}{al} = 0.$$

The four values of $\frac{d}{dt}$ which satisfy the equation are evidently impossible; let them be $\pm(-)^{\frac{1}{2}}r_1$, $\pm(-)^{\frac{1}{2}}r_2$; then the solution of (216) is

$$y = \text{E}_1\cos(r_1 t - a_1) + \text{F}_1\cos(r_2 t - a_2);$$

where E_1, F_1, a_1, a_2 are constants depending on the initial or other circumstances of the beam and string. The other variables have the following values:

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$$y' = \text{E}_1' \cos(r_1 t - a_1) + \text{F}_1' \cos(r_2 t - a_2),$$
$$x = \text{E}_2 \cos(r_1 t - \beta_1) + \text{F}_2 \cos(r_2 t - \beta_2),$$
$$x' = \text{E}_2' \cos(r_1 t - \beta_1) + \text{F}_2' \cos(r_2 t - \beta_2);$$

and each of these variables involves the same two circular functions, of which the periodic times are respectively $\frac{2\pi}{r_1}$ and $\frac{2\pi}{r_2}$, although the amplitudes of vibration and the commencement of the periodic times are different for each. All the undetermined variables can be found in terms of the initial circumstances of the rods.

212.] In connection with this theory of small oscillations another problem, which arises out of the formulæ of the preceding Chapter, requires investigation; for the circumstances of it at the time t may be expressed by means of variables which vanish when the system is in a position of equilibrium; and for small values of which the system in motion is in a state approximate to that of equilibrium. The problem is this:

A body which has one point in it fixed is in motion about an instantaneous axis, the angle of inclination of which to a principal axis of the body at the fixed point is always small; it is required to determine the motion when the body makes small oscillations about its mean position.

Let the principal axis of the body with which the instantaneous always nearly coincides be the ζ-axis; so that ω_1 and ω_2 are always small quantities, the products and powers of which above the second we shall omit in our approximations. Also ω_3 is nearly constant: we shall take n to represent its mean value, and shall replace ω_3 by n in small terms. As the motion of the body is small L, M, N are also supposed to be small. Under these circumstances Euler's equations are

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C} - \text{B})\, n\, \omega_2 &= \text{L}, \\ \text{B}\frac{d\omega_2}{dt} + (\text{A} - \text{C})\, n\, \omega_1 &= \text{M}, \\ \text{C}\frac{d\ \ _3}{dt} \qquad\qquad\qquad &= \text{N}; \end{aligned}\right\} \tag{217}$$

Let us refer the motion to the axes of (x, y, z), fixed in space by means of the direction-cosines given in 39 and 40, Art. 2, of the present volume. Let the mean position of the ζ-principal axis be the z-axis, so that the angle between the z- and the ζ-axes is

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always small; consequently c_3, which is the cosine of the angle contained between these axes, is always nearly equal to 1, and may be replaced by 1 in small terms. Hence also it follows, that c_1, c_2, a_3, b_3 are always small quantities. And since

$$\omega_z = a_3\,\omega_1 + b_3\,\omega_2 + c_3\,\omega_3,$$

we may replace ω_z by ω_3, that is, by n in small terms. Now replacing ω_x and ω_y by their values given in (83) and (84), Art. 40, we have

$$\begin{aligned}
\omega_1 = \quad & a_1\,\omega_x + a_2\,\omega_y + a_3\,\omega_z \\
= \quad & a_1\left\{a_2\frac{da_3}{dt} + b_2\frac{db_3}{dt} + c_2\frac{dc_3}{dt}\right\} \\
& - a_2\left\{a_1\frac{da_3}{dt} + b_1\frac{db_3}{dt} + c_1\frac{dc_3}{dt}\right\} \\
& + a_3\,n \\
= \quad & (a_1\,b_2 - a_2\,b_1)\frac{db_3}{dt} + (a_1\,c_2 - a_2\,c_1)\frac{dc_3}{dt} + a_3 n \\
= \quad & c_3\frac{db_3}{dt} - b_3\frac{dc_3}{dt} + a_3\,n;
\end{aligned}$$

now $c_3 = 1$ in small terms, and its variation is so small that $\frac{dc_3}{dt}$ must be omitted;

$$\therefore \quad \omega_1 = \frac{db_3}{dt} + a_3\,n. \tag{218}$$

Similarly,

$$\omega_2 = -\frac{da_3}{dt} + b_3\,n. \tag{219}$$

Let us substitute these values in (217); and we have, omitting the subscript 3,

$$\left.\begin{aligned}
\text{A}\frac{d^2b}{dt^2} + (\text{A}+\text{B}-\text{C})n\frac{da}{dt} + n^2(\text{C}-\text{B})b &= \text{L}, \\
-\text{B}\frac{d^2a}{dt^2} + (\text{A}+\text{B}-\text{C})n\frac{db}{dt} + n^2(\text{A}-\text{C})a &= \text{M}.
\end{aligned}\right\} \tag{220}$$

L and M, which are the moments of the impressed couples whose axes are the ξ- and η-axes respectively, must be expressed in terms of a and b; as these however are small quantities, and as L and M both vanish in the state of equilibrium when $a = b = 0$, these quantities are of the following forms;

$$\left.\begin{aligned}
\text{L} &= p\,a + p'\,b, \\
\text{M} &= q\,a + q'\,b;
\end{aligned}\right\} \tag{221}$$

so that finally the differential equations in terms of a and b are of the forms

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$$\left.\begin{aligned}\frac{d^2b}{dt^2}+\alpha\frac{da}{dt}+\beta^2a+\beta'^2b&=0,\\ \frac{d^2a}{dt^2}-\alpha\frac{db}{dt}+\gamma^2a+\gamma'^2b&=0;\end{aligned}\right\}\qquad(222)$$

which are two simultaneous differential equations* of the second order, and are integrable by the processes explained in Vol. II. Hereby a and b will be expressed in terms of t; and as they are the cosines of the angles contained between the fixed z-axis and the moving axes of ξ and η respectively, so will they determine the position of the three principal axes of the body at the time t; and (218) and (219) will give ω_1 and ω_2 in terms of t; and as $\omega_3 = n$, so will also ω be known; and the position of the instantaneous rotation-axis will be given.

213.] Let us apply these equations to the solution of a problem which we have before considered.

A heavy conical top, whose vertex is on a rough horizontal plane and does not move, rotates with its axis of figure nearly vertical; it is required to determine the circumstances of motion.

In this case the vertex of the top may be considered as a fixed point, about which the body moves. We will take it to be the origin, and the horizontal plane through it to be the plane of (x, y); also $\text{A} = \text{B}$; and C is the principal moment of inertia relative to the axis of figure. Let h = the distance of the centre of gravity of the top from the vertex: then, as the line of action of gravity is parallel to the z-axis, we have, as in Art. 167, $\text{L} = mghb$, $\text{M} = -mgha$, $\text{N} = 0$; so that (220) become

$$\left.\begin{aligned}\text{A}\frac{d^2b}{dt^2}+(2\text{A}-\text{C})n\frac{da}{dt}+n^2(\text{C}-\text{A})b&=mghb,\\ \text{A}\frac{d^2a}{dt^2}-(2\text{A}-\text{C})n\frac{db}{dt}+n^2(\text{C}-\text{A})a&=mgha;\end{aligned}\right\}\qquad(223)$$

of which the form is

$$\left.\begin{aligned}\frac{d^2b}{dt^2}+2\alpha\frac{da}{dt}+\beta^2b&=0,\\ \frac{d^2a}{dt^2}-2\alpha\frac{db}{dt}+\beta^2a&=0;\end{aligned}\right\}\qquad(224)$$

* These equations are the same in *form* as those given in "An Elementary Treatise on the Dynamics of a System of Rigid Bodies," by E. J. Routh, M. A., Cambridge, 1860; equations A, p. 174. It is indeed to a study of that treatise that I owe the thought of transforming generally the first two of Euler's equations into (222) as their equivalents.

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whence, eliminating b, we have

$$\left\{\frac{d^4}{dt^4}+2(\beta^2+2a^2)\frac{d^2}{dt^2}+\beta^4\right\}a=0. \tag{225}$$

Now the form of a which is derived from this equation depends on the nature of the roots of the equation

$$r^4+2(\beta^2+2a^2)r^2+\beta^4=0. \tag{226}$$

Solving this, we have

$$\begin{aligned} r &= \pm(-)^{\frac{1}{2}}\{a\pm(a^2+\beta^2)^{\frac{1}{2}}\} \\ &= \pm(-)^{\frac{1}{2}}\left\{\frac{2\text{A}-\text{C}}{2\text{A}}n\pm\frac{(\text{C}^2n^2-4\text{A}mgh)^{\frac{1}{2}}}{2\text{A}}\right\}; \end{aligned} \tag{227}$$

and thus we have three cases:

(1) If C^2n^2 is greater than $4\text{A}mgh$; that is, if

$$n^2 \text{ is greater than } \frac{4\text{A}mgh}{\text{C}^2}, \tag{228}$$

the four values of r are impossible; let them be

$$\pm(-)^{\frac{1}{2}}r_1, \qquad \pm(-)^{\frac{1}{2}}r_2;$$

then the solution of (225) is

$$a=c_1\sin(r_1t+\gamma_1)+c_2\sin(r_2t+\gamma_2); \tag{229}$$

where c_1, c_2, γ_1, γ_2 are four constants, which are to be determined by the initial or other circumstances of the top. In this case the motion is stable, and the top makes oscillations about its mean place. Thus the stability of the motion depends on the inequality (228), which shews that the angular velocity of the top about its own axis must be greater than a certain assigned quantity.

(2) If $\text{C}^2n^2=4\text{A}mgh$; then $r=\pm(-)^{\frac{1}{2}}a$; and

$$a=(c_1+c_2t)\sin(at+\gamma); \tag{230}$$

where c_1, c_2, and γ are constants depending on the initial circumstances. This result shews generally, that a has periodical values; but that, as c_1+c_2t increases with the time, its maximum values increase; and thus the motion of the top is still oscillatory.

(3) If C^2n^2 is less than $4\text{A}mgh$, then

$$\begin{aligned} r &= \pm\left\{\pm\frac{(4\text{A}mgh-\text{C}^2n^2)^{\frac{1}{2}}}{2\text{A}}+(-)^{\frac{1}{2}}\frac{2\text{A}-\text{C}}{2\text{A}}n\right\} \\ &= \pm\{\pm\rho+(-)^{\frac{1}{2}}r\}; \end{aligned}$$

$$\therefore\quad a=c_1e^{\rho t}\sin(rt+\gamma_1)+c_2e^{-\rho t}\sin(rt+\gamma_2); \tag{231}$$

where c_1, c_2, γ_1, and γ_2 are constants to be determined by the

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initial circumstances. This form shews that a increases without limit as t increases without limit; but a is the cosine of an angle, and cannot exceed 1; consequently this form soon ceases to represent the motion of the top.

From the value of a determined as above, that of b may be deduced by means of either of the equations (224).

214.] Another subject, which has arisen in the course of the examples in Art. 207, requires a few words of explanation. We have frequently met with a resistance or a force arising from friction; and we have assumed the force to act in a direction contrary to that of the motion: we have spoken too of a friction of *rolling* as distinct from a friction of *sliding*. This distinction, as well as the dynamical effects of the two kinds, we proceed to explain more fully; and we must begin with certain laws which have been experimentally observed; for although these are, a priori, reasonable, yet they depend upon the physical constitution of matter; and our knowledge of molecular physics is, as yet, too uncertain, so that any proof derived from that source should supersede proof drawn from observation.

Friction of sliding has been considered statically in Section 3, Chapter III, of Vol. III, and the three laws therein stated are sufficient for the dynamical effects which we have now to consider. I shall take the case of a heavy body placed on a rough inclined plane, or on a rough curved surface, and sliding down it, so that its velocity is retarded by friction.

From the laws just alluded to it appears, (1) that, so long as the weight is the same, the friction is independent of the area of the surfaces in contact; (2) that the friction varies as the normal pressure exerted by the heavy body against the surface on which it moves. It also appears that the body does not begin to move unless the inclination of the plane, or if it is on a curved surface, of the tangent plane of the point at which it is placed, is equal to or exceeds a certain angle called "the angle of repose," and that, if $\mu =$ the coefficient of friction, this angle $= \tan^{-1}\mu$. It appears also from law III of the section above cited, that the friction is independent of the velocity of sliding. I propose now to apply these laws to some problems.

Ex. 1. A particle is placed on a rough inclined plane, the angle of inclination of which to the horizon is not less than the angle of repose; it is required to determine the motion.

Let $\alpha =$ the angle of inclination of the plane; $m =$ the mass

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of the particle; R = the normal pressure on the plane; F = the retarding force of friction; μ = the coefficient of friction. Let x = the distance along the plane through which the particle has moved in the time t; then

$$m\frac{d^2x}{dt^2} = mg\sin a - F,$$

$$R = mg\cos a,$$

$$F = \mu R;$$

$$\therefore \quad \frac{d^2x}{dt^2} = g\{\sin a - \mu\cos a\};$$

from which equation the motion may be determined.

Ex. 2. A particle slides down within a rough circular cylinder whose axis is horizontal: determine the equation of motion.

Let θ = the angular distance from the lowest point at the time; then the equation of motion is evidently

$$-a\frac{d^2\theta}{dt^2} = g\{\sin\theta - \mu\cos\theta\};$$

$$\therefore \quad -a\frac{d\theta^2}{dt^2} = 2g\{\cos a - \cos\theta + \mu(\sin a - \sin\theta)\};$$

where a is the value of θ, when the particle is at rest.

215.] In these cases the friction has been that of sliding only; and although the inclination of the plane has, in the latter example, been less than the angle of repose, yet, by reason of the previously acquired momentum, the particle has still continued to move. In cases however of bodies moving in contact with rough surfaces there may be friction of rolling as well as friction of sliding. If a cube is placed on an inclined plane whose inclination to the horizon is less than 45°, the cube will not fall over, and will slide down if the angle of repose is less than that of the inclination of the plane. If however a heavy sphere is placed on a rough inclined plane it will always roll; it will moreover slide as well as roll if the angle of inclination of the plane is greater than its angle of repose. Now, if the sphere rolls only, the process taken in Ex. 1, Art. 207, determines the motion; whereas, if it slides as well as rolls, other terms are required in the equations.

We may form a tolerably precise notion of the friction of rolling by imagining a heavy cylinder or wheel rolling on a horizontal plane. By reason of the compressibility of the matter in con-

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tact, the cylinder and the plane mutually penetrate each other; and hence arise reactions, acting on the cylinder in a direction contrary to that in which it is moving, and which act as obstacles to its rolling. The friction of rolling is measured by the horizontal force which it is necessary to apply to the axis of the wheel to maintain an uniform velocity of translation of the cylinder. Experiments were made in this subject by Coulomb; and he discovered the following law: the force of rolling friction for a heavy cylinder and a given plane varies directly as the pressure, and inversely as the radius of the cylinder. Thus, if R = the pressure of the cylinder on the plane, r = the radius of the cylinder, F = the rolling friction,

$$F = \nu \frac{R}{r}, \tag{232}$$

where ν is a constant, called the coefficient of rolling friction, which depends on the nature of the surfaces in contact.

Rolling friction as a retarding force is much less than sliding friction, and may be neglected when the latter acts.

In mechanical problems the difficulty frequently is to determine whether a body will slide and roll, or only roll: now let F = the friction, and R = the normal pressure; then if the ratio of F to R, which is equal to μ, is equal to, or greater than, $\tan \alpha$, the body only rolls; in which case a geometrical condition will exist, which may take the form of a relation between the space of translation described by the centre of gravity and that due to the rotation about the instantaneous axis passing through the centre of gravity: the use of this relation is evident in the examples of Art. 207. If however the ratio of F to R is less than $\tan \alpha$, the body will slide as well as roll, and the geometrical relation just alluded to ceases to hold good. The method of solution then to be applied will be evident from the following examples:

Ex. 1. A heavy sphere moves down a rough inclined plane, whose angle of inclination to the horizon is greater than that of repose; it is required to determine the motion.

Let α = the angle of inclination; μ = the coefficient of sliding friction; F = the sliding friction; R = the normal pressure; the equations by which the motion is determined are evidently

$$m \frac{d^2 s}{dt^2} = mg \sin \alpha - F,$$

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$$\mathrm{R} = mg\cos a,$$

$$\mathrm{F} = \mu\,\mathrm{R},$$

$$mk^2\frac{d^2\theta}{dt^2} = a\mathrm{F};$$

$$\left.\begin{aligned} \therefore\quad \frac{d^2s}{dt^2} &= g\{\sin a - \mu\cos a\}, \\ k^2\frac{d^2\theta}{dt^2} &= \mu g\cos a. \end{aligned}\right\} \qquad (233)$$

If the sphere only rolls, $ds = a\,d\theta$; in which case

$$\mu = \frac{\mathrm{F}}{\mathrm{R}} = \frac{k^2}{a^2+k^2}\tan a$$

$$= \frac{2}{7}\tan a;$$

consequently the sphere will slide and roll, or will only roll, according as $\tan a$ is greater than, or not greater than $\frac{7}{2}\mu$.

Ex. 2. If the body moving down the plane is a circular cylinder of radius $= a$, with its axis horizontal; then

$$3\mu = \tan a;$$

and the body will slide and roll, or roll only, according as a is greater or not greater than $\tan^{-1} 3\mu$.

Ex. 3. A heavy body whose bounding surface is a circular cylinder, but whose centre of gravity is not in the axis of the surface, makes small titubations on a rough horizontal plane. Determine the limits of the angle of titubation so that it may not slide on the plane.

Take the figure and symbols of Ex. 7, Art. 207; and let $\mu =$ the coefficient of sliding friction between the rocking body and the plane. Then, as the angle through which the body rocks is small, I shall neglect powers of θ higher than the second. Hence we have

$$\frac{d^2x}{dt^2} = (a-c)\frac{d^2\theta}{dt^2};$$

$$\therefore\quad -k^2\mathrm{F} = (a-c)^2\mathrm{F} - \mathrm{R}\,c\,(a-c)\,\theta;$$

$$\therefore\quad \frac{\mathrm{F}}{\mathrm{R}} = \mu = \frac{c\,(a-c)\,\theta}{(a-c)^2+k^2};$$

and consequently, if the body rolls and does not slide, θ must not be greater than $\dfrac{(a-c)^2+k^2}{c\,(a-c)}\mu$.

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216.] The following examples of motion of rigid bodies, free and constrained, involve various modes of application of preceding principles, and are inserted in illustration, as well as because many of them are in themselves of considerable interest.

A heavy homogeneous solid ellipsoid is struck by a blow whose momentum is Q, in a line parallel to one of its principal axes, and subsequently moves freely under the action of its weight; it is required to determine the motion.

Let M = the mass of the ellipsoid; and let the equation to the bounding surface relative to the centre as origin, and its three principal axes as coordinate axes, be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

and let us suppose the line of blow to be parallel to the z-axis, and to intersect the plane of (x, y) in the point (x_0, y_0).

The centre of gravity, which is the centre of the ellipsoid, will move as if it were a particle of mass = M; and consequently its path is a parabola; and if V is the initial velocity,

$$V = \frac{Q}{M};$$

and its initial line is that of the blow. Thus all the elements of its path are known.

The ellipsoid also rotates under the action of Q, as if the centre were a fixed point. Consequently, if A and B are the principal moments of inertia of the solid ellipsoid about the x- and y-axes respectively, the initial instantaneous axis through the centre is, see (73), Art. 194,

$$\frac{x\,x_0}{B} + \frac{y\,y_0}{A} = 0; \tag{234}$$

and if Ω is the initial angular velocity, see (72), Art. 194,

$$\Omega = Q\left\{\frac{y_0^2}{A^2} + \frac{x_0^2}{B^2}\right\}^{\frac{1}{2}}; \tag{235}$$

the combined effect of these two motions is indeed a rotation about an initial spontaneous axis, whose equation relative to the moving ellipsoid is

$$\frac{x\,x_0}{B} + \frac{y\,y_0}{A} + \frac{1}{M} = 0. \tag{236}$$

Since the initial rotation-axis, given by (234), is not a principal axis of the body it is not a permanent axis; consequently

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it continually moves both in the body and in space; that determined above being only its initial position. Its motion will be determined by Euler's three equations, simplified by the condition, that momenta are impressed only at the origin; so that we have

$$\left.\begin{aligned} \mathrm{A}\frac{d\omega_1}{dt} + (\mathrm{C}-\mathrm{B})\,\omega_2\,\omega_3 &= 0, \\ \mathrm{B}\frac{d\omega_2}{dt} + (\mathrm{A}-\mathrm{C})\,\omega_3\,\omega_1 &= 0, \\ \mathrm{C}\frac{d\omega_3}{dt} + (\mathrm{B}-\mathrm{A})\,\omega_1\,\omega_2 &= 0; \end{aligned}\right\} \qquad (237)$$

the initial values of ω_1, ω_2, ω_3 being respectively $\frac{\mathrm{Q}\,y_0}{\mathrm{A}}$, $-\frac{\mathrm{Q}\,x_0}{\mathrm{B}}$, and 0. These equations however have been so fully discussed in the preceding Chapter that it is unnecessary to say more on the subject.

If either x_0 or $y_0 = 0$, the initial instantaneous axis is a principal axis, and therefore is a permanent axis; and the ellipsoid during its motion in space uniformly revolves about this axis. Thus, if $x_0 = 0$, the line of the blow Q is in the plane of (y, z) and is parallel to the z-axis, and the x-axis is the instantaneous rotation-axis, which is also the permanent rotation-axis; and the permanent angular velocity of the body about it

$$= \frac{\mathrm{Q}\,y_0}{\mathrm{A}}$$

$$= \frac{5\,\mathrm{Q}\,y_0}{\mathrm{M}\,(b^2+c^2)}.$$

217.] A right cone is placed with its slant side on a perfectly rough inclined plane, and rolls on it by the action of its weight; it is required to determine its motion.

Let $2\,a$ = the vertical angle of the cone; M = its mass; a = its height; and let β = the inclination of the plane to the horizon.

The forces acting on the cone are, its weight, the rolling friction of the cone on the inclined plane, and the normal reaction of the plane. As the plane is perfectly rough, and the cone rolls on its convex surface without sliding, the place of the vertex of the cone is always the same, and the motion is that of a rotating body which has a fixed point in its axis. We shall therefore investigate it by means of Euler's three equations. Now the line of contact of the cone with the plane

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is evidently always the instantaneous rotation-axis; and as the force of rolling friction, as well as the normal reaction of the plane, acts through this line, they produce relatively to it no angular velocity: it will be convenient to derive from Euler's equations the equation of rotation relative to this line.

Let C be the principal moment of inertia of the cone relative to its own axis; and let A be that relative to an axis perpendicular to the axis of the cone and passing through the vertex; so that Euler's equations are

$$\left.\begin{aligned} \mathrm{A}\frac{d\omega_1}{dt} + (\mathrm{C}-\mathrm{A})\,\omega_2\,\omega_3 &= \mathrm{L}, \\ \mathrm{A}\frac{d\omega_2}{dt} - (\mathrm{C}-\mathrm{A})\,\omega_3\,\omega_1 &= \mathrm{M}, \\ \mathrm{C}\frac{d\omega_3}{dt} &= \mathrm{N}; \end{aligned}\right\} \tag{238}$$

Let us suppose the x-axis, to which A corresponds, to be initially in the inclined plane, and θ to be the angle through which this axis, and consequently the cone, rotates in the time t. Let ω be the instantaneous angular velocity at the time t; then, relative to the three principal axes of the body, the direction-cosines of the instantaneous axis are evidently $\sin a \cos\theta$, $\sin a \sin\theta$, and $\cos a$; so that

$$\omega_1 = \omega \sin a \cos\theta, \qquad \omega_2 = \omega \sin a \sin\theta, \qquad \omega_3 = \omega \cos a; \tag{239}$$

and if G is the moment of the impressed forces relative to the instantaneous rotation-axis,

$$\mathrm{G} = \mathrm{L}\sin a\cos\theta + \mathrm{M}\sin a\sin\theta + \mathrm{N}\cos a. \tag{240}$$

Also
$$\omega = \frac{d\theta}{dt}. \tag{241}$$

Now, substituting these values in (240), we have

$$\{\mathrm{A}\,(\sin a)^2 + \mathrm{C}\,(\cos a)^2\}\,\frac{d^2\theta}{dt^2} = \mathrm{G}.$$

Let ϕ be the angle at the time t contained between the line of contact of the cone with the plane, and a straight line on the plane perpendicular to a horizontal line. Then, as the cone rolls on the plane, evidently
$$-\,a\,d\phi = a\tan a\,dt; \tag{242}$$

and as the weight, which acts at the centre of gravity of the cone, is the only force which impresses angular velocity on the body relative to the instantaneous rotation-axis, and tends to increase θ,
$$\mathrm{G} = \mathrm{M}\,g\,\frac{3}{4}\,a\sin a\sin\beta\sin\phi. \tag{243}$$

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Also $\text{A}(\sin a)^2 + \text{C}(\cos a)^2 = \frac{3\text{M}}{20}\{6 + (\tan a)^2\} a^2 (\sin a)^2$;

so that $$a\frac{d^2\phi}{dt^2} = -5g\frac{(\cos a)^2 \sin\beta}{(\sin a)^3\{6+(\tan a)^2\}}\sin\phi; \qquad (244)$$

which equation determines the motion.

If $\phi = \phi_0$ when the cone is at rest, then, integrating (244), we have

$$a\left(\frac{d\phi}{dt}\right)^2 = 10g\frac{(\cos a)^2 \sin\beta}{(\sin a)^3\{6+(\tan a)^2\}}\{\cos\phi - \cos\phi_0\}. \qquad (245)$$

If the cone makes small oscillations on the plane, the time of an oscillation

$$= \pi\left\{\frac{a(\sin a)^3\{6+(\tan a)^2\}}{5g(\cos a)^2\sin\beta}\right\}^{\frac{1}{2}}.$$

If the cone is fixed by its vertex to a point in a rough perpendicular wall, and rolls on the wall, then the above equations determine the motion, when $\beta = 90°$.

218.] Determine the motion of a top whose apex moves on a smooth horizontal plane.

The motion of a top has already come twice into consideration; viz., in Art. 167—172, where we have generally investigated the motion of a heavy rigid body with a point fixed, and having two equal principal moments of inertia relative to that point; and also again in Art. 213, where the small motions of it about a mean position have been investigated in illustration of the general law of small oscillations. In both these cases the apex of the top has been assumed to be a fixed point; and this condition is approximately satisfied when the top moves on a perfectly rough plane. If however the plane is smooth, the apex moves in the plane; and we propose now to investigate its motion, and the motion of the top.

I shall assume the centre of gravity to be in the geometrical axis of the top, and at a distance equal to l from the apex or peg of the top. And I shall also use the same symbols in the same significance as in Art. 167, except that the origin of the several axes to which the rotation is referred will be taken at the centre of gravity.

As the plane is perfectly smooth, the only forces acting on the body are the vertical reaction of the plane $= \text{R}$ (say), and the weight of the top; so that if (ξ, η, ζ) is the place of the centre of gravity relative to a system of axes fixed in space, of

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which that of ζ is vertical, and those of ξ and η are in the given plane, then

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} = \frac{d^2\eta}{dt^2} = 0, \\ m\frac{d^2\zeta}{dt^2} = \text{R} - mg; \end{aligned}\right\} \tag{246}$$

from the first two of which it is plain that in horizontal motion the centre of gravity either remains at rest or moves uniformly in a rectilineal path, the elements of which depend on the initial impulsion.

Also, since $\zeta = l\cos\theta$,

$$\text{R} = m\left\{\frac{d^2.l\cos\theta}{dt^2} + g\right\}. \tag{247}$$

Next let us consider the motion of the top relative to the centre of gravity. The only force which produces a moment about that point is the pressure of the plane at the apex, of which the value is given in (247); and we have, as in Art. 167,

$$\left.\begin{aligned} \text{L} &= \text{R}\, l\sin\theta\cos\phi, \\ \text{M} &= -\text{R}\, l\sin\theta\sin\phi, \\ \text{N} &= 0. \end{aligned}\right\} \tag{248}$$

Now $\text{A} = \text{B}$; and consequently the third of Euler's equations, by reason of the third of (248), gives

$$\omega_3 = \text{a constant} = n \text{ (say)}; \tag{249}$$

and thus the first two of Euler's equations are

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})\, n\,\omega_2 &= \text{R}\, l\sin\theta\cos\phi, \\ \text{A}\frac{d\omega_2}{dt} - (\text{C}-\text{A})\, n\,\omega_1 &= -\text{R}\, l\sin\theta\sin\phi. \end{aligned}\right\} \tag{250}$$

Let us suppose the whole initial angular velocity of the body to be about the axis of the top, and to be n; so that initially $\omega_1 = \omega_2 = 0$, and consequently $\frac{d\theta}{dt} = \frac{d\psi}{dt} = 0$; and let us suppose the initial values of θ to be θ_0, and of ϕ and ψ to be zero. From (250) we have, as in Art. 167,

$$\begin{aligned} \text{A}\left\{\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt}\right\} &= \text{R}\, l\sin\theta\,\{\omega_1\cos\phi - \omega_2\sin\phi\} \\ &= \text{R}\, l\sin\theta\frac{d\theta}{dt} \\ &= ml\sin\theta\frac{d\theta}{dt}\left\{g - l\cos\theta\left(\frac{d\theta}{dt}\right)^2 - l\sin\theta\frac{d^2\theta}{dt^2}\right\}; \end{aligned}$$

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$$\therefore \quad \text{A}(\omega_1^2+\omega_2^2) = 2mlg(\cos\theta_0-\cos\theta)-ml^2(\sin\theta)^2\left(\frac{d\theta}{dt}\right)^2;$$

therefore

$$\{\text{A}+ml^2(\sin\theta)^2\}\left(\frac{d\theta}{dt}\right)^2+\text{A}(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 = 2mlg(\cos\theta_0-\cos\theta). \quad (251)$$

Also, multiplying the first of (250) by $\sin\phi$, and the second by $\cos\phi$, adding, and integrating, as in Art. 167, we have

$$\text{A}(\sin\theta)^2\frac{d\psi}{dt} = \text{c}\,n(\cos\theta_0-\cos\theta); \quad (252)$$

and eliminating $\frac{d\psi}{dt}$ by means of (251) and (252), we have

$$dt = \pm\frac{\{\text{A}^2+ml^2\text{A}(\sin\theta)^2\}^{\frac{1}{2}}\sin\theta\,d\theta}{\{\cos\theta_0-\cos\theta\}^{\frac{1}{2}}\{2mlg\text{A}(\sin\theta)^2-\text{c}^2n^2(\cos\theta_0-\cos\theta)\}^{\frac{1}{2}}}. \quad (253)$$

In these equations the following results are implied. Since the left hand member of (251) is necessarily positive, the right hand member is also positive; so that θ is never less than θ_0: thus the angle at which the axis of the top is inclined to the vertical is never less than its initial value, and θ increases until it reaches a value, say θ_1, at which the second radical in the denominator of (253) vanishes; then $\frac{d\theta}{dt}=0$, and the inclination of the axis to the vertical is a maximum; θ_1 is always less than π, because the expression which determines it is positive when $\theta=\theta_0$, and is negative when $\theta=\pi$: also the time in which the value of θ passes from θ_0 to θ_1 is finite, as we have proved in Art. 168. Thus the axis of the top makes isochronal oscillations in a vertical plane, as that plane revolves about the vertical axis of z.

That vertical plane however does not revolve uniformly; in other words, its precessional velocity, which $=\frac{d\psi}{dt}$, is not constant; (252) shews this.

According as n is positive or negative, so is the precession direct or retrograde; that is, the line of intersection of the equatorial plane of the top with the horizontal plane revolves in the same direction as the top rotates.

And the variations of the precessional velocity are periodic, having the same period as those of the inclination of the axis to the vertical; now the precessional velocity vanishes when $\theta=\theta_0$, and becomes a maximum when $\theta=\theta_1$; and continues to make these periodical oscillations. This is explained at greater length

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in Art. 168. Thus, if the centre of gravity of the top does not move, the apex of the top describes on the horizontal plane the curve delineated in Fig. 35, where the radius of the interior and exterior circles are respectively $l \sin \theta_0$, and $l \sin \theta_1$; and where the arcs of the path respectively touch the exterior circle when $\frac{d\psi}{dt}$ is a maximum; and meet the interior at right angles when $\frac{d\psi}{dt} = 0$.

It is evident, by the principle of vis viva, that the angular velocity of the top is a maximum when $\theta = \theta_1$, and is a minimum when $\theta = \theta_0$.

If n is very large, so that θ_1 is very little greater than θ_0, the values of θ are confined within very small limits. In this case we can, as in Art. 169, integrate (253) approximately, and obtain results which give an accurate representation of the motion of the top.

219.] On the motion of a heavy homogeneous spherical ball, (an ivory billiard ball,) on a rough horizontal table.

I shall take M to be the mass of the sphere, a to be its radius, A to be the moment of inertia about a diameter.

I shall suppose the ball to be put into motion initially by means of a blow, of which the intensity, line of action, and point of application are known; all these circumstances being given in billiards by the stroke of the cue. Thus, the initial velocity of translation of the centre of gravity, and the angular velocity relative to the initial rotation-axis will be known. During the subsequent motion, the ball will both roll and slide, so that retarding forces of rolling and sliding friction will be in action on it. That of sliding friction acts at the point of contact of the ball with the table, and in the line along which the point of contact slides on the table. That of rolling friction acts in the line along which at the time the centre of gravity is moving. This latter friction however is, in ordinary cases, only a very small fraction of the former, and may consequently be neglected. As by the roughness of the table the sliding motion of the ball is diminished much more rapidly than the rolling motion, so the sliding motion soon ceases, and the ball only rolls. The following equations will determine with accuracy the time and the place at which this cessation of sliding takes place.

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Let the plane in which the centre of the ball moves be that of (x, y); this plane is consequently parallel to that of the table, and is horizontal; let (x, y) be the place of the centre of the ball at the time t; so that at that time $(x, y, -a)$ is the place of the point of contact. Let F be the force of sliding friction which acts at the point of contact, and let α be the angle at which its line of action is inclined to the axis of x. Let R $(= \text{M}g)$ be the pressure of the ball on the plane, and let $\mu =$ the coefficient of sliding friction; so that $\text{F} = \mu\text{R} = \mu\text{M}g$.

The equations of motion of the centre of gravity are

$$\left.\begin{aligned} \text{M}\frac{d^2x}{dt^2} &= -\text{F}\cos\alpha, \\ \text{M}\frac{d^2y}{dt^2} &= -\text{F}\sin\alpha, \\ \text{M}\frac{d^2z}{dt^2} &= -\text{M}g + \text{R} = 0. \end{aligned}\right\} \tag{254}$$

Let us consider the rotation in reference to a system of axes originating at the centre of gravity, and parallel to the fixed axes of (x, y, z); then we have

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} &= -a\text{F}\sin\alpha, \\ \text{A}\frac{d\omega_2}{dt} &= a\text{F}\cos\alpha, \\ \text{A}\frac{d\omega_3}{dt} &= 0; \end{aligned}\right\} \tag{255}$$

$$\left.\begin{aligned} \therefore\quad \text{A}\frac{d\omega_1}{dt} &= a\text{M}\frac{d^2y}{dt^2}, \\ \text{A}\frac{d\omega_2}{dt} &= -a\text{M}\frac{d^2x}{dt^2}; \end{aligned}\right\} \tag{256}$$

$$\left.\begin{aligned} \therefore\quad \text{A}(\omega_1 - \Omega_1) &= a\text{M}\left(\frac{dy}{dt} - v_0\right), \\ \text{A}(\omega_2 - \Omega_2) &= -a\text{M}\left(\frac{dx}{dt} - u_0\right), \\ \omega_3 - \Omega_3 &= 0; \end{aligned}\right\} \tag{257}$$

where u_0, v_0 are the axial components of the initial velocities of the centre of gravity, and Ω_1, Ω_2, Ω_3 are the initial angular velocities about three axes originating at the centre of gravity and parallel to the fixed axes.

These equations connect the instantaneous angular velocity

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with the velocity of the centre of gravity of the ball. Thus ω_3 is constant, and ω_1 and ω_2 only vary when $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vary; that is, when F acts. And therefore if the centre of gravity moves uniformly in a straight line, the angular velocity of the ball and the direction of the rotation-axis do not vary, and there is no sliding friction. And since

$$A = \frac{2a^2}{5} M;$$

these become
$$\left.\begin{aligned} \omega_1 - \Omega_1 &= \frac{5}{2a}\left(\frac{dy}{dt} - v_0\right), \\ \omega_2 - \Omega_2 &= -\frac{5}{2a}\left(\frac{dx}{dt} - u_0\right). \end{aligned}\right\} \tag{258}$$

220.] By means of these we can determine the path of the centre of gravity of the ball, so long as the ball continues to slide. The line of action of F, as we have said, is that of the motion of the point of contact. Now the projections on the x- and y-axes respectively of the space described by this point in the time dt are $dx - a\omega_2 dt$, and $dy + a\omega_1 dt$; and by (258),

$$dx - a\omega_2 dt = \frac{7dx}{2} - \left(a\Omega_2 + \frac{5u_0}{2}\right)dt,$$

$$dy + a\omega_1 dt = \frac{7dy}{2} + \left(a\Omega_1 - \frac{5v_0}{2}\right)dt;$$

And these are proportional to $\cos a$ and to $\sin a$, which enter into (254). To simplify these expressions however let

$$\left.\begin{aligned} u_1 &= \frac{2}{7}\left(\frac{5u_0}{2} + a\Omega_2\right), \\ v_1 &= \frac{2}{7}\left(\frac{5v_0}{2} - a\Omega_1\right), \end{aligned}\right\} \tag{259}$$

so that
$$\left.\begin{aligned} dx - a\omega_2 dt &= \frac{7}{2}(dx - u_1 dt), \\ dy + a\omega_1 dt &= \frac{7}{2}(dy - v_1 dt); \end{aligned}\right\} \tag{260}$$

and consequently from (254),

$$\frac{\frac{d^2x}{dt^2}}{\frac{dx}{dt} - u_1} = \frac{\frac{d^2y}{dt^2}}{\frac{dy}{dt} - v_1}; \tag{261}$$

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and integrating,
$$\frac{\frac{dx}{dt}-u_1}{u_0-u_1}=\frac{\frac{dy}{dt}-v_1}{v_0-v_1}; \tag{262}$$
hence also we have
$$\frac{\frac{d^2x}{dt^2}}{u_0-u_1}=\frac{\frac{d^2y}{dt^2}}{v_0-v_1}; \tag{263}$$
and consequently,
$$\frac{\cos a}{u_0-u_1}=\frac{\sin a}{v_0-v_1}; \tag{264}$$
and thus not only is F, the force of friction which retards the ball, constant in magnitude, being equal to μ M g, but the line of action of it has a constant direction. And therefore the centre of gravity of the ball describes a parabolic path, like a heavy projectile, the axis of the parabola being parallel to the line of action of the constant force of friction.

And a is the angle at which the line of action of F is inclined to the axis of x; and
$$\tan a=\frac{v_0-v_1}{u_0-u_1}; \tag{265}$$
u_1 and v_1 are given by (259) in terms of initial quantities. They are the axial components of the velocity of the centre of gravity of the ball at the instant when it ceases to slide, for then
$$\frac{dx}{dt}=a\omega_2=u_1, \qquad \frac{dy}{dt}=-a\omega_1=v_1;$$
the last terms of these equations following by reason of (260). This is also evident from (261); for when the ball only rolls F = 0; and consequently
$$\frac{d^2x}{dt^2}=\frac{d^2y}{dt^2}=0, \quad \text{and} \quad \frac{dx}{dt}=u_1, \quad \frac{dy}{dt}=v_1.$$
It is observed, that u_1 and v_1 depend not on the friction, but on the initial circumstances of motion.

Let
$$(u_0-u_1)^2+(v_0-v_1)^2=s^2; \tag{266}$$
so that (254) become
$$\frac{d^2x}{dt^2}=-\mu g\frac{u_0-u_1}{s},$$
$$\frac{d^2y}{dt^2}=-\mu g\frac{v_0-v_1}{s};$$
and therefore, if the initial place of the centre of gravity of the ball is taken as the origin,

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$$\left.\begin{aligned}\frac{dx}{dt} &= u_0 - \frac{\mu g\,(u_0-u_1)}{s}\,t,\\ \frac{dy}{dt} &= v_0 - \frac{\mu g\,(v_0-v_1)}{s}\,t\,;\end{aligned}\right\} \tag{267}$$

$$\left.\begin{aligned}x &= u_0 t - \frac{\mu g\,(u_0-u_1)}{2\,s}\,t^2,\\ y &= v_0 t - \frac{\mu g\,(v_0-v_1)}{2\,s}\,t^2;\end{aligned}\right\} \tag{268}$$

and eliminating t,

$$\{x(v_0-v_1)-y(u_0-u_1)\}^2 + \frac{2\,s}{\mu g}(u_1 v_0 - v_1 u_0)(u_0 y - v_0 x) = 0\,; \tag{269}$$

which is the equation to the parabolic path of the centre of gravity of the sphere.

221.] Now this equation becomes $y = x$, that is, represents a straight line when u_1, v_1 are respectively proportional to u_0, v_0; in which case we have, from (259),

$$u_0\,\Omega_1 + v_0\,\Omega_2 = 0\,; \tag{270}$$

which shews that the initial rotation-axis is in a vertical plane at right angles to the line of initial velocity of the centre of gravity.

The equation (269) expresses the path of the centre of the ball, so long as the ball slides as well as rolls; at the instant however when the sliding ceases, and the ball only rolls,

$$\frac{dx}{dt} = u_1 = a\,\omega_2, \qquad \frac{dy}{dt} = v_1 = -\,a\,\omega_1\,; \tag{271}$$

and from either of (267) we have

$$t = \frac{s}{\mu g}\,; \tag{272}$$

at which time, from (268), we have

$$x = \frac{s}{\mu g}\,\frac{u_0+u_1}{2}, \qquad y = \frac{s}{\mu g}\,\frac{v_0+v_1}{2}. \tag{273}$$

After this instant, if there were no friction of rolling, the ball would continue to move uniformly in a straight line, with a velocity of which u_1 and v_1 are the axial components. But as a friction of rolling acts to retard it, the ball continues its rectilineal course with a decreasing velocity, until it finally comes to rest; and the equation to the line along which the centre of gravity moves, and which is also the path of the point of contact with the table, is

$$2\,\mu g\,\{u_1 y - v_1 x\} + s\,(u_0 v_1 - v_0 u_1) = 0\,; \tag{274}$$

this line is evidently a tangent to the parabola at the point given by (273).

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222.] Thus we have arrived at the exact course which a billiard ball takes on a table, in the most general circumstances of a stroke of a cue. The motion is at first a mixed one of sliding and of rolling; and the centre of the ball moves in a parabola so long as the ball slides on the table, which causes a sliding friction at the point of contact; this sliding however eventually ceases, and before the ball comes to rest; and the centre of the ball then takes a rectilineal path, which is a tangent to the parabola at the point where the sliding has ceased.

Let us now give to the equations determined in the preceding Articles those interpretations which arise from initial circumstances produced by the stroke of a cue.

Let Q be the momentum with which the cue strikes the ball; and let α be the angle at which its line of action is inclined to the plane of the table. Let h be the horizontal distance from the centre of the ball to the vertical plane containing the axis of the cue, which is the line of blow; and let k be the perpendicular distance on this line from a horizontal line through the centre; let us moreover take the plane of (x, z) to be parallel to the axis of the cue. Let F be the friction which is brought into action between the ball and the table by the blow of the cue, and let β be the angle between the line along which F acts and the axis of x; let $\mu =$ the coefficient of friction; then, as in Art. 205, the equations of motion of the centre of gravity are,

$$\left.\begin{aligned} \text{M}\,u_0 &= \text{Q}\cos\alpha - \text{F}\cos\beta, \\ \text{M}\,v_0 &= -\text{F}\sin\beta, \\ 0 &= -\text{Q}\sin\alpha + \text{R} - \text{M}\,g\,; \end{aligned}\right\} \tag{275}$$

and the equations of rotation about the centre of gravity are

$$\left.\begin{aligned} \text{A}\,\Omega_1 &= -\text{Q}h\sin\alpha - a\,\text{F}\sin\beta, \\ \text{A}\,\Omega_2 &= \text{Q}k + a\,\text{F}\cos\beta, \\ \text{A}\,\Omega_3 &= -\text{Q}h\cos\alpha. \end{aligned}\right\} \tag{276}$$

Also $$\text{F} = \mu\,\text{R}, \qquad \text{A} = \frac{2a^2\,\text{M}}{5};$$

$$\frac{\cos\beta}{u_0 - a\,\Omega_2} = \frac{\sin\beta}{v_0 + a\,\Omega_1}. \tag{277}$$

From these equations we have

$$\tan\beta = \frac{5\,h\sin\alpha}{5k - 2a\cos\alpha}; \tag{278}$$

which assigns the direction in which the point of contact of the

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ball with the plane begins to move. This also is the direction of the constant retarding force of friction, and of the axis of the parabola in which the ball moves until sliding friction ceases. Hence also we can determine $u_0, v_0, \Omega_1, \Omega_2$; also we have

$$u_1 = \frac{5\,\text{Q}\,(a\cos\alpha + k)}{7\,a\,\text{M}}, \qquad v_1 = \frac{5\,\text{Q}\,h\sin\alpha}{7\,a\,\text{M}}; \tag{279}$$

which values are independent of the friction, as we have before observed in Art. 220.

If the path of the ball is rectilineal, and in the direction of the stroke of the cue, $\tan\beta = 0$, and $v_1 = 0$; hence, either $h = 0$, or $\sin\alpha = 0$; in the former case, the centre of the ball is in the vertical plane containing the axis of the cue; in the latter, the axis of the cue is horizontal. Under either of these circumstances therefore, and under these only, is the path of the ball rectilineal; in all other cases the path is parabolic.

If the path of the centre of the ball is not rectilinear, the value of v_1, given in (279), shews that the deviation lies on that side of the centre in its initial position, on which the stroke is given, for v_1 and h are of the same sign.

These several circumstances are represented in Fig. 59. O, the centre of the ball at the instant of the blow, is the origin; and the horizontal plane containing O is that of (x, y). The plane of the table is parallel to it, and at a distance below it equal to a, the radius of the ball. P is the point of contact of the ball with the table. QRT is the axis of the cue, the vertical plane parallel to which, and containing O, is the plane of (x, z). The circle is the section of the ball by the plane of (x, z). QRT pierces the plane of (y, z) at R, and the plane of the table at T, and α is the angle which it makes with the table. OH $= h$ is the perpendicular distance from O to the vertical plane RLT, containing the axis of the cue; and HK $= k$ is the perpendicular from H to the axis of the cue; so that h, k, and α determine the position of the line of blow. Let PP_1 be the curve which is the parabolic path of the point of contact of the ball with the table; P_1 being the point given by the coordinates (273), at which sliding friction ceases, and after which the path becomes rectilineal; let $\text{P}_1\,\text{S}_1$ be that rectilineal path; let RL $= l$; so that

$$k = (l - a)\cos\alpha; \tag{280}$$

thus $$u_1 = \frac{5\,\text{Q}\,l\cos\alpha}{7\,a\,\text{M}}, \qquad v_1 = \frac{5\,\text{Q}\,h\sin\alpha}{7\,a\,\text{M}};$$

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therefore
$$\frac{v_1}{u_1} = \frac{h}{l \cot a} = \frac{\text{MT}}{\text{MP}}$$
$$= \tan \text{MPT}. \tag{281}$$

but $\frac{v_1}{u_1}$ is the tangent of the angle between the axis of x and the rectilineal path taken by the ball when friction ceases. Consequently the rectilineal path P_1S_1 is parallel to PT. Hence the final direction of the ball when friction ceases is easily determined; it is parallel to the line drawn from the point of contact of the ball with the table to the point at which the axis of the cue pierces the table.

Hence it follows, that if the axis of the cue is inclined to the plane of (x, y), at an angle so large that the point T falls on the negative side of the line PL, the ball in its final state moves in a direction opposite to that of the stroke.

Explanation of these formulæ, in further application of them to the game of billiards, will be found in "Théorie Mathématique des effets du jeu de Billiard," par G. Coriolis; Paris, 1835.

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CHAPTER VIII.

RELATIVE MOTION OF A MATERIAL SYSTEM.

SECTION 1.—*Investigation of the general equations.*

223.] IN the preceding parts of this work the motion of a material system has been investigated relatively to a fixed origin, and to a fixed system of coordinate axes. The coordinate axes indeed, to which we have found it convenient to refer the motion primarily, have not always been fixed; for in the last Chapter two systems were used, one of which was fixed in the body and moved with it; the other system however was absolutely fixed, and to it ultimately the motion of the material system was referred; and its incidents were deduced from the position of the parts of it relatively to that system. Now I propose to consider a more general case; and to investigate the motion of a material system relatively to a moving origin, and a moving system of rectangular axes, all the incidents of motion of these latter, as well as those of the material system, being given, and the material system also moving relatively to it. This general case is that of *relative motion;* that is, of the motion of a material system relatively to moving coordinate axes, to which allusion has already been made in Vol. III, Art. 259, in the case of a single particle; under circumstances however in which a single material particle moves, and the motion of the rectangular axes is simply that of translation. As the formulæ which express the circumstances of relative motion are long and complicated, it will be convenient to consider primarily the motion of one particle; and thus omit the signs of summation; but the result will be capable of the most general application, because, by D'Alembert's principle, they can, mutatis mutandis, be extended to systems of moving particles. Our process will be purely analytical. Doubtless hereby we are in danger of overlooking the full meaning of the symbols; we may perhaps lose sight in them of the mechanical truths which they represent; and thus our apprehension of things in their actual state may be indistinct. Consequently I

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shall interpret the equations, and shall shew that they are the expression of results arrived at from the consideration of relative motion and its incidents in their first principles, and from (as it is commonly said) general reasoning. The cinematics of relative motion first require investigation.

224.] Let m be the mass of the particle whose motion is to be considered; and let (x, y, z) be its place at the time t relatively to an origin and to a system of axes fixed absolutely in space. At the same time let (x_0, y_0, z_0) be the place of another origin, which moves; and relatively to it, and to a moving system of axes originating at it, let (ξ, η, ζ) be the place of m. Let the system of direction-cosines connecting these two systems of axes be that of Art. 2; then

$$\left.\begin{aligned} x &= x_0 + a_1\xi + b_1\eta + c_1\zeta, \\ y &= y_0 + a_2\xi + b_2\eta + c_2\zeta, \\ z &= z_0 + a_3\xi + b_3\eta + c_3\zeta; \end{aligned}\right\} \qquad (1)$$

and, as all the quantities in the right hand members are functions of t,

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_0}{dt} + \xi\frac{da_1}{dt} + \eta\frac{db_1}{dt} + \zeta\frac{dc_1}{dt} + a_1\frac{d\xi}{dt} + b_1\frac{d\eta}{dt} + c_1\frac{d\zeta}{dt}, \\ \frac{dy}{dt} &= \frac{dy_0}{dt} + \xi\frac{da_2}{dt} + \eta\frac{db_2}{dt} + \zeta\frac{dc_2}{dt} + a_2\frac{d\xi}{dt} + b_2\frac{d\eta}{dt} + c_2\frac{d\zeta}{dt}, \\ \frac{dz}{dt} &= \frac{dz_0}{dt} + \xi\frac{da_3}{dt} + \eta\frac{db_3}{dt} + \zeta\frac{dc_3}{dt} + a_3\frac{d\xi}{dt} + b_3\frac{d\eta}{dt} + c_3\frac{d\zeta}{dt}. \end{aligned}\right\} \qquad (2)$$

Let us examine the several terms of these equations. The left hand members are the axial components of the velocity of m relatively to the three fixed axes of (x, y, z); that is, of the absolute velocity of m. The first four terms of the right hand members are the value of these components when ξ, η, ζ do not vary; that is, when the place of m is fixed relatively to the moving origin and the moving axes; and thus they are the axial components of the velocity of m considered as a fixed point of the coordinate system of reference; the first terms of the several equations being due to the motion of translation of the origin; and the following three to the rotation of that system about an axis passing through that origin; and the last three terms in each equation are the projections of the fixed axes of the axial components of the velocity of m relatively to the moving axes and the moving origin. Equations (2) then yield the following theorem:

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The absolute velocity of the particle m is the resultant of its velocity relatively to the system of moving axes and of the velocity of it, considered as a fixed point of that system; or,

The velocity of a particle relative to a system of moving axes is the excess of its absolute velocity over the velocity of the system of axes.

If v_ξ, v_η, v_ζ are the components of the absolute velocity of m along the moving axes; that is, along the ξ-, η-, ζ-axes respectively; then

$$\begin{aligned} v_\xi &= a_1\frac{dx}{dt} + a_2\frac{dy}{dt} + a_3\frac{dz}{dt} \\ &= a_1\frac{dx_0}{dt} + a_2\frac{dy_0}{dt} + a_3\frac{dz_0}{dt} \\ &\qquad + \eta\left\{a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right\} \\ &\qquad + \zeta\left\{a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt}\right\} + \frac{d\xi}{dt}; \end{aligned} \tag{3}$$

$$\begin{aligned} v_\eta &= b_1\frac{dx_0}{dt} + b_2\frac{dy_0}{dt} + b_3\frac{dz_0}{dt} \\ &\qquad + \zeta\left\{b_1\frac{dc_1}{dt} + b_2\frac{dc_2}{dt} + b_3\frac{dc_3}{dt}\right\} \\ &\qquad + \xi\left\{b_1\frac{da_1}{dt} + b_2\frac{da_2}{dt} + b_3\frac{da_3}{dt}\right\} + \frac{d\eta}{dt}; \end{aligned} \tag{4}$$

$$\begin{aligned} v_\zeta &= c_1\frac{dx_0}{dt} + c_2\frac{dy_0}{dt} + c_3\frac{dz_0}{dt} \\ &\qquad + \xi\left\{c_1\frac{da_1}{dt} + c_2\frac{da_2}{dt} + c_3\frac{da_3}{dt}\right\} \\ &\qquad + \eta\left\{c_1\frac{db_1}{dt} + c_2\frac{db_2}{dt} + c_3\frac{db_3}{dt}\right\} + \frac{d\zeta}{dt}; \end{aligned} \tag{5}$$

in which equivalents for v_ξ, v_η, v_ζ all the terms except the last of each arise from the motion of the coordinate system of reference, and the last arise from the motion of m relatively to that system.

If ω_ξ, ω_η, ω_ζ are the axial components of the angular velocity of m at (ξ, η, ζ), considered as a point fixed relatively to the moving system, then, substituting from (88), Art. 40, we have

$$\left.\begin{aligned} v_\xi &= a_1\frac{dx_0}{dt} + a_2\frac{dy_0}{dt} + a_3\frac{dz_0}{dt} + \zeta\omega_\eta - \eta\omega_\zeta + \frac{d\xi}{dt}, \\ v_\eta &= b_1\frac{dx_0}{dt} + b_2\frac{dy_0}{dt} + b_3\frac{dz_0}{dt} + \xi\omega_\zeta - \zeta\omega_\xi + \frac{d\eta}{dt}, \\ v_\zeta &= c_1\frac{dx_0}{dt} + c_2\frac{dy_0}{dt} + c_3\frac{dz_0}{dt} + \eta\omega_\xi - \xi\omega_\eta + \frac{d\zeta}{dt}. \end{aligned}\right\} \tag{6}$$

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And thus along the axes of the moving system the axial components of the absolute velocity of m are the resultants of the axial components of (1) the absolute velocity of the origin, (2) of the velocity of m at (ξ, η, ζ) due to the angular velocities of the coordinate axes, (3) of the velocity of m relative to the moving axes.

If the origin of the moving system is fixed, and the axes only move, these become

$$\left.\begin{aligned} v_\xi &= \zeta\,\omega_\eta - \eta\,\omega_\zeta + \frac{d\xi}{dt}, \\ v_\eta &= \xi\,\omega_\zeta - \zeta\,\omega_\xi + \frac{d\eta}{dt}, \\ v_\zeta &= \eta\,\omega_\xi - \xi\,\omega_\eta + \frac{d\zeta}{dt}. \end{aligned}\right\} \tag{7}$$

255.] Suppose now that forces act on the system so that the absolute velocity of m varies, then, taking the t-differentials of (2), we have

$$\begin{aligned} \frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} &+ \xi\frac{d^2a_1}{dt^2} + \eta\frac{d^2b_1}{dt^2} + \zeta\frac{d^2c_1}{dt^2} \\ &+ 2\left\{\frac{d\xi}{dt}\frac{da_1}{dt} + \frac{d\eta}{dt}\frac{db_1}{dt} + \frac{d\zeta}{dt}\frac{dc_1}{dt}\right\} \\ &+ a_1\frac{d^2\xi}{dt^2} + b_1\frac{d^2\eta}{dt^2} + c_1\frac{d^2\zeta}{dt^2}; \end{aligned} \tag{8}$$

$$\begin{aligned} \frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} &+ \xi\frac{d^2a_2}{dt^2} + \eta\frac{d^2b_2}{dt^2} + \zeta\frac{d^2c_2}{dt^2} \\ &+ 2\left\{\frac{d\xi}{dt}\frac{da_2}{dt} + \frac{d\eta}{dt}\frac{db_2}{dt} + \frac{d\zeta}{dt}\frac{dc_2}{dt}\right\} \\ &+ a_2\frac{d^2\xi}{dt^2} + b_2\frac{d^2\eta}{dt^2} + c_2\frac{d^2\zeta}{dt^2}; \end{aligned} \tag{9}$$

$$\begin{aligned} \frac{d^2z}{dt^2} = \frac{d^2z_0}{dt^2} &+ \xi\frac{d^2a_3}{dt^2} + \eta\frac{d^2b_3}{dt^2} + \zeta\frac{d^2c_3}{dt^2} \\ &+ 2\left\{\frac{d\xi}{dt}\frac{da_3}{dt} + \frac{d\eta}{dt}\frac{db_3}{dt} + \frac{d\zeta}{dt}\frac{dc_3}{dt}\right\} \\ &+ a_3\frac{d^2\xi}{dt^2} + b_3\frac{d^2\eta}{dt^2} + c_3\frac{d^2\zeta}{dt^2}; \end{aligned} \tag{10}$$

which are severally the equivalents of the expressed velocity-increments along the fixed coordinate axes in terms of the elements of the moving system of axes.

Let $v_\xi', v_\eta', v_\zeta'$ be the axial components of the expressed velocity-increment along the moving axes of ξ, η, ζ, then

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$$
\begin{aligned}
v'_\xi = \; & a_1 \frac{d^2 x}{dt^2} + a_2 \frac{d^2 y}{dt^2} + a_3 \frac{d^2 z}{dt^2} \\
= \; & a_1 \left\{ \frac{d^2 x_0}{dt^2} + \xi \frac{d^2 a_1}{dt^2} + \eta \frac{d^2 b_1}{dt^2} + \zeta \frac{d^2 c_1}{dt^2} \right\} \\
& + a_2 \left\{ \frac{d^2 y_0}{dt^2} + \xi \frac{d^2 a_2}{dt^2} + \eta \frac{d^2 b_2}{dt^2} + \zeta \frac{d^2 c_2}{dt^2} \right\} \\
& + a_3 \left\{ \frac{d^2 z_0}{dt^2} + \xi \frac{d^2 a_3}{dt^2} + \eta \frac{d^2 b_3}{dt^2} + \zeta \frac{d^2 c_3}{dt^2} \right\} \\
& + 2 \left\{ a_1 \frac{db_1}{dt} + a_2 \frac{db_2}{dt} + a_3 \frac{db_3}{dt} \right\} \frac{d\eta}{dt} \\
& \qquad + 2 \left\{ a_1 \frac{dc_1}{dt} + a_2 \frac{dc_2}{dt} + a_3 \frac{dc_3}{dt} \right\} \frac{d\zeta}{dt} + \frac{d^2 \xi}{dt^2}; \quad (11)
\end{aligned}
$$

$$
\begin{aligned}
v'_\eta = \; & b_1 \frac{d^2 x}{dt^2} + b_2 \frac{d^2 y}{dt^2} + b_3 \frac{d^2 z}{dt^2} \\
= \; & b_1 \left\{ \frac{d^2 x_0}{dt^2} + \xi \frac{d^2 a_1}{dt^2} + \eta \frac{d^2 b_1}{dt^2} + \zeta \frac{d^2 c_1}{dt^2} \right\} \\
& + b_2 \left\{ \frac{d^2 y_0}{dt^2} + \xi \frac{d^2 a_2}{dt^2} + \eta \frac{d^2 b_2}{dt^2} + \zeta \frac{d^2 c_2}{dt^2} \right\} \\
& + b_3 \left\{ \frac{d^2 z_0}{dt^2} + \xi \frac{d^2 a_3}{dt^2} + \eta \frac{d^2 b_3}{dt^2} + \zeta \frac{d^2 c_3}{dt^2} \right\} \\
& + 2 \left\{ b_1 \frac{dc_1}{dt} + b_2 \frac{dc_2}{dt} + b_3 \frac{dc_3}{dt} \right\} \frac{d\zeta}{dt} \\
& \qquad + 2 \left\{ b_1 \frac{da_1}{dt} + b_2 \frac{da_2}{dt} + b_3 \frac{da_3}{dt} \right\} \frac{d\xi}{dt} + \frac{d^2 \eta}{dt^2}; \quad (12)
\end{aligned}
$$

$$
\begin{aligned}
v'_\zeta = \; & c_1 \frac{d^2 x}{dt^2} + c_2 \frac{d^2 y}{dt^2} + c_3 \frac{d^2 z}{dt^2} \\
= \; & c_1 \left\{ \frac{d^2 x_0}{dt^2} + \xi \frac{d^2 a_1}{dt^2} + \eta \frac{d^2 b_1}{dt^2} + \zeta \frac{d^2 c_1}{dt^2} \right\} \\
& + c_2 \left\{ \frac{d^2 y_0}{dt^2} + \xi \frac{d^2 a_2}{dt^2} + \eta \frac{d^2 b_2}{dt^2} + \zeta \frac{d^2 c_2}{dt^2} \right\} \\
& + c_3 \left\{ \frac{d^2 z_0}{dt^2} + \xi \frac{d^2 a_3}{dt^2} + \eta \frac{d^2 b_3}{dt^2} + \zeta \frac{d^2 c_3}{dt^2} \right\} \\
& + 2 \left\{ c_1 \frac{da_1}{dt} + c_2 \frac{da_2}{dt} + c_3 \frac{da_3}{dt} \right\} \frac{d\xi}{dt} \\
& \qquad + 2 \left\{ c_1 \frac{db_1}{dt} + c_2 \frac{db_2}{dt} + c_3 \frac{db_3}{dt} \right\} \frac{d\eta}{dt} + \frac{d^2 \zeta}{dt^2}; \quad (13)
\end{aligned}
$$

Let X, Y, Z be the axial components of the impressed velocity-increments on m parallel to the fixed coordinate axes of x, y, z;

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and let X', Y', Z' be the axial components of the same impressed velocity-increments parallel to the moving axes of ξ, η, ζ; then

$$\left.\begin{aligned} X' &= a_1 X + a_2 Y + a_3 Z, \\ Y' &= b_1 X + b_2 Y + b_3 Z, \\ Z' &= c_1 X + c_2 Y + c_3 Z; \end{aligned}\right\}$$

and the equations which express the motion of m are

$$v'_\xi = X', \qquad v'_\eta = Y', \qquad v'_\zeta = Z'; \tag{14}$$

and from these equations all the incidents of its motion are to be deduced.

226.] Before however they are applied to the solution of particular problems, let us examine them more closely, and detect the origin and the meaning of their several terms. This course has already been taken by Clairaut and Coriolis*; and has been fully explained by M. Bertrand (Journal de l'Ecole Polytechnique; XXXII Cahier, p. 149), so that little need be added to their labours. In the values for v'_ξ, v'_η, v'_ζ, given in (11), (12), and (13), it is evident that if the coordinate system of ξ, η, ζ, to which the place of m at the time t is referred, were fixed and immoveable, then the several right hand members would be reduced to their last terms $\frac{d^2\xi}{dt^2}, \frac{d^2\eta}{dt^2}, \frac{d^2\zeta}{dt^2}$; and the equations of motion would be for m the same as those which have been found in Vol. III; and for a material system, the same as those which are explained in chapter III of the present volume; consequently all the other terms arise from the motion of the coordinate system of reference. If therefore we consider the motion of m, relatively to the moving system of reference, to be absolute, just as motion has heretofore been considered and referred, then we must suppose m to be under the action of certain fictitious forces, as well as the actual forces, by which certain velocity-increments are impressed exactly equal to those which arise from the motion of the system of reference, and of which the mathematical expressions are given in the preceding equations. This is the view of the subject taken by Clairaut, Coriolis, and Bertrand. Now the velocity-increments which arise from these fictitious forces are of two kinds; (1) we have those which are

* See Mémoires de l'Académie des Sciences de Paris, 1742, p. 1. Journal de l'Ecole Polytechnique; XXI and XXIV Cahiers. Traité de la Mécanique des corps solides et du calcul de l'effet des Machines par G. Coriolis; 2nd edit., Paris, 1844, p. 46.

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independent of the motion of m relatively to the moving origin and axes; and which therefore are independent of the t-differtials of ξ, η, ζ. Coriolis has called the force whence these arise, "*force d'entrainement.*" We shall call them the velocity-increments of *transference;* meaning of course the transference of the coordinate system. These velocity-increments are contained in the first twelve terms in the right hand members of (11), (12), (13), severally. Let them be denoted by X_t, Y_t, Z_t respectively; so that

$$\begin{aligned}\mathrm{X}_t = \; & a_1\left\{\frac{d^2x_0}{dt^2}+\xi\frac{d^2a_1}{dt^2}+\eta\frac{d^2b_1}{dt^2}+\zeta\frac{d^2c_1}{dt^2}\right\}\\ & +a_2\left\{\frac{d^2y_0}{dt^2}+\xi\frac{d^2a_2}{dt^2}+\eta\frac{d^2b_2}{dt^2}+\zeta\frac{d^2c_2}{dt^2}\right\}\\ & +a_3\left\{\frac{d^2z_0}{dt^2}+\xi\frac{d^2a_3}{dt^2}+\eta\frac{d^2b_3}{dt^2}+\zeta\frac{d^2c_3}{dt^2}\right\}; \qquad (15)\end{aligned}$$

$$\begin{aligned}\mathrm{Y}_t = \; & b_1\left\{\frac{d^2x_0}{dt^2}+\xi\frac{d^2a_1}{dt^2}+\eta\frac{d^2b_1}{dt^2}+\zeta\frac{d^2c_1}{dt^2}\right\}\\ & +b_2\left\{\frac{d^2y_0}{dt^2}+\xi\frac{d^2a_2}{dt^2}+\eta\frac{d^2b_2}{dt^2}+\zeta\frac{d^2c_2}{dt^2}\right\}\\ & +b_3\left\{\frac{d^2z_0}{dt^2}+\xi\frac{d^2a_3}{dt^2}+\eta\frac{d^2b_3}{dt^2}+\zeta\frac{d^2c_3}{dt^2}\right\}; \qquad (16)\end{aligned}$$

$$\begin{aligned}\mathrm{Z}_t = \; & c_1\left\{\frac{d^2x_0}{dt^2}+\xi\frac{d^2a_1}{dt^2}+\eta\frac{d^2b_1}{dt^2}+\zeta\frac{d^2c_1}{dt^2}\right\}\\ & +c_2\left\{\frac{d^2y_0}{dt^2}+\xi\frac{d^2a_2}{dt^2}+\eta\frac{d^2b_2}{dt^2}+\zeta\frac{d^2c_2}{dt^2}\right\}\\ & +c_3\left\{\frac{d^2z_0}{dt^2}+\xi\frac{d^2a_3}{dt^2}+\eta\frac{d^2b_3}{dt^2}+\zeta\frac{d^2c_3}{dt^2}\right\}; \qquad (17)\end{aligned}$$

(2) we have the groups which stand fourth and fifth in (11), (12), (13) respectively; let us express them in terms of the angular velocities of m at (ξ, η, ζ) about the three moving coordinate axes, by means of the equivalents given in (88), Art. 40; they become respectively

$$2\left\{\omega_\eta\frac{d\zeta}{dt}-\omega_\zeta\frac{d\eta}{dt}\right\},\quad 2\left\{\omega_\zeta\frac{d\xi}{dt}-\omega_\xi\frac{d\zeta}{dt}\right\},\quad 2\left\{\omega_\xi\frac{d\eta}{dt}-\omega_\eta\frac{d\xi}{dt}\right\}; \qquad (18)$$

The form of these quantities shews that they are the axial components of a certain velocity-increment; this we proceed to determine; and omitting the numerical coefficient, we will denote it by F; let α, β, γ be the direction-angles of its line of action; then

$$\mathrm{F} = \frac{\omega_\eta\frac{d\zeta}{dt}-\omega_\zeta\frac{d\eta}{dt}}{\cos\alpha} = \frac{\omega_\zeta\frac{d\xi}{dt}-\omega_\xi\frac{d\zeta}{dt}}{\cos\beta} = \frac{\omega_\xi\frac{d\eta}{dt}-\omega_\eta\frac{d\xi}{dt}}{\cos\gamma}; \qquad (19)$$

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therefore $$\omega_\xi \cos\alpha + \omega_\eta \cos\beta + \omega_\zeta \cos\gamma = 0; \tag{20}$$

$$\cos\alpha \frac{d\xi}{dt} + \cos\beta \frac{d\eta}{dt} + \cos\gamma \frac{d\zeta}{dt} = 0; \tag{21}$$

which shew that the line of action of F is perpendicular both to the rotation-axis of the system of reference which passes through the moving origin, and to the line of the relative velocity of m. Let θ be the angle between these lines; let ω be the angular velocity of the coordinate system, and let v be the velocity of m relative to the moving system; then from (19) we have

$$\begin{aligned} F &= \left\{\omega^2 v^2 - \left(\omega_\xi \frac{d\xi}{dt} + \omega_\eta \frac{d\eta}{dt} + \omega_\zeta \frac{d\zeta}{dt}\right)^2\right\}^{\frac{1}{2}} \\ &= \{\omega^2 v^2 - \omega^2 v^2 (\cos\theta)^2\}^{\frac{1}{2}} \\ &= \omega v \sin\theta. \end{aligned} \tag{22}$$

Thus F is the product of the angular velocity of the moving coordinate system about its rotation-axis, and the projection of the relative velocity of m on a plane perpendicular to this rotation-axis. And the line of action of this force is perpendicular to the rotation-axis of the system, and to the line of the relative velocity of m. F has been called by Coriolis the *compound centrifugal force,* and in foreign treatises on mechanics is frequently cited by that name.

Thus the equations of relative motion of m, given in (14), become

$$\left.\begin{aligned} X' - X_t - 2F\cos\alpha - \frac{d^2\xi}{dt^2} &= 0, \\ Y' - Y_t - 2F\cos\beta - \frac{d^2\eta}{dt^2} &= 0, \\ Z' - Z_t - 2F\cos\gamma - \frac{d^2\zeta}{dt^2} &= 0; \end{aligned}\right\} \tag{23}$$

in which form the equations are most conveniently applied to the solution of problems.

227.] On comparing these with the equations which express absolute motion, it appears that the axial components of the impressed velocity-increments are to be diminished by similar quantities, which we may suppose to arise from two fictitious forces; one of which impresses on the particle a velocity-increment equal to that of the particle of the moving coordinate system, which has the same place as m at the time t, and by virtue of which the particle has no relative motion; the other impresses a velocity-increment equal to the product of the in-

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stantaneous angular velocity of the moving coordinate system and the projection of the relative velocity of the particle on a plane perpendicular to the rotation-axis, and the line of action of which is perpendicular to the rotation-axis of the system and to the line of the relative velocity of m.

These fictitious forces are not given in the same way as the ordinary acting forces are given; they depend on the relative motion, inasmuch as they involve ω and v; thus the problem is in its general form extremely complicated: the solution of a few particular problems will indicate the mode of their application better than any general remarks.

If the motion of the system of coordinate axes is that of translation only, then $\omega = 0$, and the second fictitious force of course vanishes; in this case, the axes of the moving system will be, or may be taken to be, always parallel to those of the fixed system, so that there will be no t-variations of the nine direction-cosines; and all the direction-cosines will vanish except a_1, b_2, and c_3, each of which becomes equal to unity; and the equations (23) become

$$\left.\begin{aligned} \text{X}' - \frac{d^2x_0}{dt^2} - \frac{d^2\xi}{dt^2} &= 0, \\ \text{Y}' - \frac{d^2y_0}{dt^2} - \frac{d^2\eta}{dt^2} &= 0, \\ \text{Z}' - \frac{d^2z_0}{dt^2} - \frac{d^2\zeta}{dt^2} &= 0; \end{aligned}\right\} \tag{24}$$

which are the equations already found in Art. 259, Vol. III. These are sufficient for the determination of the relative motion of a planet, and have been applied to that problem in Arts. 293 and 295, Vol. III.

If moreover the moving origin travels along a straight line with uniform velocity, then also $\frac{d^2x_0}{dt^2} = \frac{d^2y_0}{dt^2} = \frac{d^2z_0}{dt^2}$; and the equations (24) have the same form as those which express the ordinary absolute motion. It is evident however from (16), which express the relative velocities, that these will not be the same as the absolute velocities; and consequently some quantities will be introduced in the integration-process which depend on the elements of the line of motion and on the velocity of the moving origin.

228.] One particular form of the preceding general results requires especial mention; viz., that in which the origin moves in

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the plane of (x, y), and the moving axes of (ξ, η) are always in that plane, so that the axes of z and ζ are always parallel. The equations which express this motion might of course be deduced from the general equations (23); but it is convenient to derive them from first principles, because the origin of the several terms and the meaning of the fictitious forces of which we have spoken will thereby become more palpable.

At the time t, let (x_0, y_0) be the origin, see Fig. 60, and let θ be the angle at which the ξ-axis is inclined to that of x; so that

$$\left.\begin{aligned} x &= x_0 + \xi\cos\theta - \eta\sin\theta, \\ y &= y_0 + \xi\sin\theta + \eta\cos\theta, \\ z &= \qquad \zeta; \end{aligned}\right\} \tag{25}$$

therefore

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_0}{dt} + \frac{d\xi}{dt}\cos\theta - \frac{d\eta}{dt}\sin\theta - (\xi\sin\theta + \eta\cos\theta)\frac{d\theta}{dt}, \\ \frac{dy}{dt} &= \frac{dy_0}{dt} + \frac{d\xi}{dt}\sin\theta + \frac{d\eta}{dt}\cos\theta + (\xi\cos\theta - \eta\sin\theta)\frac{d\theta}{dt}; \end{aligned}\right\} \tag{26}$$

Let v_ξ and v_η be the components of the absolute velocity parallel to the axes of ξ and η: then

$$\begin{aligned} v_\xi &= \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta \\ &= \frac{dx_0}{dt}\cos\theta + \frac{dy_0}{dt}\sin\theta - \eta\frac{d\theta}{dt} + \frac{d\xi}{dt}; \end{aligned} \tag{27}$$

$$\begin{aligned} v_\eta &= -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta \\ &= -\frac{dx_0}{dt}\sin\theta + \frac{dy_0}{dt}\cos\theta + \xi\frac{d\theta}{dt} + \frac{d\eta}{dt}; \end{aligned} \tag{28}$$

in which values of v_ξ and v_η all the terms except the last of each are due to the motion of the coordinate system of reference, and the last expresses the axial component of the velocity of m relatively to the moving axes.

Let us next take the t-differentials of (26): then

$$\begin{aligned} \frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} &- 2\left(\frac{d\xi}{dt}\sin\theta + \frac{d\eta}{dt}\cos\theta\right)\frac{d\theta}{dt} - (\xi\cos\theta - \eta\sin\theta)\left(\frac{d\theta}{dt}\right)^2 \\ &- (\xi\sin\theta + \eta\cos\theta)\frac{d^2\theta}{dt^2} + \frac{d^2\xi}{dt^2}\cos\theta - \frac{d^2\eta}{dt^2}\sin\theta; \end{aligned} \tag{29}$$

$$\begin{aligned} \frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} &+ 2\left(\frac{d\xi}{dt}\cos\theta - \frac{d\eta}{dt}\sin\theta\right)\frac{d\theta}{dt} - (\xi\sin\theta + \eta\cos\theta)\left(\frac{d\theta}{dt}\right)^2 \\ &+ (\xi\cos\theta - \eta\sin\theta)\frac{d^2\theta}{dt^2} + \frac{d^2\xi}{dt^2}\sin\theta + \frac{d^2\eta}{dt^2}\cos\theta; \end{aligned} \tag{30}$$

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Let v'_ξ and v'_η be the axial components of the expressed velocity-increment parallel to the axes of ξ and η: then

$$\begin{aligned} v'_\xi &= \frac{d^2x}{dt^2}\cos\theta + \frac{d^2y}{dt}\sin\theta \\ &= \frac{d^2x_0}{dt}\cos\theta + \frac{d^2y_0}{dt^2}\sin\theta - 2\frac{d\eta}{dt}\frac{d\theta}{dt} - \xi\left(\frac{d\theta}{dt}\right)^2 - \eta\frac{d^2\theta}{dt^2} + \frac{d^2\xi}{dt^2} \\ &= \frac{d^2x_0}{dt^2}\cos\theta + \frac{d^2y_0}{dt^2}\sin\theta - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) + \frac{d^2\xi}{dt^2}; \quad (31) \end{aligned}$$

$$\begin{aligned} v'_\eta &= -\frac{d^2x}{dt^2}\sin\theta + \frac{d^2y}{dt^2}\cos\theta \\ &= -\frac{d^2x_0}{dt^2}\sin\theta + \frac{d^2y_0}{dt^2}\cos\theta + 2\frac{d\xi}{dt}\frac{d\theta}{dt} - \eta\left(\frac{d\theta}{dt}\right)^2 + \xi\frac{d^2\theta}{dt^2} + \frac{d^2\eta}{dt^2} \\ &= -\frac{d^2x_0}{dt^2}\sin\theta + \frac{d^2y_0}{dt^2}\cos\theta - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) + \frac{d^2\eta}{dt^2}; \quad (32) \end{aligned}$$

of which expressions for v'_ξ and v'_η all the terms except the last in each are due to the motion of the moving coordinate system of reference; and the last expresses the axial component of the expressed relative velocity-increment.

229.] On examining equations (31) and (32), it appears that the first two terms in the right hand member of each express those parts of the velocity-increment which arise from the motion of translation of the moving origin, and that the last three in each arise from the angular motion of the coordinate system of reference and from the relative motion of m in reference to that moving system. Now these last are in accordance with the results of radial and transversal resolution which have been discussed in Vol. III, Art. 256. The axial components of the expressed velocity-increment of m at P are the sums of the components of the expressed velocity-increments of equal particles at L and N, which are the projections of P on the axes of ξ and η; for as $\frac{d\theta}{dt}$ is the angular velocity of these axes about the point O′,

the radial component of L $= \frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2$,

the transversal component of L $= \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right)$,

the radial component of N $= \frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2$,

the transversal component of N $= \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right)$;

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and consequently of the expressed velocity-increment,

$$\text{the } \xi\text{-component} = \frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right), \tag{33}$$

$$\text{the } \eta\text{-component} = \frac{d^2\eta}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right). \tag{34}$$

Let X' and Y' be the axial components along the moving axes of the impressed velocity-increments at the time t: then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + \frac{d^2x_0}{dt^2}\cos\theta + \frac{d^2y_0}{dt^2}\sin\theta - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) &= \mathrm{X}', \\ \frac{d^2\eta}{dt^2} - \frac{d^2x_0}{dt^2}\sin\theta + \frac{d^2y_0}{dt^2}\cos\theta - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) &= \mathrm{Y}'. \end{aligned}\right\} \tag{35}$$

These are the general equations of relative motion of a particle moving in the plane of (ξ, η); and by means of them ξ and η may be expressed in terms of t and known quantities. If t is eliminated from these last two equations, the resulting equation in terms of ξ and η will be that of the relative path in which the particle moves. The equation to the absolute path will be found from (25), when ξ and η are expressed in terms of x and y.

If the origin of the moving axes does not move, and the axes revolve with an uniform angular velocity ω, then (35) become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= \mathrm{X}', \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= \mathrm{Y}'. \end{aligned}\right\} \tag{36}$$

These equations however refer to a very special case of the general motion.

SECTION 2.—*The relative motion of a material particle.*

230.] Although our object is the discovery of equations which represent the relative motion of a material system, and the investigation of the preceding equations which apply to a single material particle has been subordinate to that object; yet it is desirable to shew their applicability to the solution of certain problems in which only a particle moves; because we shall hereby obtain a clearer insight into the several parts of them, and shall also solve some problems of considerable interest.

Let us first take the case of a relative constrained motion; viz., that of a particle moving within a tube, which also itself

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moves, and carries with it the system of axes to which it is referred; I shall consider the tube to be perfectly smooth, so that it offers no resistance to the moving particle in the direction of its motion; but presents a reaction acting along the normal, if the tube is a plane curve; and along the principal normal, if it is a curve of double curvature; and I shall also consider the small bore of the tube, and the size of the particle, to be such that the particle may exactly fill the tube.

In the most simple form of the problem the tube is a plane curve lying in the plane of (x, y), and rotating with a constant angular velocity ω, about the axis of ζ or z, the origin of the moving axes being fixed at the fixed origin, and no external force acts on the particle. Let R be the normal reacting pressure of the tube against the particle; let m be the mass of the particle, and ds = the length-element of the tube at the point (ξ, η), which is the place of m at the time t; then the equations of relative motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= \pm\frac{R}{m}\frac{d\eta}{ds}, \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= \mp\frac{R}{m}\frac{d\xi}{ds}. \end{aligned}\right\} \tag{37}$$

From these equations the following general theorems are deduced;

$$\frac{d\xi\, d^2\xi + d\eta\, d^2\eta}{dt^2} - \omega^2\{\xi\, d\xi + \eta\, d\eta\} = 0;$$

let $\xi^2 + \eta^2 = r^2$, so that r is the distance of the particle from the fixed origin at the time t; then, assuming the relative velocities of the particle to be v and v_0, when the distances of its place from the origin are respectively r and r_0, this equation gives by integration

$$v^2 - v_0^2 = \omega^2(r^2 - r_0^2); \tag{38}$$

and thus assigns the relative velocity of the particle.

Also from (37) we have

$$\frac{d\eta\, d^2\xi - d\xi\, d^2\eta}{dt^2} - \omega^2(\xi\, d\eta - \eta\, d\xi) - 2\omega\frac{ds^2}{dt} = \pm\frac{R}{m}ds;$$

now if ρ = the radius of curvature of the curve of the tube at the point (ξ, η), and p = the perpendicular on the tangent from the origin, this becomes

$$\pm\frac{R}{m} = \frac{v^2}{\rho} + \omega^2 p - 2\omega v; \tag{39}$$

which assigns the pressure on the tube.

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231.] The following are simple problems in illustration of these equations.

Ex. 1. A particle is placed within a rectilinear tube which revolves with a constant angular velocity about an axis which intersects it at right angles: it is required to determine the motion of the particle.

Let the line of the tube be the moving ξ-axis; so that always $\eta = 0$; let $\omega =$ the constant angular velocity of the tube. Then (37) become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi &= 0, \\ 2\omega\frac{d\xi}{dt} &= \pm\frac{\mathrm{R}}{m}. \end{aligned}\right\}$$

Let us assume the particle to be at relative rest at a distance a from the origin, when $t=0$; then from the first of these we have

$$\left(\frac{d\xi}{dt}\right)^2 = \omega^2(\xi^2 - a^2);$$

$$\therefore \quad \xi = \frac{a}{2}\{e^{\omega t} + e^{-\omega t}\};$$

$$\pm\frac{\mathrm{R}}{m} = a\omega^2\{e^{\omega t} - e^{-\omega t}\};$$

and if θ is the angle described by the tube during the time t, then $\theta = \omega t$; and the relative equation to the path of the particle is

$$\xi = \frac{a}{2}\{e^{\theta} + e^{-\theta}\}.$$

If r is the absolute radius vector, $r = \xi$, and

$$r = \frac{a}{2}\{e^{\theta} + e^{-\theta}\};$$

and this is the absolute polar equation of the path of the particle.

Ex. 2. A particle is placed within a rectilinear tube which revolves about an axis intersecting it at right angles, with an angular velocity such that the tangent of the angle described in a given time is proportional to the time: it is required to determine the motion of the particle.

Let, as in the preceding problem, the line of the tube be the ξ-axis, so that always $\eta = 0$; let θ be the angle through which the tube has moved in the time t; then, by the conditions of the problem, if $\theta = 0$, when $t = 0$, and if k is a constant,

$$\tan\theta = kt;$$

and (35) become

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$$\left.\begin{aligned} &\frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 = 0, \\ &\frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) = \pm\frac{\text{R}}{m}. \end{aligned}\right\}$$

Let us eliminate t by means of the first two of these three equations, and take θ to be an equicrescent variable; then we have

$$\frac{d^2\xi}{d\theta^2} - 2\tan\theta\frac{d\xi}{d\theta} - \xi = 0;$$

$$\therefore \quad \cos\theta\frac{d^2\xi}{d\theta^2} - 2\sin\theta\frac{d\xi}{d\theta} - \xi\cos\theta = 0.$$

When $t=0$, let us suppose the tube to lie along the x-axis, and the particle to be projected with a velocity u along the tube from a point whose distance from the origin $= a$; then from above, by integration, we have

$$\cos\theta\frac{d\xi}{d\theta} - \xi\sin\theta = \frac{u}{k};$$

$$\therefore \quad \xi\cos\theta = a + \frac{u}{k}\theta;$$

which is the relative equation of the path of the particle. And in reference to the fixed axes of x and y,

$$x = a + \frac{u}{k}\theta;$$

which gives the absolute path of the particle.

If we substitute from above in the third of the preceding equations, we have

$$\pm\frac{\text{R}}{m} = 2uk(\cos\theta)^3.$$

Ex. 3. A circular tube revolves uniformly about a vertical axis which is perpendicular to its plane and passes through a point in its circumference: it is required to determine the relative motion of a particle within the tube.

Let r and p refer to the place of the particle at the time t; and let a be the radius of the circular tube. Then the equations to the tube are $r = 2a\cos\phi$, and $r^2 = 2ap$;

$$\therefore \quad \frac{ds}{dt} = 2a\frac{d\phi}{dt};$$

and therefore from (38) we have

$$4a^2\left(\frac{d\phi}{dt}\right)^2 = v_0^2 + 4a^2\omega^2(\cos\phi)^2 - \omega^2 r_0^2; \qquad (40)$$

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which equation, being integrated, would give ϕ, and consequently r, in terms of t. The integral however can only generally be expressed as an elliptic function.

But if $v_0 = \omega r_0$, then

$$\frac{d\phi}{dt} = \omega \cos \phi;$$

so that $\frac{d\phi}{dt} = \omega$, when $\phi = 0$; in which case

$$\log \frac{1+\sin\phi}{1-\sin\phi} = 2\omega t; \tag{41}$$

the limits of integration being such that $t = 0$, when $\phi = 0$; that is, when the particle is in that point of the tube which is directly opposite to the origin;

$$\left.\begin{aligned} \therefore \quad \sin\phi &= \frac{e^{\omega t} - e^{-\omega t}}{e^{\omega t} + e^{-\omega t}}, \\ \therefore \quad r &= \frac{4a}{e^{\omega t} + e^{-\omega t}}; \end{aligned}\right\} \tag{42}$$

which determine the place of the particle at the time t relative to the moving axes of the circular tube.

When $t = \infty$, $\phi = \frac{\pi}{2}$; so that the particle falls into the origin only after an infinite time.

For this particular value of the constant we have, from (39),

$$\pm \frac{R}{m} = 2a\omega^2 \cos\theta\,(3\cos\theta - 2). \tag{43}$$

Ex. 4. A particle is placed within a thin tube which is of the form of an equiangular spiral; the tube revolves with an uniform angular velocity about a vertical axis passing through its pole, the plane of the tube being horizontal: it is required to determine the relative path of the particle.

Let a be the constant angle at which the tangent is inclined to the radius vector at every point of the curve. Then

$$p = r\sin a, \qquad dr = ds\cos a,$$

are equations to the curve. And (38) gives

$$\left(\frac{dr}{dt}\right)^2 (\sec a)^2 = \omega^2 r^2 - \omega^2 r_0^2 + v_0^2.$$

Let $$\omega^2 r_0^2 - v_0^2 = k^2;$$

then if $t = 0$, when $r = k$, this equation gives by integration

$$r = \frac{k}{2}\left\{e^{t\cos a} + e^{-t\cos a}\right\};$$

which assigns the relative motion of the particle in the tube.

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232.] For an example, in which the origin of coordinates itself moves, let us consider the motion of a particle within a circular tube, which revolves about an axis through its centre perpendicular to its plane with an uniform angular velocity ω; and the centre of the circular tube also describes a circle in the plane of the tube with an uniform angular velocity Ω.

Let us suppose the particle and tube to be situated at the time t, as they are placed in Fig. 61; wherein AQB is the circle in which Q, the centre of the circular tube, moves, and P is the place of the particle. Let us also suppose the centre of the tube to have been at A, on the axis of x, when $t = 0$, and the particle at that time to have been at relative rest at C, C being at C′ at the time t. Let OA $= a$, AC $= c$; then

$$\left.\begin{aligned} x_0^2 + y_0^2 &= a^2, \\ \xi^2 + \eta^2 &= c^2, \\ \theta &= (\Omega+\omega)\,t\,; \end{aligned}\right\} \tag{44}$$

and we have also

$$\left.\begin{aligned} x_0 &= a\cos\Omega t, \\ y_0 &= a\sin\Omega t\,; \end{aligned}\right\} \tag{45}$$

from (35), we have

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - a\Omega^2\cos\omega t - (\Omega+\omega)^2\,\xi - 2\,(\Omega+\omega)\frac{d\eta}{dt} &= \pm\frac{\text{R}}{m}\frac{d\eta}{ds}, \\ \frac{d^2\eta}{dt^2} + a\Omega^2\sin\omega t - (\Omega+\omega)^2\,\eta + 2\,(\Omega+\omega)\frac{d\xi}{dt} &= \mp\frac{\text{R}}{m}\frac{d\xi}{ds}; \end{aligned}\right\} \tag{46}$$

which are the required equations of motion, and do not generally admit of integration.

233.] Another simple problem, to which the preceding principles are applicable, is that in which a heavy particle moves in a smooth tube, which rotates about a vertical axis with an uniform angular velocity.

Let us first suppose the tube to be of single curvature, and the axis about which it rotates to lie in its plane; let this axis be taken for the axis of z; and let the positive direction of it be measured from the origin in a direction opposite to that of gravity. Let this axis also be taken as the ζ-axis to which the curve is referred, and let a perpendicular to this line through the origin be the ξ-axis; so that the equation to the curve of the tube is

$$\zeta = f(\xi).$$

Let ω be the constant angular velocity with which the tube

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rotates, the plane of the tube being in the plane of (z, x) when $t = 0$. Then

$$x = \xi \cos \omega t, \qquad y = \xi \sin \omega t, \qquad z = \zeta;$$

$$\therefore \quad \frac{d^2 x}{dt^2} = \cos \omega t \frac{d^2 \xi}{dt^2} - 2\omega \sin \omega t \frac{d\xi}{dt} - \omega^2 \xi \cos \omega t,$$

$$\frac{d^2 y}{dt^2} = \sin \omega t \frac{d^2 \xi}{dt^2} + 2\omega \cos \omega t \frac{d\xi}{dt} - \omega^2 \xi \sin \omega t$$

$$\frac{d^2 z}{dt^2} = \frac{d^2 \zeta}{dt^2};$$

$$\therefore \quad v'_\xi = \frac{d^2 \xi}{dt^2} - \omega^2 \xi;$$

and consequently the equations of motion are

$$\left.\begin{aligned} \frac{d^2 \xi}{dt^2} - \omega^2 \xi &= \pm \frac{R}{m} \frac{d\zeta}{ds}, \\ \frac{d^2 \zeta}{dt^2} &= \mp \frac{R}{m} \frac{d\xi}{ds} - g; \end{aligned}\right\} \tag{47}$$

$$\therefore \quad v^2 - v_0^2 = \omega^2 (\xi^2 - \xi_0^2) - 2g(\zeta - \zeta_0).$$

If the velocity is given in terms of the coordinates of the place of m at the time t, the equation of the curve of the tube may be found; and if the equation of the curve of the tube is given, the velocity of m and the other circumstances of motion of it at any time may be found.

234.] The following are problems solved by means of these equations.

Ex. 1. Determine the motion of a heavy particle within a rectilinear tube which describes with an uniform angular velocity the surface of a right cone, the axis of which is vertical.

Let a = the semi-vertical angle of the cone, and let r be the distance of m at the time t from the vertex of the cone: then

$$\xi = r \sin a, \quad \zeta = r \cos a;$$

and substituting these in (47) we have

$$\sin a \left\{ \frac{d^2 r}{dt^2} - \omega^2 r \right\} = \pm \frac{R}{m} \cos a,$$

$$\cos a \frac{d^2 r}{dt^2} = \mp \frac{R}{m} \sin a - g,$$

$$\therefore \quad \frac{d^2 r}{dt^2} - (\omega \sin a)^2 r + g \cos a = 0.$$

Let us suppose m to be projected from the vertex of the cone with a velocity u; so that $\frac{dr}{dt} = u$, when $t = 0$, and $r = 0$; then

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from the above by integration we have

$$\left(\frac{dr}{dt}\right)^2 - u^2 - (\omega \sin a)^2 r^2 + 2gr\cos a = 0,$$

therefore

$$r = \frac{g\cos a}{\omega^2(\sin a)^2} + \frac{u\omega\sin a - g\cos a}{2\omega^2(\sin a)^2} e^{\omega t \sin a} - \frac{u\omega\sin a + g\cos a}{2\omega^2(\sin a)^2} e^{-\omega t \sin a};$$

which equation determines the circumstances of motion of the particle.

If u is greater than $\frac{g\cot a}{\omega}$, r increases without limit as t increases; and thus m moves along the tube farther and farther from the vertex.

If $u = \frac{g\cot a}{\omega}$, as t increases, r increases; and when t becomes infinite, $r = \frac{g\cos a}{\omega^2(\sin a)^2}$; so that m moves farther from the vertex along the tube, but never passes the limit assigned by this value of r.

If $a = 90°$, we have the same results as those of Ex. 1, Art. 231.

Ex. 2. By means of these equations we may also determine the position of the two heavy balls in Watt's centrifugal governor of the steam engine. The arrangement of this contrivance will be understood from Fig. 62, where O is the fixed point on the vertical axis at which the rods carrying the heavy balls cross each other. We shall take the plane of the rods and the balls to be that of (ξ, ζ) and shall take the vertical line drawn downwards from O to be the positive direction of ζ: we shall assume the weight of the rods to be so small in comparison of that of the balls, that the former may be neglected without sensible error. Let OP, the length of the rod, $= a$; and let its inclinations to the vertical be θ and θ_0 at the times t and 0 respectively; and let us suppose θ to be greater than θ_0 and $\frac{d\theta}{dt}$ to be zero, when $t = 0$; let ω be the angular velocity with which the plane containing the balls and rods rotates about Oz; and let T be the tension of the rods. Then the equations of motion are

$$\frac{d^2\xi}{dt^2} - \omega^2\xi = -\mathrm{T}\sin\theta,$$

$$\frac{d^2\zeta}{dt^2} = -\mathrm{T}\cos\theta\, g;$$

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$$\therefore \quad \left(\frac{ds}{dt}\right)^2 = \omega^2(\xi^2 - \xi_0^2) + 2g(\zeta - \zeta_0);$$

$$a^2\left(\frac{d\theta}{dt}\right)^2 = a^2\omega^2\{(\cos\theta_0)^2 - (\cos\theta)^2\} + 2ga\{\cos\theta - \cos\theta_0\},$$

$$a\left(\frac{d\theta}{dt}\right)^2 = (\cos\theta_0 - \cos\theta)\{a\omega^2(\cos\theta_0 + \cos\theta) - 2g\},$$

$$\therefore \quad dt = \frac{a^{\frac{1}{2}}d\theta}{\{\cos\theta_0 - \cos\theta\}^{\frac{1}{2}}\{a\omega^2(\cos\theta_0 + \cos\theta) - 2g\}^{\frac{1}{2}}};$$

in which equation the variables are separated; and if the integration can be effected, θ will be given in terms of t.

$$\frac{d\theta}{dt} = 0, \text{ when } (1)\ \theta = \theta_0;\ (2)\ \cos\theta = \frac{2g}{a\omega^2} - \cos\theta_0;$$

so that θ varies between the angles given by these two limits.

Let ω_0 be the angular velocity with which the plane of the balls revolves, when the angle at which they are inclined to the vertical axis is θ_0, and does not vary. In this case, as they have no change of ξ or ζ, the preceding equations of motion give

$$g = \omega_0^2 a\cos\theta_0;$$

so that

$$dt = \frac{d\theta}{(\cos\theta_0 - \cos\theta)^{\frac{1}{2}}\{(\omega^2 - 2\omega_0^2)\cos\theta_0 + \omega^2\cos\theta\}^{\frac{1}{2}}};$$

and if $\text{OB} = \text{BO} = b$; so that $\text{OQ} = 2b\cos\theta$,

$$d.\,\text{OQ} = -2b\sin\theta\frac{d\theta}{dt};$$

which assigns the vertical displacement of Q due to the change of angle of inclination of OP to the vertical.

235.] Next let us suppose the curve of the tube in which the particle moves to be of double curvature; and to rotate about the z- or ζ-axis with the same constant angular velocity ω; then the equations given in (35) are true, when $x_0 = y_0 = 0$, and θ is replaced by ωt; and we have

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= \frac{\text{R}}{m}\cos\alpha, \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= \frac{\text{R}}{m}\cos\beta, \\ \frac{d^2\zeta}{dt^2} &= \frac{\text{R}}{m}\cos\gamma - g; \end{aligned}\right\} \qquad (48)$$

wherein α, β, γ are direction-angles of the principal normal at the point (ξ, η, ζ).

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From these equations we have

$$v^2 - \omega^2(\xi^2 + \eta^2) = -2g\zeta + \mathrm{C},$$

where C is a constant depending on the initial values of the several quantities.

If we suppose the relative velocity of the particle to be constant or to be zero, then the preceding equation expresses a paraboloid of revolution; and we infer that the curve traced in any way on a paraboloid of revolution satisfies the given conditions.

236.] Lastly, let us consider the motion of a heavy particle moving in contact with a surface which rotates with an uniform angular velocity about its ζ-axis, which is vertical and is the fixed z-axis.

Let the equation to the surface be

$$\mathrm{F}(\xi, \eta, \zeta) = 0;$$

of which let the partial derived functions be U, V, W; also let

$$\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2 = \mathrm{Q}^2;$$

then we may suppose the particle to move in a thin space contained between two parallel surfaces infinitesimally near to each other; in which case the equations of relative motion are

$$\frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} = \frac{\mathrm{R}}{m}\frac{\mathrm{U}}{\mathrm{Q}},$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} = \frac{\mathrm{R}}{m}\frac{\mathrm{V}}{\mathrm{Q}},$$

$$\frac{d^2\zeta}{dt^2} = \frac{\mathrm{R}}{m}\frac{\mathrm{W}}{\mathrm{Q}} - g;$$

if we multiply these respectively by $d\xi$, $d\eta$, and $d\zeta$, and add and integrate, we have

$$v^2 - \omega^2(\xi^2 + \eta^2) = -2g\zeta + \mathrm{C};$$

where C is a constant depending on the initial values of the several quantities.

If v is constant, or if the particle remains at rest wherever it is put so that $v = 0$, this is the equation to a paraboloid of revolution.

If v varies as the distance from the fixed origin, the surface is a surface of the second degree.

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Suppose the surface to be a plane in which the z-axis is; let us also assume the ξ-axis to be in the plane, so that always $\eta=0$; then the equations of motion are

$$\frac{d^2\xi}{dt^2} - \omega^2\xi = 0, \qquad 2\omega\frac{d\xi}{dt} = \frac{R}{m}, \qquad \frac{d^2\zeta}{dt^2} = -g;$$

from which the motion may be easily determined.

237.] The principles and equations of the preceding articles are applicable to the solution of a problem of considerable interest; viz. to the motion of a particle, either free or constrained, near to the earth's surface, relative to a system of axes originating on the earth's surface and moving with it.

We may without error assume the centre of gravity of the earth to be fixed, if we impress forces on the moving particle which are equal to the excess of those which act on it over those which act on the earth at its centre of gravity: but as the sun, which is the main force acting on the earth, impresses velocity-increments nearly equal on both the earth and the particle, we may suppose this excess, either positive or negative, to be so small that it may be neglected without sensible error. We may also suppose the position of the rotation-axis of the earth to be fixed and the angular velocity to be constant.

The two systems of axes are imagined to have that arrangement which is drawn in Fig. 63. O is the centre of the earth; the axis of z is measured from O towards C the north pole; the axes of x and y are taken in the plane of the equator. Let ω be the angular velocity of the earth, with which indeed the earth rotates from the y-axis to the x-axis: it will be convenient however to take it at present in the contrary direction, and to change the sign in the final equations ere we apply them to the particular problem.

Let R be the place of observation, and let us suppose it to be in the northern hemisphere of the earth. Let P be the origin of the moving system of rectangular axes to which the motion of m is referred: let the axis of ζ be the vertical line at P measured upwards from the earth towards the zenith of P; this may be assumed, without sensible error, to pass through the earth's centre. Let axes of ξ and η be in the horizontal plane at P, and be respectively N and S, and E and W; the positive direction of ξ being taken towards the south, and that of η towards

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the west. Let the latitude of P, viz. POQ, $= \lambda$. Let the plane of the meridian of P, when $t = 0$, be that of (x, z); and let the earth's radius be r. Then PON $= \lambda$, MON $= \omega t$: and

$$\left.\begin{aligned} \text{OM} &= x_0 = r\cos\lambda\cos\omega t, \\ \text{MN} &= y_0 = r\cos\lambda\sin\omega t, \\ \text{NP} &= z_0 = r\sin\lambda; \end{aligned}\right\}$$

$$\therefore \quad \left.\begin{aligned} \frac{d^2x_0}{dt^2} &= -\omega^2 r\cos\lambda\cos\omega t, \\ \frac{d^2y_0}{dt^2} &= -\omega^2 r\cos\lambda\sin\omega t, \\ \frac{d^2z_0}{dt^2} &= 0. \end{aligned}\right\} \tag{49}$$

Also resolving ω along the axes of ξ, η, ζ, we have

$$\omega_\xi = -\omega\cos\lambda, \qquad \omega_\eta = 0, \qquad \omega_\zeta = \omega\sin\lambda; \tag{50}$$

of which the first is the component about the line running due S and N in the horizontal plane, and is the only component in that plane: and the last is the component about the vertical at P.

Now if the place of m at the time t is (x, y, z) relatively to the fixed axes, and is (ξ, η, ζ) relatively to the moving axes which originate at P; then

$$\left.\begin{aligned} x &= r\cos\lambda\cos\omega t + \xi\sin\lambda\cos\omega t - \eta\sin\omega t + \zeta\cos\lambda\cos\omega t, \\ y &= r\cos\lambda\sin\omega t + \xi\sin\lambda\sin\omega t + \eta\cos\omega t + \zeta\cos\lambda\sin\omega t, \\ z &= r\sin\lambda \qquad\qquad - \xi\cos\lambda \qquad\qquad\qquad + \zeta\sin\lambda. \end{aligned}\right\} \tag{51}$$

On comparing these with (2) Art. 2, we have

$$\left.\begin{aligned} a_1 &= \sin\lambda\cos\omega t, & b_1 &= -\sin\omega t, & c_1 &= \cos\lambda\cos\omega t, \\ a_2 &= \sin\lambda\sin\omega t, & b_2 &= \cos\omega t & c_2 &= \cos\lambda\sin\omega t, \\ a_3 &= -\cos\lambda; & b_3 &= 0; & c_3 &= \sin\lambda; \end{aligned}\right\} \tag{52}$$

and differentiating

$$\left.\begin{aligned} \frac{d^2a_1}{dt^2} &= -\omega^2\sin\lambda\cos\omega t, & \frac{d^2b_1}{dt^2} &= \omega^2\sin\omega t, & \frac{d^2c_1}{dt^2} &= -\omega^2\cos\lambda\cos\omega t, \\ \frac{d^2a_2}{dt^2} &= -\omega^2\sin\lambda\sin\omega t, & \frac{d^2b_2}{dt^2} &= -\omega^2\cos\omega t, & \frac{d^2c_2}{dt^2} &= -\omega^2\cos\lambda\sin\omega t, \\ \frac{d^2a_3}{dt^2} &= 0; & \frac{d^2b_3}{dt^2} &= 0; & \frac{d^2c_3}{dt^2} &= 0; \end{aligned}\right\}$$

and substituting these quantities in (15), (16), and (17), we have

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$$\left.\begin{aligned} X_t &= -\omega^2 r \sin\lambda\cos\lambda - \xi\omega^2(\sin\lambda)^2 - \zeta\omega^2\sin\lambda\cos\lambda, \\ Y_t &= -\eta\omega^2, \\ Z_t &= -\omega^2 r(\cos\lambda)^2 - \xi\omega^2\sin\lambda\cos\lambda - \zeta\omega^2(\cos\lambda)^2; \end{aligned}\right\} \quad (53)$$

and substituting from (50) in (19), we have

$$\left.\begin{aligned} F\cos\alpha &= -\omega\sin\lambda\frac{d\eta}{dt}, \\ F\cos\beta &= \omega\sin\lambda\frac{d\xi}{dt} + \omega\cos\lambda\frac{d\zeta}{dt}, \\ F\cos\gamma &= -\omega\cos\lambda\frac{d\eta}{dt}. \end{aligned}\right\} \quad (54)$$

When these several quantities are substituted in the equations of motion given in (23), these last equations become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2 r\sin\lambda\cos\lambda - \xi\omega^2(\sin\lambda)^2 - \zeta\omega^2\sin\lambda\cos\lambda - 2\omega\sin\lambda\frac{d\eta}{dt} &= X', \\ \frac{d^2\eta}{dt^2} - \eta\omega^2 + 2\omega\left(\sin\lambda\frac{d\xi}{dt} + \cos\lambda\frac{d\zeta}{dt}\right) &= Y', \\ \frac{d^2\zeta}{dt^2} - \omega^2 r(\cos\lambda)^2 - \xi\omega^2\sin\lambda\cos\lambda - \zeta\omega^2(\cos\lambda)^2 - 2\omega\cos\lambda\frac{d\eta}{dt} &= Z'. \end{aligned}\right\} \quad (55)$$

These equations may be deduced directly from (51) without the intervention of the general process, which has been investigated in the preceding Articles. For we may take the second t-differentials of x, y, and z, and equate the sum of their several components along the axes of ξ, η, ζ to the impressed velocity-increments acting on those axes. In particular problems this is the most convenient method.

238.] To adapt these equations to the actual circumstances of the earth, the sign of ω must be changed, because the earth revolves from west to east, which is a direction opposite to that taken in the preceding Article. To determine its value, we will take a second for the unit of time; then, since a mean sidereal day contains 86164.09 seconds,

$$\omega = \frac{2\pi}{86164.09} = \frac{1}{13713} = .00007292,$$

which is a small fraction; and consequently ω^2, which enters into the preceding equations, is a very small quantity. Also, in the problems to which we shall apply the equations, ξ, η, ζ will be always very small parts of the earth's radius; and thus we may at first, without sensible error, neglect those terms in the left

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hand members of the equations which involve products of these coordinates and of ω^2; and the equations become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2 r \sin\lambda\cos\lambda + 2\,\omega\sin\lambda\frac{d\eta}{dt} &= \text{X}', \\ \frac{d^2\eta}{dt^2} - 2\,\omega\left(\sin\lambda\frac{d\xi}{dt} + \cos\lambda\frac{d\zeta}{dt}\right) &= \text{Y}', \\ \frac{d^2\zeta}{dt^2} - \omega^2 r\,(\cos\lambda)^2 + 2\,\omega\cos\lambda\frac{d\eta}{dt} &= \text{Z}'; \end{aligned}\right\} \tag{56}$$

where X′, Y′, Z′ are the components along the moving axes of all the absolute velocity-increments impressed on m.

239.] Now I propose to apply these equations in the first place to the motion of a particle projected with a given velocity and in a given direction from P, the place of observation, which is also the origin of the moving system of axes. Although the power of our weapons of projection has been very greatly increased of late, yet still, for all points of the path, ξ, η, ζ are but small parts of the earth's radius; consequently $\omega^2\xi$, $\omega^2\eta$, $\omega^2\zeta$ are small quantities which we may omit, and (56) are applicable.

In the right hand members, for the same reasons, I shall assume the earth's attraction to be the same at all points of the particle's path, and to be what it is at P, the place of observation. Although gravity varies at different points of the earth's surface, according to a law which is accordant with Clairaut's theorem, yet I shall take it to be the same at all latitudes; and no sensible error will, within the compass of our approximations, thereby be introduced into the results. I shall also consider the projectile to move in vacuo, and shall consequently neglect the resistance of the air. Thus the particle moves under the action of gravity only; and the force which acts on it during its motion is the same as that which acted on it in its original state of rest. Consequently we may determine the values of X′, Y′, Z′ in (56) by their values when m is at relative rest at P; thus,

$$\xi = \eta = \zeta = 0;\ \frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0;\ \frac{d^2\xi}{dt^2} = \frac{d^2\eta}{dt^2} = 0;\ \frac{d^2\zeta}{dt^2} = -g.$$

Thus

$$\left.\begin{aligned} \text{X}' &= -\,\omega^2 r\sin\lambda\cos\lambda, \\ \text{Y}' &= 0, \\ \text{Z}' &= -\,\omega^2 r\,(\cos\lambda)^2 - g; \end{aligned}\right\} \tag{57}$$

and (56) become

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$$\left.\begin{aligned}
&\frac{d^2\xi}{dt^2}+2\omega\ \sin\lambda\frac{d\eta}{dt} &&= 0,\\
&\frac{d^2\eta}{dt^2}-2\omega\left(\sin\lambda\frac{d\xi}{dt}+\cos\lambda\frac{d\zeta}{dt}\right) &&= 0,\\
&\frac{d^2\zeta}{dt^2}+2\omega\ \cos\lambda\frac{d\eta}{dt} &&= -g.
\end{aligned}\right\}\qquad(58)$$

If $\omega = 0$, these equations express the ordinary case of a projectile's motion which has already been solved in Vol. III.

Now (58) admit of integration. Let u = the velocity of projection, and let α, β, γ be the direction-angles of the line of projection;

$$\therefore\quad\left.\begin{aligned}
&\frac{d\xi}{dt}+2\omega\eta\sin\lambda = u\cos\alpha,\\
&\frac{d\eta}{dt}-2\omega(\xi\sin\lambda+\zeta\cos\lambda) = u\cos\beta,\\
&\frac{d\zeta}{dt}+2\omega\eta\cos\lambda = u\cos\gamma-gt;
\end{aligned}\right\}\qquad(59)$$

which assign the components of the velocity at any point of the path.

Again, if we substitute for $\frac{d\xi}{dt}$ and $\frac{d\zeta}{dt}$ from the first and last of these equations in the second of (58), and omit the terms involving the product of ω^2 and of one of the relative coordinates, then we have

$$\frac{d^2\eta}{dt^2}-2u\omega(\cos\alpha\sin\lambda+\cos\gamma\cos\lambda)+2\omega gt\cos\lambda = 0;$$

therefore

$$\eta = ut\cos\beta+u\omega(\cos\alpha\sin\lambda+\cos\gamma\cos\lambda)t^2-\omega g\cos\lambda\frac{t^3}{3};\quad(60)$$

and replacing η in the first and last of (59) by this value, and omitting terms involving products of ω^2 and of one of the relative coordinates, and integrating,

$$\xi = ut\cos\alpha-u\omega\sin\lambda\cos\beta\, t^2,\qquad(61)$$

$$\zeta = ut\cos\gamma-\left(\frac{g}{2}+u\omega\cos\lambda\cos\beta\right)t^2;\qquad(62)$$

which three equations express the motion of the projectile to the degree of approximation attainable by the preceding equations of motion.

If $\omega = 0$, the results are the same as those which have already been found in Art. 280, Vol. III; viz.,

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$$\xi = u\,t\cos\alpha,$$
$$\eta = u\,t\cos\beta,$$
$$\zeta = u\,t\cos\gamma - \frac{1}{2}g\,t^2.$$

On comparing these quantities with the preceding equations, it appears, that if the particle or ball is projected from a place in the northern hemisphere, in a direction westwards of the meridian, both the vertical height of it and its distance southwards from the parallel of latitude are diminished by the earth's rotation; and that if it is projected eastwards of the meridian, that is, in the direction in which the earth is going, both these quantities are increased. As to the three terms of which η consists, only the first, viz., $u\,t\cos\beta$, depends on the line of projection being eastwards or westwards; and consequently the increase or diminution of η will depend also on the sign of the other two terms which involve t.

The apparent path of the projectile may be determined by the elimination of t; which will give the equations to two surfaces, the line of intersection of which is its path: it is evident that the path will generally be a curve of double curvature.

240.] Let us however consider certain particular cases and results of these equations.

(1) Let the body fall, as e. g. down a mine, without any initial velocity; then $u = 0$; $\cos\alpha = \cos\beta = 0$; $\cos\gamma = -1$;

$$\therefore\quad \left.\begin{aligned} \xi &= 0, \\ \eta &= -\,\omega g\cos\lambda\,\frac{t^3}{3}, \\ \zeta &= -\,\frac{1}{2}g\,t^2. \end{aligned}\right\} \qquad (63)$$

The first equation shews that there is no deviation in the line of the meridian; from the second we infer a deviation towards the east; that is, in the direction towards which the earth is moving, which varies as the cube of the time of falling; and that this deviation is greatest at the equator, where $\lambda = 0$; and the last equation shews that the earth's rotation does not produce any alteration in the time of falling.

If we eliminate t, and take ζ downwards to be positive,

$$\eta^2 = \frac{8\,\omega^2(\cos\lambda)^2}{9\,g}\,\zeta^3;$$

which is the equation to a semicubical parabola; and shews that

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the square of the deviation towards the east varies as the cube of the space through which the particle has fallen.

(2) Let the particle be projected vertically upwards; then $\cos\alpha = \cos\beta = 0$; $\cos\gamma = 1$; and

$$\left.\begin{aligned} \xi &= 0, \\ \eta &= u\omega\cos\lambda\, t^2 - \omega g\cos\lambda\,\frac{t^3}{3}, \\ \zeta &= ut - \frac{1}{2}g t^2; \end{aligned}\right\} \qquad (64)$$

the last equation shews that the vertical motion is the same as it would be if the earth did not rotate on its axis; and consequently if h is the height to which the particle ascends, and τ is the whole time of ascent and descent, $u^2 = 2gh$, and $\tau = \frac{2u}{g}$. The first equation shews that there is no deviation in the line of the meridian; the second shews that the deviation is always westwards; for the greatest value of τ is $\frac{2u}{g}$ (unless the particle, after having descended to its original place, continues to fall down a mine), and consequently η is always positive. When the particle, after its ascent, strikes the earth,

$$\eta = \frac{4\omega u^3\cos\lambda}{3g^2};$$

which is the deviation westwards of the point of impact on the ground; and varies as the cube of the velocity of projection.

(3) Let the particle be projected due westwards at an angle of elevation equal to θ; then $\cos\alpha = 0$, $\cos\beta = \cos\theta$, $\cos\gamma = \sin\theta$; and

$$\left.\begin{aligned} \xi &= -u\omega\sin\lambda\cos\theta\, t^2, \\ \eta &= ut\cos\theta + u\omega\sin\theta\cos\lambda\, t^2 - \omega g\cos\lambda\,\frac{t^3}{3}, \\ \zeta &= ut\sin\theta - \left(\frac{g}{2} + u\omega\cos\lambda\cos\theta\right)t^2; \end{aligned}\right\} \qquad (65)$$

the first of which equations shews that the projectile generally deviates northwards; when the projectile strikes the ground, $\zeta = 0$; in which case

$$\begin{aligned} t &= \frac{2u\sin\theta}{g + 2u\omega\cos\theta\cos\lambda} \\ &= \frac{2u\sin\theta}{g}\left\{1 - \frac{2u\omega}{g}\cos\theta\cos\lambda\right\}, \end{aligned}$$

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omitting those terms which involve ω^2: in this case

$$\xi = -\frac{4u^3\omega\sin\lambda(\sin\theta)^2\cos\theta}{g^2}, \tag{66}$$

$$\eta = \frac{u^2}{g}\sin 2\theta + \frac{4u^3\omega\cos\lambda}{3g}\{(\sin\theta)^2 - 3(\cos\theta)^2\}; \tag{67}$$

which are the approximate coordinates of the point of impact on the ground. The terms involving ω denote the effects due to the earth's rotation: the former gives the deviation northwards; and the latter shews that the range measured westward is increased or diminished according as θ is greater or less than 60°.

(4) If the particle is projected due eastwards at an angle of elevation equal to θ, all the preceding results are true if we replace θ by $180° - \theta$; so that (66) and (67) become

$$\xi = \frac{4u^3\omega\sin\lambda(\sin\theta)^2\cos\theta}{g^2}; \tag{68}$$

$$\eta = -\frac{u^2}{g}\sin 2\theta + \frac{4u^3\omega\cos\lambda}{3g}\{(\sin\theta)^2 - 3(\cos\theta)^2\}; \tag{69}$$

so that in this case the deviation of the projectile is southwards; and the range is increased or diminished according as the angle of elevation is less than or greater than 60°.

(5) Let the particle be projected due southwards at an angle of elevation equal to θ; then

$$\cos\alpha = \cos\theta, \quad \cos\beta = 0, \quad \cos\gamma = \sin\theta;$$

and

$$\left.\begin{aligned} \xi &= ut\cos\theta, \\ \eta &= u\omega\sin(\theta+\lambda)t^2 - \omega g\cos\lambda\frac{t^3}{3}, \\ \zeta &= ut\sin\theta - \frac{gt^2}{2}; \end{aligned}\right\} \tag{70}$$

from the first and the last of these equations we infer, that neither the time nor the range on the meridian is altered by the rotation of the earth. But when $\zeta = 0$, that is, when the projectile strikes the ground, $t = \dfrac{2u\sin\theta}{g}$; in which case

$$\eta = \frac{4u^3\omega(\sin\theta)^2}{3g^2}\{\sin\theta\cos\lambda + 3\cos\theta\sin\lambda\}; \tag{71}$$

and therefore the point where the projectile strikes the ground is always west of the meridian; and a similar result will be true when the particle is projected due northwards.

Now we shall hereafter prove that these results, which have

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herein been applied to the motion of a material particle, are also true of that of the centre of gravity of a body. Neglecting therefore the resistance of the air, and the action due to the rotation of a ball or bolt, we have the following results as to rifle and cannon practice:

When the shot is fired due north or south, the range in that direction is not altered, but there is always a deviation of the shot westwards, the value of which at the point of impact on the ground is given in (71).

When the shot is fired due east, the range eastwards is increased or diminished according as the angle of elevation of the gun is less than or greater than 60°; and there is deviation southwards for all places in the northern hemisphere, and northwards for all places in the southern hemisphere, the value of which is given in (68).

When the shot is fired due west, the range is increased or diminished according as the angle of elevation is greater than or less than 60°; and there is a deviation northwards for all places in the northern hemisphere, and southwards for all places in the southern hemisphere.

So that for firing from a place in a direction coincident with the parallel of latitude, and with an elevation less than 60°, the range is increased or diminished according as we fire eastwards or westwards; and the difference between the two ranges

$$= \frac{8u^3\omega\cos\lambda}{3g}\{3(\cos\theta)^2-(\sin\theta)^2\};$$

and if the place is in the northern hemisphere, the deviation parallel to the meridian is north or south, according as we fire west or east.

And for places in the northern hemisphere for all directions lying west of the meridian, the deviation parallel to the meridian is northwards; and for all directions lying east of the meridian, the deviation parallel to the meridian is southwards.

241.] The expressions (60), (61), and (62), which have been explained in the preceding Article, are deduced from equations of motion, whose form is simplified on the assumption that products of ω^2, and one of the relative coordinates of m, are small quantities, and are to be neglected. Let us now retain these quantities in the equations of motion, and assume that products of ω^3 and of a small variable are to be neglected; and

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that all small quantities of a lower order are to be retained. In this case the equations of motion are

$$\left.\begin{aligned}
&\frac{d^2\xi}{dt^2}-\xi\omega^2(\sin\lambda)^2-\zeta\omega^2\sin\lambda\cos\lambda+2\omega\sin\lambda\frac{d\eta}{dt}=0,\\
&\frac{d^2\eta}{dt^2}-\eta\omega^2-2\omega\left(\sin\lambda\frac{d\xi}{dt}+\cos\lambda\frac{d\zeta}{dt}\right)=0,\\
&\frac{d^2\zeta}{dt^2}-\xi\omega^2\sin\lambda\cos\lambda-\zeta\omega^2(\cos\lambda)^2+2\omega\cos\lambda\frac{d\eta}{dt}=-g.
\end{aligned}\right\}\quad(72)$$

Of these equations, the values of ξ, η, ζ, given in (60), (61), (62) are approximate solutions of the first order; viz.,

$$\left.\begin{aligned}
&\xi=ut\cos a-u\omega\sin\lambda\cos\beta\, t^2,\\
&\eta=ut\cos\beta+u\omega(\cos a\sin\lambda+\cos\gamma\cos\lambda)t^2-\omega g\cos\lambda\frac{t^3}{3},\\
&\zeta=ut\cos\gamma-\frac{1}{2}gt^2-u\omega\cos\lambda\cos\beta\, t^2;
\end{aligned}\right\}\quad(73)$$

and these may be employed to find approximate solutions of (72).

In the second of (72), in the term $\omega^2\eta$, let η be replaced by $ut\cos\beta$ from (73); then integrating, we have

$$\frac{d\eta}{dt}-u\cos\beta-u\omega^2\cos\beta\frac{t^2}{2}-2\omega(\xi\sin\lambda+\zeta\cos\lambda)=0;$$

and substituting for ξ and ζ their values given in (73), and integrating again,

$$\eta=ut\cos\beta+u\omega(\cos a\sin\lambda+\cos\gamma\cos\lambda)t^2$$
$$-\omega g\cos\lambda\frac{t^3}{3}-u\omega^2\cos\beta\frac{t^3}{2}.\quad(74)$$

Again, in the first and third of (72), in the terms involving $\omega^2\xi$ and $\omega^2\zeta$, let ξ and ζ be respectively replaced by

$$ut\cos a,\quad\text{and}\quad ut\cos\gamma-\frac{1}{2}gt^2;$$

then integrating, we have

$$\frac{d\xi}{dt}-u\cos a-u\omega^2\sin\lambda(\cos a\sin\lambda+\cos\gamma\cos\lambda)\frac{t^2}{2}$$
$$+\omega^2 g\sin\lambda\cos\lambda\frac{t^3}{6}+2\omega\sin\lambda\,\eta=0;$$

$$\frac{d\zeta}{dt}-u\cos\gamma-u\omega^2\cos\lambda(\cos a\sin\lambda+\cos\gamma\cos\lambda)\frac{t^2}{2}$$
$$+\omega^2 g(\cos\lambda)^2\frac{t^3}{6}+2\omega\cos\lambda\,\eta=-gt;$$

substituting in the last terms of these the value of η, given in (74), and integrating, we have

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$$\xi = ut\cos a - u\omega\sin\lambda\cos\beta\, t^2$$
$$-u\omega^2\sin\lambda(\cos a\sin\lambda+\cos\gamma\cos\lambda)\frac{t^3}{2}+g\omega^2\sin\lambda\cos\lambda\frac{t^4}{8}; \quad (75)$$
$$\zeta = ut\cos\gamma-\frac{1}{2}gt^2-u\omega\cos\lambda\cos\beta\, t^2$$
$$-u\omega^2\cos\lambda(\cos a\sin\lambda+\cos\gamma\cos\lambda)\frac{t^3}{2}+g\,\omega^2(\cos\lambda)^2\frac{t^4}{8}; \quad (76)$$

which expressions for ξ, η, ζ are correct as far as terms involving ω^2 inclusive.

Explanations might be given of particular cases of these equations, similar to those of the last Article. I will only take two cases :

(1) Let the body fall without any initial velocity; then $u=0$, $\cos a = \cos\beta = 0$; $\cos\gamma = -1$;

$$\left.\begin{aligned}\xi &= \omega^2 g\sin\lambda\cos\lambda\frac{t^4}{8},\\ \eta &= -\omega g\cos\lambda\frac{t^3}{3},\\ \zeta &= -\frac{1}{2}gt^2+\omega^2 g(\cos\lambda)^2\frac{t^4}{8}.\end{aligned}\right\}$$

The first equation shews that there is a deviation of the falling particle in the plane of the meridian towards the south; and the second shews that the deviation in the parallel of latitude is towards the east; so that the resulting deviation of the falling body is towards the south-east. From the last equation it appears, that the space due to a given time is less than it would be if there were no rotation. Hereby then we have corrections of the results explained in (1) of the preceding Article.

(2) Let the body be projected due southwards at an angle of elevation equal to θ, so that $\cos a = \cos\theta$; $\cos\beta = 0$; $\cos\gamma = \sin\theta$; then

$$\left.\begin{aligned}\xi &= ut\cos\theta-u\omega^2\sin\lambda\sin(\lambda+\theta)\frac{t^3}{2}+g\,\omega^2\sin\lambda\cos\lambda\frac{t^4}{8},\\ \eta &= u\omega\sin(\lambda+\theta)\,t^2-\omega g\cos\lambda\frac{t^3}{3},\\ \zeta &= ut\sin\theta-\frac{gt^2}{2}-u\omega^2\cos\lambda\sin(\lambda+\theta)\frac{t^3}{2}+g\omega^2(\cos\lambda)^2\frac{t^4}{8};\end{aligned}\right\}$$

when the projectile strikes the ground, $\zeta=0$; and approximately $t = \dfrac{2u\sin\theta}{g}$; in which case

$$\eta = \frac{4u^3\omega(\sin\theta)^2}{3g^2}\{\sin\theta\cos\lambda+3\cos\theta\sin\lambda\};$$

so that the deviation along the parallel of latitude is westwards.

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In the investigation of this problem, given by M. Poisson, Journal de l'Ecole Polytechnique, Cahier 26, p. 1, terms are introduced representing the resistance of the air. The equations, thus enlarged, do not admit of direct integration; the effect however of the resistance of the air is determined by the method of variation of parameters. The student desirous of knowing the extent to which mathematical analysis has been applied to balistics, must consult three memoirs of M. Poisson, contained in Cahiers 26, 27 of the aforesaid Journal.

242.] Another problem, also of considerable physical interest, may be solved by means of these equations. For although the fact of the rotation of the earth was satisfactorily demonstrated by proofs drawn from astronomical observations, still Astronomers, as well as Geometers, were desirous of an ocular proof of a less abstract nature. Now we have observed in Art. 249, Vol. III, that if a heavy ball falls from the top of a lofty vertical tower; and if we suppose the earth to rotate from west to east, and take that rotation into account; then, if the ball falls on the east side of the tower, it strikes the ground at a certain distance from the foot; and if it falls on the west, it strikes the tower before it reaches the ground. This fact is evident from general reasoning, as we have before stated, but we have yet to investigate the law of the deviation. This was determined by Laplace; see Mécanique Céleste, 2nde partie, Livre X, Ch. V. 15: we can however deduce the law from the general equations (56). Our problem is;

A heavy ball falls from a height h to the earth: it is required to determine the circumstances of motion.

The equations of motion are (58); and the initial circumstances are these: when $t = 0$,

$$\frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\xi}{dt} = 0; \qquad \xi = \eta = 0, \qquad \zeta = h.$$

$$\therefore \left.\begin{aligned} &\frac{d\xi}{dt} + 2\,\omega\,\eta \sin\lambda = 0, \\ &\frac{d\eta}{dt} - 2\,\omega\,(\xi \sin\lambda + \zeta \cos\lambda) = 0, \\ &\frac{d\zeta}{dt} + 2\,\omega\,\eta \cos\lambda = -g\,t; \end{aligned}\right\} \qquad (77)$$

substituting the first and last of these in the second of (58), and omitting terms involving ω^2, we have

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$$\frac{d^2\eta}{dt^2} + 2\,\omega g \cos\lambda\, t = 0;$$

$$\therefore \quad \eta = -\frac{\omega g \cos\lambda}{3} t^3;$$

and from the first and the last of (77), omitting the terms involving ω^2, we have

$$\zeta = h - \frac{1}{2} g t^2,$$

$$\xi = 0;$$

so that to the degree of approximation we have taken, the vertical motion of the particle is the same as if the earth did not rotate; no deviation takes place in the plane of the meridian, and the horizontal deviation is towards the east, and varies as the cube of the time during which the body has been falling. Since the time due to the height h is $\left(\frac{2h}{g}\right)^{\frac{1}{2}}$, the deviation towards the east of the point where the body strikes the ground

$$= \frac{2^{\frac{3}{2}}\,\omega\, h^{\frac{3}{2}} \cos\lambda}{3\, g^{\frac{1}{2}}},$$

and varies therefore as the square root of the cube of the height from which the body has fallen.

The student desirous of further information on the subject of these Articles, in addition to the Memoirs of M. Poisson already alluded to, will consult with advantage (1) Benzenberg, Versuche uber das Gesetz des falls, &c., Dortmund, 1804; (2) G. L. Houel, De deviatione Meridionali corporum libere cadentium, &c., Utrecht, 1839. In both these treatises he will find the investigations of Gauss, in which the resulting equations are carried to an approximation involving higher powers of ω than the second. In the latter too he will find an account of the experiments made by M. Reich in a mine near to Freiberg, in Saxony, in the year 1833.

243.] We can also by means of these equations investigate the oscillations of a pendulum, when its motion is affected by the rotation of the earth. And we shall arrive at the results which M. Foucault exhibited in his famous pendulum experiment before the Academy of Sciences in Paris on Feb. 3rd, 1851; and which have been repeated, and confirmed, in many parts of the earth.

We shall hereby have another ocular proof of the diurnal

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rotation of the earth; and perhaps a more striking one than any that had formerly existed; for our process will shew that the observed results are in accordance with the physical laws which cause them.

It will be convenient to make a slight change in the moving system of reference, and to take the point of suspension of the pendulum for the moving origin: let the axis of ζ be taken vertically downwards from it, so that the sign of it must be changed in the preceding equations; the axes of ξ and η being taken respectively southwards and westwards as heretofore; and let h be the vertical distance of the point of suspension from the earth's surface.

We shall assume the pendulum to be perfect; and shall take l to be its length, that is, to be the distance of the bob, considered as a particle of mass m, from the point of suspension.

Let (ξ, η, ζ) be the place of its bob at the time t; then

$$\xi^2 + \eta^2 + \zeta^2 = l^2; \tag{78}$$

and let the tension along the rod of the pendulum $= m\,\text{T}$; let the components of T be introduced into the equations of motion (55); and let X', Y', Z' be the axial components of the other impressed momentum-increments; then we have

$$\frac{d^2\xi}{dt^2} - \omega^2 r \sin\lambda\cos\lambda - \xi\omega^2(\sin\lambda)^2 + (\zeta - h)\omega^2\sin\lambda\cos\lambda + 2\omega\sin\lambda\frac{d\eta}{dt} = -\text{T}\frac{\xi}{l} + \text{X}',$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta - 2\omega\left(\sin\lambda\frac{d\xi}{dt} - \cos\lambda\frac{d\zeta}{dt}\right) = -\text{T}\frac{\eta}{l} + \text{Y}',$$

$$\frac{d^2\zeta}{dt^2} + \omega^2 r(\cos\lambda)^2 + \xi\omega^2\sin\lambda\cos\lambda - (\zeta - h)\omega^2(\cos\lambda)^2 - 2\omega\cos\lambda\frac{d\eta}{dt} = -\text{T}\frac{\zeta}{l} + \text{Z}'.$$

Now these equations must satisfy the mechanical circumstances of the pendulum when it hangs vertically, and is at rest; in which case $\xi = \eta = 0, \zeta = l$; $\frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0$; $\frac{d^2\xi}{dt^2} = \frac{d^2\eta}{dt^2} = 0$, $\frac{d^2\zeta}{dt^2} = g - \text{T}$; so that

$$\left.\begin{aligned} \text{X}' &= -\omega^2 r\sin\lambda\cos\lambda + (l-h)\,\omega^2\sin\lambda\cos\lambda, \\ \text{Y}' &= 0, \\ \text{Z}' &= g - \text{T} + \omega^2 r(\cos\lambda)^2 - (l-h)\,\omega^2(\cos\lambda)^2 + \text{T}; \end{aligned}\right\}$$

and the equations of motion become

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$$\frac{d^2\xi}{dt^2} - \omega^2\xi(\sin\lambda)^2 + (\zeta - l)\,\omega^2 \sin\lambda\cos\lambda + 2\omega\sin\lambda\frac{d\eta}{dt} = -\text{T}\frac{\xi}{l},$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta - 2\omega\left(\sin\lambda\frac{d\xi}{dt} - \cos\lambda\frac{d\zeta}{dt}\right) = -\text{T}\frac{\eta}{l},$$

$$\frac{d^2\zeta}{dt^2} + \omega^2\xi\sin\lambda\cos\lambda - (\zeta - l)\,\omega^2(\cos\lambda)^2 - 2\omega\cos\lambda\frac{d\eta}{dt} = -\text{T}\frac{\zeta}{l} + g.$$

These equations represent accurately the motion of the pendulum; but as they do not admit of complete integration, we must have recourse to methods of approximation, as in the preceding Articles. We shall suppose the extent of oscillation to be very small, so that ξ, η, and $l-\zeta$ are always small quantities; and as ω^2 is a very small fraction, we shall neglect products of them and it: and thus the equations of motion become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + 2\omega\sin\lambda\frac{d\eta}{dt} &= -\text{T}\frac{\xi}{l}, \\ \frac{d^2\eta}{dt^2} - 2\omega\left(\sin\lambda\frac{d\xi}{dt} - \cos\lambda\frac{d\zeta}{dt}\right) &= -\text{T}\frac{\eta}{l}, \\ \frac{d^2\zeta}{dt^2} - 2\omega\cos\lambda\frac{d\eta}{dt} &= -\text{T}\frac{\zeta}{l} + g. \end{aligned}\right\} \qquad (79)$$

244.] Various methods have been chosen by different mathematicians of dealing with these equations. If the rotation of the earth is neglected, $\omega = 0$, and the equations become those which express the motion of a conical pendulum, and which have already been discussed in Articles 369 and 370 of Vol. III. We may take the solution of these simplified equations to be *in form* the solution of our actual equations; the former will contain four undetermined constants depending on the initial values of the velocity and coordinates of the place of the bob of the pendulum; these constants may be considered variable, according to Lagrange's method of variation of parameters; and the differential equations of motion will enable us to determine these in terms of the time, whereby we shall obtain variable elements, which will at any time fix the position of the place of the pendulum. This method has been adopted by M. Quet, in a memoir of great ability in Liouville's Journal, Vol. XVIII. Paris, 1853. Other mathematicians have followed the same process under a different form: they have considered the terms involving ω to arise from a certain disturbing function, the ξ-, η-, ζ-partial differentials of which are severally,

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$$-2\omega \sin\lambda \frac{d\eta}{dt}, \qquad 2\omega\left(\sin\lambda \frac{d\xi}{dt} - \cos\lambda \frac{d\zeta}{dt}\right), \qquad 2\omega \cos\lambda \frac{d\eta}{dt};$$

and then they have pursued the method indicated by Sir W. R. Hamilton and Jacobi. This process has been developed by M. Dumas in an Academical Dissertation, "De Motu Penduli Sphærici rotatione Terræ perturbato," Königsberg, March, 1854, in which the results are expressed in terms of the higher elliptic transcendents.

Again, other mathematicians have adopted a method of approximation depending on the successive omission of small terms. The original investigation of M. Binet* was made on this principle; and it has subsequently been applied by Hansen, "Theorie der Pendelbewegung," Danzig, 1853. I have treated the equations in the following Articles by this process, because it is the most simple and the most natural, and indicates the chief results of the equations with the least labour.

245.] Let the equations (79) be multiplied respectively by $2d\xi$, $2d\eta$, $2d\zeta$, and added; then, since by (78),

$$\xi d\xi + \eta d\eta + \zeta d\zeta = 0, \tag{80}$$

we have

$$d.\left\{\frac{d\xi^2 + d\eta^2 + d\zeta^2}{dt^2}\right\} = 2g\,d\zeta, \tag{81}$$

and if we multiply the second of (79) by ξ, and the first by η, and subtract the latter from the former, we have

$$d.\left\{\frac{\xi d\eta - \eta d\xi}{dt}\right\} - \omega \sin\lambda\, d(\xi^2 + \eta^2) + 2\omega \cos\lambda\, \xi d\zeta = 0. \tag{82}$$

Now let us refer the place of the pendulum at the time t to the horizontal plane at the place of observation and to a vertical line which passes through the point of suspension. Let θ be the angle between this vertical line and the rod of the pendulum, and let ψ be the angle at which the vertical plane, in which the pendulum is at the time t, is inclined to the plane of (ξ, ζ), which is the meridian plane; ψ increasing positively as we move from the ξ-axis towards the η-axis, that is, as we revolve from south westwards, and on northwards, and so on towards the east: that is, in a direction opposite to that in which the earth rotates. Also let ρ be the perpendicular distance from the bob of the pendulum to the vertical line through the point of suspension. Then

* See Comptes Rendus de l'Académie des Sciences de Paris, 1851, p. 197.

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if the path described by the bob is projected in the horizontal plane, ρ and ψ are the polar coordinates of it, the pole being the point directly beneath the point of suspension: thus we have

$$\left.\begin{aligned} \rho &= l\sin\theta, \\ \zeta &= l\cos\theta, \\ \xi &= \rho\cos\psi = l\sin\theta\cos\psi, \\ \eta &= \rho\sin\psi = l\sin\theta\sin\psi; \end{aligned}\right\} \tag{83}$$

$$\therefore\quad d\xi^2+d\eta^2+d\zeta^2 = l^2\{(d\theta)^2+(\sin\theta)^2(d\psi)^2\}, \tag{84}$$

$$\xi\,d\eta-\eta\,d\xi = l^2(\sin\theta)^2\,d\psi; \tag{85}$$

thus (81) and (82) become

$$\left.\begin{aligned} d.\left\{\left(\frac{d\theta}{dt}\right)^2+(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2-\frac{2g}{l}\cos\theta\right\} &= 0, \\ d.\left\{\sin\theta)^2\left(\frac{d\psi}{dt}-\omega\sin\lambda\right)\right\}-2\omega\cos\lambda\,(\sin\theta)^2\cos\psi\,d\theta &= 0. \end{aligned}\right\} \tag{86}$$

As these two equations are deduced from (79) by a change of coordinates, they have lost none of their generality, and consequently they express the general motion of a pendulum to the same degree of accuracy; and that is, when terms involving the products of ω^2 and either ξ, η, or $l-\zeta$ are omitted.

246.] For our purpose however it will be sufficient to consider the oscillations of the pendulum to be small, and thus to assume the greatest angle of inclination of the pendulum to the vertical to be so small that cubes of it and all powers higher than the cubes may be omitted. Consequently θ is always such that θ^3 and higher powers of θ will be omitted; also $\dfrac{d\theta}{dt}$ is a small quantity. Let us replace $l\theta$ by ρ, as we may by means of (83); because we shall thereby obtain the polar equation of the curve in the horizontal plane into which the path described by the bob of the pendulum is projected. Then omitting θ^3 and higher powers of θ, from (83) we have

$$\rho = l\theta,$$

$$\therefore\quad \frac{d\rho}{dt} = l\frac{d\theta}{dt}.$$

Also the last terms in the second equation of (86) must be omitted, because $(\sin\theta)^2\,d\theta$ is a small term of an order higher than those which are to be retained. Thus (86) become

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$$\left.\begin{aligned} d.\left\{\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\psi}{dt}\right)^2+\frac{g}{l}\rho^2\right\} &= 0, \\ d.\left\{\rho^2\left(\frac{d\psi}{dt}-\omega\sin\lambda\right)\right\} &= 0. \end{aligned}\right\} \tag{87}$$

Now these equations are integrable; and let us suppose the pendulum to start from rest at a distance $\rho = a$, from the vertical line passing through the point of suspension; so that

$$\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\psi}{dt}\right)^2+\frac{g}{l}(\rho^2-a^2) = 0; \tag{88}$$

$$\rho^2\frac{d\psi}{dt}-\omega\sin\lambda(\rho^2-a^2) = 0; \tag{89}$$

eliminating $\frac{d\psi}{dt}$, we have

$$\rho^2\left(\frac{d\rho}{dt}\right)^2 = -\left\{\frac{g}{l}+\omega^2(\sin\lambda)^2\right\}\rho^4+a^2\left\{\frac{g}{l}+2\omega^2(\sin\lambda)^2\right\}\rho^2 - \omega^2(\sin\lambda)^2a^4, \tag{90}$$

$$= \left\{\frac{g}{l}+\omega^2(\sin\lambda)^2\right\}\left\{-\rho^4+\frac{g+2l\omega^2(\sin\lambda)^2}{g+l\omega^2(\sin\lambda)^2}a^2\rho^2 - \frac{\omega^2l(\sin\lambda)^2}{g+l\omega^2(\sin\lambda)^2}a^4\right\}. \tag{91}$$

Now the right hand member of this equation is a quadratic expression in ρ^2, which has two roots, both of which are positive, and of which one is a^2; let b^2 be the other, and let us suppose a^2 to be the greater: also, for convenience of expression, let

$$\frac{g}{l}+\omega^2(\sin\lambda)^2 = n^2; \tag{92}$$

so that

$$b = \frac{\omega\sin\lambda}{n}a. \tag{93}$$

Then (91) becomes

$$\rho^2\left(\frac{d\rho}{dt}\right)^2 = n^2(a^2-\rho^2)(\rho^2-b^2); \tag{94}$$

so that a and b are manifestly the greatest and least values of ρ. From (94) we have

$$\frac{\rho\,d\rho}{\{(a^2-\rho^2)(\rho^2-b^2)\}^{\frac{1}{2}}} = -n\,dt;$$

the negative sign being taken, because on our supposition ρ decreases as t increases. Let also $t=0$, when $\rho=a$; then integrating, we have

$$\rho^2 = \frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2nt; \tag{95}$$

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which gives the value of ρ^2 in terms of t; and shews that (1) a^2 and b^2 are respectively the greatest and least values of ρ^2; (2) their values recur periodically; and (3) the periodic time

$$= \frac{\pi}{n} = \frac{\pi l^{\frac{1}{2}}}{\{g + l\omega^2 (\sin\lambda)^2\}^{\frac{1}{2}}}: \tag{96}$$

this result evidently agrees with that of the common simple pendulum, when $\omega = 0$.

To find the relation between ρ and ψ, we have from (89)

$$\frac{d\psi}{dt} - \omega \sin\lambda = -\frac{\omega a^2 \sin\lambda}{\rho^2}; \tag{97}$$

but from (93)

$$b = \frac{\omega \sin\lambda}{n} a;$$

$$\therefore \quad \frac{d\psi}{dt} - \omega \sin\lambda = -\frac{nab}{\rho^2}; \tag{98}$$

$$\therefore \quad d\psi - \omega \sin\lambda\, dt = \frac{ab\, d\rho}{\rho\{(a^2-\rho^2)(\rho^2-b^2)\}^{\frac{1}{2}}}; \tag{99}$$

whence, by integration, with the assumption that $\psi = \psi_0$, and $\rho = a$, when $t = 0$,

$$\frac{1}{\rho^2} = \frac{a^2+b^2}{2a^2b^2} - \frac{a^2-b^2}{2a^2b^2}\cos 2(\psi - \psi_0 - \omega \sin\lambda\, t)$$

$$= \frac{\{\cos(\psi - \psi_0 - \omega\sin\lambda t)\}^2}{a^2} + \frac{\{\sin(\psi - \psi_0 - \omega\sin\lambda t)\}^2}{b^2}. \tag{100}$$

If t is constant, this is an ellipse whose principal axes are respectively $2a$ and $2b$; so that the path described by the bob of the pendulum projected on the horizontal plane is an ellipse, the whole period being that given in (96). And since $\frac{d\psi}{dt}$, which is given in (97), is negative, the pendulum revolves in a direction opposite to that in which ψ increases; that is, the direction of its revolution is the same as that of the earth. And since (93) shews that the ratio of b to a varies nearly as ω, b is small compared with a, so that the eccentricity of the ellipse is very large; if $\omega = 0$, $b = 0$, in which case the minor axis vanishes, and the pendulum moves in a plane: this however cannot be the case when account is taken of the earth's rotation.

Since however t varies, let us still consider (100) to represent an ellipse whose principal axes are $2a$ and $2b$; and whose major axis at the time t is inclined to the ξ-axis, which is measured

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southwards along the meridian at an angle equal to $\psi_0 + \omega \sin\lambda\, t$: now this angle increases as t increases; and consequently the major axis revolves in azimuth with a constant angular velocity equal to $\omega \sin\lambda$ in the same direction in which ψ increases. Thus, if the path described by the bob of the pendulum is projected on the horizontal plane, it will be a revolving ellipse, whose major axis revolves in azimuth with an angular velocity equal to $\omega \sin\lambda$, in a direction opposite to that in which the earth moves: the actual path will thus be a spiral limited by two concentric circles whose radii are a and b, of which a is the greater; the spiral never extending beyond the former, nor coming within the latter; and the point where it meets the larger circle advancing with an angular velocity equal to $\omega \sin\lambda$, in a direction opposite to that of the earth's rotation, and opposite to that in which the pendulum itself moves.

This is the law which the experiment exhibited by M. Foucault confirms. We have already given a simple explanation of it in Art. 30; and that explanation appeared to M. Poinsot (see Comptes Rendus, Tome XXXII, p. 206) to be sufficient. The preceding investigation however shews that the result follows from the equations of motion, when small terms are omitted. This therefore is only the general effect; but there are sundry deviations, owing to the omitted terms, which this dynamical process will indicate if it is carried to a higher approximation, and the other method fails to shew; but it is beyond our purpose to enter upon these small disturbances in this treatise. The several memoirs already alluded to contain further approximations, and to them I must refer the student. I should also mention that M. Poncelet, whose name must ensure attention from every mathematician, has written two memoirs on this subject, which are inserted in the Comptes Rendus de l'Académie des Sciences de Paris, Vol. LI, 1860, and has arrived at results differing in some respects from the preceding.

247.] The motion whose circumstances we have investigated has been imagined to be that of a bob of a pendulum fixed by a rod of given length to a point fixed relatively to the earth and moving with it, and the effect of that rotation has been exhibited in the preceding equations. This motion is consequently that of a material particle moving on the lower concave surface of a sphere, whose radius is l, fixed to the earth and moving with it; and the general equations are applicable to any other kind

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of constrained motion of a particle. Let us take another example.

A particle moves on a smooth inclined plane fixed to the earth and moving with it: it is required to determine the relative motion of the particle.

Let the plane pass through P, the place of observation, see Fig. 63, whose latitude is λ; and let the equation be

$$\xi \cos \alpha + \eta \cos \beta + \zeta \cos \gamma = 0; \tag{101}$$

let mR be the normal pressure of the particle on the plane; then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + 2\omega \sin\lambda \frac{d\eta}{dt} &= \text{R} \cos \alpha, \\ \frac{d^2\eta}{dt^2} - 2\omega \left(\sin\lambda \frac{d\xi}{dt} + \cos\lambda \frac{d\zeta}{dt}\right) &= \text{R} \cos \beta, \\ \frac{d^2\zeta}{dt^2} + 2\omega \cos\lambda \frac{d\eta}{dt} &= \text{R} \cos \gamma - g. \end{aligned}\right\} \tag{102}$$

Although it is convenient to retain α, β, and γ, yet we shall require their values in terms of (θ) the inclination of the plane to the horizontal plane of (ξ, η), and of the angle (ψ) between the ξ-axis, which is southwards, and the line of intersection (the line of nodes) of the plane with the horizontal plane. In reference to these

$$\left.\begin{aligned} \cos \alpha &= \sin\theta \sin\psi, \\ \cos \beta &= -\sin\theta \cos\psi, \\ \cos \gamma &= \cos\theta. \end{aligned}\right\} \tag{103}$$

Let the particle start from rest from (ξ_0, η_0, ζ_0); then, multiplying (102) severally by $d\xi$, $d\eta$, and $d\zeta$, adding, and integrating, we have

$$\frac{ds^2}{dt^2} = v^2 = 2g(\zeta_0 - \zeta), \tag{104}$$

because, by reason of (101),

$$d\xi \cos \alpha + d\eta \cos \beta + d\zeta \cos \gamma = 0. \tag{105}$$

Thus (104) shews that the velocity acquired is the same as if the rotation of the earth was not considered.

From the last two equations of (102) we have

$$\cos\gamma \frac{d^2\eta}{dt^2} - \cos\beta \frac{d^2\zeta}{dt^2} + 2\omega(\cos\lambda \cos\alpha - \sin\lambda \cos\gamma) \frac{d\xi}{dt} = g \cos\beta;$$

therefore

$$\cos\gamma \frac{d\eta}{dt} - \cos\beta \frac{d\zeta}{dt} + 2\omega(\cos\lambda \cos\alpha - \sin\lambda \cos\gamma)(\xi - \xi_0) = gt \cos\beta. \tag{106}$$

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Similarly

$$\cos a \frac{d\zeta}{dt} - \cos\gamma \frac{d\xi}{dt} + 2\omega(\cos\lambda\cos a - \sin\lambda\cos\gamma)(\eta - \eta_0) = -gt\cos a; \quad (107)$$

$$\cos\beta \frac{d\xi}{dt} - \cos a \frac{d\eta}{dt} + 2\omega(\cos\lambda\cos a - \sin\lambda\cos\gamma)(\zeta - \zeta_0) = 0. \quad (108)$$

Again, multiplying (102) severally by $\cos a$, $\cos\beta$, and $\cos\gamma$; adding, and omitting the terms which vanish by reason of the differential of (105), we have

$$-2\omega\sin\lambda\left(\cos\beta\frac{d\xi}{dt} - \cos a\frac{d\eta}{dt}\right) + 2\omega\cos\lambda\left(\cos\gamma\frac{d\eta}{dt} - \cos\beta\frac{d\zeta}{dt}\right) = \text{R} - g\cos\gamma; \quad (109)$$

substituting in which from (106) and (108), and omitting terms involving ω^2, we have

$$\text{R} = g\cos\gamma + 2\omega\cos\lambda\cos\beta\, gt, \quad (110)$$

$$= g\cos\theta - 2\omega g t\cos\lambda\sin\theta\cos\psi; \quad (111)$$

which assigns the pressure on the plane; and shews that it is diminished or increased by the earth's rotation according as the line of nodes lies in the S. W. and N. E. quadrants, or in the N. W. and S. E. quadrants; and that this increase or diminution vanishes when the line of nodes lies E. and W. It vanishes at the pole, and is, cæteris paribus, a maximum at the equator; and it also vanishes when the plane is horizontal. It also varies as the time during which the particle has been moving. Since $\omega\cos\lambda$ is the component of the earth's angular velocity along the tangent to the meridian, that is, along the N. and S. line on the horizontal plane, the change of pressure on the plane is due to that component only, and not to the component along the vertical.

Substituting in the first and third equations of (102) the value of R, given in (110), and integrating, we have

$$\left.\begin{aligned} \frac{d\xi}{dt} + 2\omega\sin\lambda(\eta - \eta_0) &= \quad gt\cos a\cos\gamma + \omega\cos\lambda\cos a\cos\beta\, gt^2, \\ \frac{d\zeta}{dt} + 2\omega\cos\lambda(\eta - \eta_0) &= -gt\ (\sin\gamma)^2 + \omega\cos\lambda\cos\beta\cos\gamma\, gt^2; \end{aligned}\right\} \quad (112)$$

and substituting these values in the second of (102), and omitting the terms involving ω^2, we have

$$\frac{d^2\eta}{dt^2} + 2\omega\cos a\{\cos\lambda\cos a - \sin\lambda\cos\gamma\}gt = g\cos\beta\cos\gamma;$$

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therefore

$$\eta-\eta_0=\cos\beta\cos\gamma\frac{gt^2}{2}-\omega\cos a\{\cos\lambda\cos a-\sin\lambda\cos\gamma\}\frac{gt^3}{3}; \quad (113)$$

and substituting this value of $\eta-\eta_0$ in (112), and integrating,

$$\xi-\xi_0=\cos a\cos\gamma\frac{gt^2}{2}+\omega\cos\beta\{\cos\lambda\cos a-\sin\lambda\cos\gamma\}\frac{gt^3}{3}; \quad (114)$$

$$\zeta-\zeta_0=-(\sin\gamma)^2\frac{gt^2}{2}; \quad (115)$$

so that in terms of θ and ψ,

$$\left.\begin{aligned}\xi&=\xi_0+\sin\theta\cos\theta\sin\psi\frac{gt^2}{2}-\omega\sin\theta\cos\psi\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}\frac{gt^3}{3},\\ \eta&=\eta_0-\sin\theta\cos\theta\cos\psi\frac{gt^2}{2}-\omega\sin\theta\sin\psi\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}\frac{gt^3}{3},\\ \zeta&=\zeta_0-(\sin\theta)^2\frac{gt^2}{2};\end{aligned}\right\}(116)$$

which assign the position of the particle at the time t. Whence it appears, that if we omit powers of ω higher than the first, the vertical distance through which m falls in the time t is not affected by the earth's rotation.

To determine the curve which the particle describes, let us refer its place at the time t to the point (ξ_0, η_0, ζ_0) as origin; and to two axes in the plane, one of which, that of ξ', is parallel to, and the other, that of η', is perpendicular to the line of nodes; so that

$$\left.\begin{aligned}(\xi_0-\xi)\cos\psi+(\eta_0-\eta)\sin\psi&=\xi',\\ \zeta_0-\zeta&=\eta'\sin\theta;\end{aligned}\right\} \quad (117)$$

$$\left.\begin{aligned}\therefore\quad \xi'&=\omega\sin\theta\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}\frac{gt^3}{3},\\ \eta'&=\sin\theta\frac{gt^2}{2}.\end{aligned}\right\} \quad (118)$$

Thus, if $\omega=0$, $\xi'=0$, and the particle falls down the plane in a straight line perpendicular to the line of nodes; but if the rotation of the earth is considered, there is a lateral deviation from the rectilineal path, which varies as the cube of the time of falling; and if we eliminate t, we obtain the equation to the path, which is

$$\xi'^2=\frac{8\omega^2}{9g\sin\theta}\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}^2\eta'^3; \quad (119)$$

this is the equation to the semicubical parabola, which is the path of the particle. It will be observed that I have assumed

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the particle to start from rest from (ξ_0, η_0, ζ_0); if it were projected from that point on the plane with a given velocity, other terms, which can easily be found, would be introduced into the preceding equations. And if $\omega = 0$, the resulting equations would of course represent a parabola.

248.] Let us suppose the plane on which the particle moves to be horizontal; then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + 2\,\omega \sin\lambda \frac{d\eta}{dt} &= 0, \\ \frac{d^2\eta}{dt^2} - 2\,\omega \sin\lambda \frac{d\xi}{dt} &= 0; \end{aligned}\right\}$$

and let us suppose the particle to be projected from the origin along the plane with a velocity u in a line inclined at an angle β to the axis of ξ; then, integrating the preceding equations, we have

$$\left.\begin{aligned} \frac{d\xi}{dt} + 2\,\omega \sin\lambda\, \eta &= u \cos\beta, \\ \frac{d\eta}{dt} - 2\,\omega \sin\lambda\, \xi &= u \sin\beta; \end{aligned}\right\}$$

which equations assign the relative velocity of the particle at the point (ξ, η), and by subsequent integration we have

$$\left(\xi + \frac{u \sin\beta}{2\,\omega \sin\lambda}\right)^2 + \left(\eta - \frac{u \cos\beta}{2\,\omega \sin\lambda}\right)^2 = \frac{u^2}{4\,\omega^2 (\sin\lambda)^2};$$

which is the equation to a circle. Consequently the particle moves in a circle whose radius is $\frac{u}{2\omega \sin\lambda}$; whose centre is at the point $\left(-\frac{u \sin\beta}{2\omega \sin\lambda}, \frac{u \cos\beta}{2\omega \sin\lambda}\right)$; and the periodic time $= \frac{\pi}{\omega \sin\lambda} = \frac{43082''}{\sin\lambda}$ = a mean solar day divided by twice the sine of the latitude.

Another problem, which may be solved by these general equations, is the motion of a particle on the surface of a right circular cone, whose vertex is at P, the place of observation, and whose axis coincides with the vertical.

SECTION 3.—*The relative motion of a material system.*

249.] THE equations of relative motion which have been found refer only to the motion of a single material particle.

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Those however of a material system may be deduced from them by means of D'Alembert's principle.

Suppose m to be the type-particle of a system, to the motion of which equations (23) refer; and suppose I to be the type of an impressed momentum-increment due to an internal force, see Art. 45, acting on M, of which $\text{I}\cos\lambda$, $\text{I}\cos\mu$, $\text{I}\cos\nu$ are the axial components; and let us suppose the system to be free from all constraint except that which exists amongst its own members; so that every particle is free to move as it is affected by the external forces acting on it, and by the internal forces of the system; then the equations of motion of the system in their most general forms are

$$\left.\begin{aligned}
\Sigma.m\left\{\text{X}'-\text{X}_t-2\,\text{F}\cos\alpha-\frac{d^2\xi}{dt^2}\right\}-\Sigma.\text{I}\cos\lambda&=0,\\
\Sigma.m\left\{\text{Y}'-\text{Y}_t-2\,\text{F}\cos\beta-\frac{d^2\eta}{dt^2}\right\}-\Sigma.\text{I}\cos\mu&=0,\\
\Sigma.m\left\{\text{Z}'-\text{Z}_t-2\,\text{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\right\}-\Sigma.\text{I}\cos\nu&=0.
\end{aligned}\right\}\quad(120)$$

If the material system is a rigid body, or is invariable in form, or otherwise is such that the internal forces taken throughout it disappear, then these equations become

$$\left.\begin{aligned}
\Sigma.m\left\{\text{X}'-\text{X}_t-2\,\text{F}\cos\alpha-\frac{d^2\xi}{dt^2}\right\}&=0,\\
\Sigma.m\left\{\text{Y}'-\text{Y}_t-2\,\text{F}\cos\beta-\frac{d^2\eta}{dt^2}\right\}&=0,\\
\Sigma.m\left\{\text{Z}'-\text{Z}_t-2\,\text{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\right\}&=0;
\end{aligned}\right\}\quad(121)$$

and it is the motion of a system of this kind which for the most part we shall consider.

From these the equations of the axial components of the moments of the couples are to be formed: let us take that whose axis is the moving axis of ξ; then we have

$$\Sigma.m\left\{\eta\left(\text{Z}'-\text{Z}_t-2\,\text{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\right)-\zeta\left(\text{Y}'-\text{Y}_t-2\,\text{F}\cos\beta-\frac{d^2\eta}{dt^2}\right)\right\}=0;$$

and replacing $\text{F}\cos\beta$ and $\text{F}\cos\gamma$ by their values given in (19), this becomes

$$\Sigma.m\left\{\eta\left(\text{Z}'-\text{Z}_t-\frac{d^2\zeta}{dt^2}\right)-\zeta\left(\text{Y}'-\text{Y}_t-\frac{d^2\eta}{dt^2}\right)\right\}$$
$$-2\,\omega_\xi\,\Sigma.m\,\frac{\eta\,d\eta+\zeta\,d\zeta}{dt}+2\,\omega_\eta\,\Sigma.m\,\eta\,\frac{d\xi}{dt}+2\,\omega_\zeta\,\Sigma.m\,\zeta\,\frac{d\xi}{dt}=0;$$

or

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$$\frac{d}{dt}\Sigma.m\left(\eta\frac{d\zeta}{dt}-\zeta\frac{d\eta}{dt}\right)=\Sigma.m\{\eta(\mathrm{Z}'-\mathrm{Z}_t)-\zeta(\mathrm{Y}'-\mathrm{Y}_t)\}$$

$$-\omega_\xi\frac{d}{dt}\Sigma.m(\eta^2+\zeta^2)+2\omega_\eta\Sigma.m\eta\frac{d\xi}{dt}+2\omega_\zeta\Sigma.m\zeta\frac{d\xi}{dt}=0;\quad(122)$$

and the similar equations for the other axes are

$$\frac{d}{dt}\Sigma.m\left(\zeta\frac{d\xi}{dt}-\xi\frac{d\zeta}{dt}\right)=\Sigma.m\{\zeta(\mathrm{X}'-\mathrm{X}_t)-\xi(\mathrm{Z}'-\mathrm{Z}_t)\}$$

$$-\omega_\eta\frac{d}{dt}\Sigma.m(\zeta^2+\xi^2)+2\omega_\zeta\Sigma.m\zeta\frac{d\eta}{dt}+2\omega_\xi\Sigma.m\xi\frac{d\eta}{dt}=0;\quad(123)$$

$$\frac{d}{dt}\Sigma.m\left(\xi\frac{d\eta}{dt}-\eta\frac{d\xi}{dt}\right)=\Sigma.m\{\xi(\mathrm{Y}'-\mathrm{Y}_t)-\eta(\mathrm{X}'-\mathrm{X}_t)\}$$

$$-\omega_\zeta\frac{d}{dt}\Sigma.m(\xi^2+\eta^2)+2\omega_\xi\Sigma.m\xi\frac{d\zeta}{dt}+2\omega_\eta\Sigma.m\eta\frac{d\zeta}{dt}=0;\quad(124)$$

and by means of these six equations the relative motion of a material system of invariable form may be determined.

250.] These six equations of relative motion may be combined into a single equation by means of the principle of virtual velocities. For suppose δs to be any arbitrary geometrical displacement of the place of m at the time t, which is consistent with the geometrical relations of the system; and let $\delta\xi$, $\delta\eta$, $\delta\zeta$ be the axial projections of δs; and let all these quantities be type-quantities; then the equations of motion may be expressed by means of the single equation,

$$\Sigma.m\left\{\left(\mathrm{X}'-\mathrm{X}_t-2\mathrm{F}\cos\alpha-\frac{d^2\xi}{dt^2}\right)\delta\xi+\left(\mathrm{Y}'-\mathrm{Y}_t-2\mathrm{F}\cos\beta-\frac{d^2\eta}{dt^2}\right)\delta\eta\right.$$

$$\left.+\left(\mathrm{Z}'-\mathrm{Z}_t-2\mathrm{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\right)\delta\zeta\right\}=0.\quad(125)$$

This equation is indeed equivalent to the six equations by reason of the arbitrariness of $\delta\xi$, $\delta\eta$, $\delta\zeta$; for these quantities in their most general forms involve six displacements, which are independent of each other; viz., three of translation and three of rotation; and the coefficients of these separately vanish. If the relative motion of one or more of the particles of the system is constrained, these displacements are thus far subject to certain conditions, and consequently are not independent; and all that has been said in Arts. 49, 50, and 51 is, mutatis mutandis, to be applied to this case.

251.] Let us suppose that the conditions to which the system

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is subject do not involve the time explicitly; then we may take for the virtual arbitrary displacement of the place of m that which actually takes place in the time dt by reason of the motion of the system, and of the forces acting on it; so that in equation (125) we may put

$$\delta\xi = d\xi, \qquad \delta\eta = d\eta, \qquad \delta\zeta = d\zeta; \tag{126}$$

then, since from (19),

$$\text{F}\{\cos\alpha\, d\xi + \cos\beta\, d\eta + \cos\gamma\, d\zeta\} = 0, \tag{127}$$

(125) becomes

$$\Sigma.m\left\{\frac{d^2\xi}{dt^2}d\xi + \frac{d^2\eta}{dt^2}d\eta + \frac{d^2\zeta}{dt^2}d\zeta\right\}$$
$$= \Sigma.m\{(\text{X}'-\text{X}_t)d\xi + (\text{Y}'-\text{Y}_t)d\eta + (\text{Z}'-\text{Z}_t)d\zeta\}; \tag{128}$$

so that if v is the relative velocity of m at the time t,

$$d.\Sigma.mv^2 = 2\,\Sigma.m\{(\text{X}'-\text{X}_t)d\xi + (\text{Y}'-\text{Y}_t)d\eta + (\text{Z}'-\text{Z}_t)d\zeta\}; \tag{129}$$

therefore

$$\Sigma.mv^2 - \Sigma.mv_0^2 = 2\int_0^1 \Sigma.m\{(\text{X}'-\text{X}_t)d\xi + (\text{Y}'-\text{Y}_t)d\eta + (\text{Z}'-\text{Z}_t)d\zeta\}; \tag{130}$$

wherein v_0 is the initial value of v, and 1 and 0 denote the limiting values of the relative coordinates of the place of m, corresponding to the terminal and the initial values of the left-hand member of the equation.

Equation (130) is that of the relative vis viva of the material system; and if we consider it in its elemental form in (129), it shews that the increment of the relative vires vivæ of all the particles of the system in the time dt is equal to the excess of twice the sum of the products of the impressed momentum-increment of each particle and the space through which it has acted over the sum of the products of the momentum-increment due to the force of transference of the coordinate system (see Art. 226), and the space through which this latter force has acted.

It will be seen that F, the compound centrifugal force, has wholly disappeared in (129) and (130); and rightly so; because its line of action is perpendicular to that of the relative velocity of m; whereas into the equation of vis viva only those forces enter whose lines of action are in, or have a component along, that of the motion of m at the time t.

252.] It is expedient to mention certain particular forms which the preceding general equations take in special cases,

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because in these simplified forms they are frequently applicable to the solution of problems.

(1) Let us suppose the origin of the moving system to be always in the plane of (x, y), and to move according to a given law; and the system to have no motion of rotation: then $a_1 = b_2 = c_3 = 1$, and all the other direction-cosines vanish; so that from (15), (16), and (17),

$$\mathrm{X}_t = \frac{d^2 x_0}{dt^2}, \qquad \mathrm{Y}_t = \frac{d^2 y_0}{dt^2}, \qquad \mathrm{Z}_t = \frac{d^2 z_0}{dt^2} = 0;$$

and (124) becomes

$$\frac{d}{dt}\Sigma.m\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) = \Sigma.m(\xi\mathrm{Y}' - \eta\mathrm{X}') - \Sigma.m\xi\frac{d^2 y_0}{dt^2} + \Sigma.m\eta\frac{d^2 x_0}{dt^2}$$

$$= \Sigma.m(\xi\mathrm{Y}' - \eta\mathrm{X}') - \frac{d^2 y_0}{dt^2}\Sigma.m\xi + \frac{d^2 x_0}{dt^2}\Sigma.m\eta \quad (131)$$

$$= \mathrm{L} - \frac{d^2 y_0}{dt^2}\Sigma.m\xi + \frac{d^2 x_0}{dt^2}\Sigma.m\eta; \quad (132)$$

if L is the moment of the couple of the impressed forces whose axis is the moving ζ-axis.

If the material system is of invariable form, and is fixed to the moving origin; and if r is the distance of m from that origin, and ω is the angular velocity of the body at the time t; then (132) becomes

$$\frac{d\omega}{dt}\Sigma.mr^2 = \mathrm{L} - \frac{d^2 y_0}{dt^2}\Sigma.m\xi + \frac{d^2 x_0}{dt^2}\Sigma.m\eta; \quad (133)$$

by which equation the relative angular velocity of the body may be determined.

(2) Let us suppose the origin of the moving axes to be fixed at the fixed origin, and the moving axes to revolve about the z-axis with an uniform angular velocity $= \omega$ (say); let θ be the angle between the axes of x and ξ at the time t; so that

$$\left.\begin{array}{lll} a_1 = \cos\theta, & b_1 = -\sin\theta, & c_1 = 1, \\ a_2 = \sin\theta, & b_2 = \cos\theta, & c_2 = 0, \\ a_3 = 0; & b_3 = 0; & c_3 = 0; \end{array}\right\} \quad (134)$$

and $\qquad \mathrm{X}_t = -\omega^2\xi, \quad \mathrm{Y}_t = -\omega^2\eta, \quad \mathrm{Z}_t = 0;$

then (124) becomes

$$\frac{d}{dt}\Sigma.m\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) = \Sigma.m(\xi\mathrm{Y}' - \eta\mathrm{X}') - \omega\frac{d}{dt}\Sigma.m(\xi^2 + \eta^2) \quad (135)$$

$$= \mathrm{L} - \omega\frac{d}{dt}\Sigma.m(\xi^2 + \eta^2); \quad (136)$$

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where L is the relative moment of the impressed couple whose axis is that of z.

The equations (132) and (136) may also be derived directly from (35), without the intervention of the general forms given in (124).

If the body is rigid, and the origin is a fixed point of it, then $\Sigma . m(\xi^2+\eta^2)$ is independent of the time, and (136) becomes

$$\frac{d\omega}{dt}\Sigma . m r^2 = \text{L}; \tag{137}$$

which is the same equation as that which expresses the rotation of a rigid body about a fixed or an instantaneous axis.

It is also to be observed that (132) is reduced to (137): (1) when (x_0, y_0), the place of the moving origin, is fixed absolutely: (2) when this moving origin has an uniform motion; so that $\frac{d^2x_0}{dt^2} = \frac{d^2y_0}{dt^2} = 0$; (3) when the moving origin is the centre of gravity of the body, because in that case $\Sigma . m\xi = \Sigma . m\eta = 0$.

253.] From these general equations we may deduce theorems similar to those of absolute motion which have been already demonstrated in Chap. II. Section 2, of the present Volume. In the first place, the relative motion of the centre of gravity of a material system of invariable form, or in which the internal forces mutually destroy each other, is the same as if the whole mass of the system were collected into it, and all the momentum due to the external forces, the forces of transference, and the compound centrifugal forces, the last two with their directions changed, was thereat applied in lines parallel to the actual lines of action.

Let $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ be the place of the centre of gravity of the system at the time t relatively to the moving axes; and let (ξ', η', ζ') be the place of m at the same time relatively to a system of parallel axes originating at the centre of gravity: then we have

$$\left.\begin{aligned} \xi &= \bar{\xi} + \xi', \\ \eta &= \bar{\eta} + \eta', \\ \zeta &= \bar{\zeta} + \zeta'. \end{aligned}\right\} \tag{138}$$

Also let M denote the mass of the whole moving system: then the newly introduced coordinates are subject to the following conditions:

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$$\Sigma.m\xi' = \Sigma.m\eta' = \Sigma.m\zeta' = 0; \tag{139}$$

$$\Sigma.m\xi = \text{M}\bar{\xi}, \quad \Sigma.m\eta = \text{M}\bar{\eta}, \quad \Sigma.m\zeta = \text{M}\bar{\zeta}. \tag{140}$$

On referring to the analytical values of the momentum due to the forces of transference given in (15), (16), and (17), it appears that the values of $\Sigma.m\text{X}_t$, $\Sigma.m\text{Y}_t$, $\Sigma.m\text{Z}_t$ are not changed; but they may be expressed as

$$\text{M}\bar{\text{X}}_t, \qquad \text{M}\bar{\text{Y}}_t, \qquad \text{M}\bar{\text{Z}}_t;$$

where $\bar{\text{X}}_t$, $\bar{\text{Y}}_t$, $\bar{\text{Z}}_t$ are the values of X_t, Y_t, Z_t, when ξ, η, ζ are replaced by $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$ so that the momentum due to the forces of transference may be applied to a mass M at the centre of gravity, along lines parallel to their original lines of action. A similar theorem is also true of $\Sigma.m\text{F}\cos\alpha$, $\Sigma.m\text{F}\cos\beta$, $\Sigma.m\text{F}\cos\gamma$, which may be replaced by $\text{M}\bar{\text{F}}\cos\alpha$, $\text{M}\bar{\text{F}}\cos\beta$, $\text{M}\bar{\text{F}}\cos\gamma$; so that the equations (121) become, after all reductions,

$$\left.\begin{aligned}
\frac{\Sigma.m\text{X}'}{\text{M}} - \bar{\text{X}}_t - 2\bar{\text{F}}\cos\alpha - \frac{d^2\bar{\xi}}{dt^2} &= 0,\\
\frac{\Sigma.m\text{Y}'}{\text{M}} - \bar{\text{Y}}_t - 2\bar{\text{F}}\cos\beta - \frac{d^2\bar{\eta}}{dt^2} &= 0,\\
\frac{\Sigma.m\text{Z}'}{\text{M}} - \bar{\text{Z}}_t - 2\bar{\text{F}}\cos\gamma - \frac{d^2\bar{\zeta}}{dt^2} &= 0;
\end{aligned}\right\} \tag{141}$$

and these equations prove the theorem which has been enuntiated. The theorem of relative motion analogous to that of Art. 55 may be framed in the same manner. Thus the relative motion of a material system, such as we have considered, may be resolved into the motion of translation of its centre of gravity, all the forces being supposed to act on the whole mass condensed into that point, and into a motion of rotation about an axis passing through the centre of gravity. Consequently the investigations of the preceding section are not limited to the motion of a material particle: they are also applicable to that of the centre of gravity of a material system, of which the internal forces vanish. Thus they apply to the relative motion of translation of the centre of gravity of planets, of shot, of pendulums with large balls, &c.; except that in these cases the resistance of the medium through which the bodies pass must be taken account of; so that other terms enter into the equations beside those which we have considered. It remains then only to investigate the rotation of the body about an axis passing through the centre of gravity considered as a fixed point in

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reference to a system of moving axes. It is true, as we have heretofore remarked, that the point through which the rotation-axis passes need not be the centre of gravity; for the general motion may always be resolved into a motion of translation of *any* point, and a motion of rotation about an axis passing through that point: but the centre of gravity is the only point at which the mass may be supposed to be condensed and the forces may be applied each in its own intensity and direction, and the translation will be the same as it is in the motion of the whole system. In the following Articles I shall take the general case, and shall suppose the fixed point, about which the rotation is estimated, not necessarily to be the centre of gravity.

254.] At this point I shall assume three systems of reference, and subsequently of coordinate-axes, to originate. (1) A system the lines of which are parallel to the analogous lines in the system absolutely fixed, so that all angles will be the same in both; and this system may also be regarded as fixed: (2) the system of axes to which the motion of the body is to be referred; this is a moving system, and its motion with reference to the fixed system is given, and the elements of it are, as heretofore, data of the problem; these two systems are connected by the scheme of cosines &c. which are involved in (1) of the present Chapter: (3) another system of rectangular axes, fixed in the body and moving with it, which I shall take to be a principal system at the point.

In reference to these three systems respectively I shall take the place of m to be (x, y, z), (ξ, η, ζ), (ξ', η', ζ'); and these last two I shall take to be connected by the following scheme of direction-cosines;

	ξ	η	ζ
ξ'	α_1	α_2	α_3
η'	β_1	β_2	β_3
ζ'	γ_1	γ_2	γ_3

(142)

so that

$$\left.\begin{aligned} \xi &= \alpha_1\xi' + \beta_1\eta' + \gamma_1\zeta', \\ \eta &= \alpha_2\xi' + \beta_2\eta' + \gamma_2\zeta', \\ \zeta &= \alpha_3\xi' + \beta_3\eta' + \gamma_3\zeta'; \end{aligned}\right\} \qquad (143)$$

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To determine the relative motion, these nine direction-cosines must be expressed in terms of t: as only three are independent, it will be eventually more convenient to determine the position &c. of the body by means of Euler's three angles, θ, ϕ, ψ, according to the process of Articles 3 and 4: so that

$$\left.\begin{aligned}
\alpha_1 &= \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta,\\
\beta_1 &= -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta,\\
\gamma_1 &= \sin\psi\sin\theta;\\
\alpha_2 &= \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta,\\
\beta_2 &= -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta,\\
\gamma_2 &= -\cos\psi\sin\theta;\\
\alpha_3 &= \sin\phi\sin\theta,\\
\beta_3 &= \cos\phi\sin\theta,\\
\gamma_3 &= \cos\theta.
\end{aligned}\right\} \tag{144}$$

255.] Now of the body we have two angular velocities, of which one is absolute, and the other is relative to the moving system of axes. Let us resolve these along the principal axes at the time t; let ω_1', ω_2', ω_3' be the axial components of the former, and let ω_1, ω_2, ω_3 be those of the latter. The difference between them is evidently due to the angnlar velocity of the moving system: and consequently if we resolve this latter along the principal axes, we may equate each component to the corresponding excess of the absolute over the relative angular velocity. Thus we have

$$\left.\begin{aligned}
\omega_1' &= \omega_1 + \alpha_1\,\omega_\xi + \alpha_2\,\omega_\eta + \alpha_3\,\omega_\zeta,\\
\omega_2' &= \omega_2 + \beta_1\,\omega_\xi + \beta_2\,\omega_\eta + \beta_3\,\omega_\zeta,\\
\omega_3' &= \omega_3 + \gamma_1\,\omega_\xi + \gamma_2\,\omega_\eta + \gamma_3\,\omega_\zeta;
\end{aligned}\right\} \tag{145}$$

as ω_1, ω_2, ω_3, θ, ϕ, ψ are all employed relatively to the moving system of axes, they are connected by the equations given in Article 42; and we have

$$\left.\begin{aligned}
\omega_1 &= \cos\phi\,\frac{d\theta}{dt} + \sin\phi\sin\theta\,\frac{d\psi}{dt},\\
\omega_2 &= -\sin\phi\,\frac{d\theta}{dt} + \cos\phi\sin\theta\,\frac{d\psi}{dt},\\
\omega_3 &= \frac{d\phi}{dt} + \cos\theta\,\frac{d\psi}{dt}.
\end{aligned}\right\} \tag{146}$$

Now ω_1', ω_2', ω_3' depend on the constitution of the body, on its

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initial circumstances, and on the forces which act on it: they must therefore be determined from equations of motion, in terms of θ, ϕ, ψ, and t, and their values substituted in (145): hereby we shall have three equations in terms of θ, ϕ, ψ, and t, and their differentials: from these, by integration, θ, ϕ, and ψ may be expressed as functions of t, and the relative position of the body will be given.

Since ω_1', ω_2', ω_3' are the components along the principal axes of the absolute angular velocity, they may be determined by Euler's three equations of motion: and we have

$$\left.\begin{aligned} \text{A}\frac{d\omega_1'}{dt} + (\text{C}-\text{B})\,\omega_2'\omega_3' &= \text{L}, \\ \text{B}\frac{d\omega_2'}{dt} + (\text{A}-\text{C})\,\omega_3'\omega_1' &= \text{M}, \\ \text{C}\frac{d\omega_3'}{dt} + (\text{B}-\text{A})\,\omega_1'\omega_2' &= \text{N}; \end{aligned}\right\} \tag{147}$$

where A, B, C are the principal moments, and L, M, N are the moments of the couples of the whole impressed momentum-increments on the body.

256.] One first integral of the general equations of motion is the equation of relative vis viva. Let us investigate the form which it takes under the present circumstances of motion. For this purpose take the t-differentials of (143), bearing in mind that ξ', η', ζ' do not vary with the time: then

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \xi'\frac{d\alpha_1}{dt} + \eta'\frac{d\beta_1}{dt} + \zeta'\frac{d\gamma_1}{dt}, \\ \frac{d\eta}{dt} &= \xi'\frac{d\alpha_2}{dt} + \eta'\frac{d\beta_2}{dt} + \zeta'\frac{d\gamma_2}{dt}, \\ \frac{d\zeta}{dt} &= \xi'\frac{d\alpha_3}{dt} + \eta'\frac{d\beta_3}{dt} + \zeta'\frac{d\gamma_3}{dt}; \end{aligned}\right\} \tag{148}$$

squaring, adding them, and taking the sum of them for every element of the moving system,

$$\begin{aligned} \Sigma.m\frac{d\xi^2+d\eta^2+d\zeta^2}{dt^2} = {} & \Sigma.m\,\xi'^2\left\{\left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2 + \left(\frac{d\alpha_3}{dt}\right)^2\right\} \\ & + \Sigma.m\,\eta'^2\left\{\left(\frac{d\beta_1}{dt}\right)^2 + \left(\frac{d\beta_2}{dt}\right)^2 + \left(\frac{d\beta_3}{dt}\right)^2\right\} \\ & + \Sigma.m\,\zeta'^2\left\{\left(\frac{d\gamma_1}{dt}\right)^2 + \left(\frac{d\gamma_2}{dt}\right)^2 + \left(\frac{d\gamma_3}{dt}\right)^2\right\}; \end{aligned} \tag{149}$$

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for $\Sigma.m\eta'\zeta' = \Sigma.m\zeta'\xi' = \Sigma.m\xi'\eta' = 0$, because the coordinate axes, to which ξ', η', ζ' refer, are principal axes. The quantities in the right-hand members of these equations are subject to equivalences similar to those given in (80), Art. 39; so that

$$\Sigma.m\frac{d\xi^2+d\eta^2+d\zeta^2}{dt^2} = \mathrm{A}\omega_1^2+\mathrm{B}\omega_2^2+\mathrm{C}\omega_3^2; \qquad (150)$$

and thus the equation of relative *vis viva* given in (129) becomes

$$d.(\mathrm{A}\omega_1^2+\mathrm{B}\omega_2^2+\mathrm{C}\omega_3^2) = 2\Sigma.m\{(\mathrm{X}'-\mathrm{X}_t)d\xi+(\mathrm{Y}'-\mathrm{Y}_t)d\eta+(\mathrm{Z}'-\mathrm{Z}_t)d\zeta\}. \qquad (151)$$

If ω_1, ω_2, ω_3 are replaced by their values given in (146), the equation of vis viva will be expressed in terms of θ, ϕ, and ψ.

One other remark may be made as to another method which we might have chosen. We might have substituted the values given in (148) in the general equations of moments given in (122), (123), and (124); and these might have been transformed into their equivalents in terms of angular velocities: we should hereby have found equations analogous to Euler's, but having more terms, on account of the compound centrifugal forces and the forces of transference. The student will find them in a form equivalent to that which we should have arrived at in M. Quet's Memoir in Liouville's Journal, Vol. XVIII. I have however chosen the present method, because it is more direct, and because our processes are simplified by the use of Euler's equations, which have already been investigated.

257.] I propose now to apply these equations to some problems of considerable interest. Suppose a body at or near to the earth's surface to rotate about an axis which passes through a fixed point, but which is otherwise unconstrained, the apparent relative motion to an observer will doubtless be affected by the diurnal rotation of the earth. It is this effect which I propose to investigate.

We shall take the fixed point to be the origin of the systems of axes, and shall suppose the moving system of axes to be fixed relatively to the earth, so that the angular velocity of this system is due to the diurnal rotation alone.

Let us moreover suppose a plane (for the present taken arbitrarily, so as to admit of subsequent determination) passing through the fixed point to be that of (ξ, η): this plane is of course fixed to and moves with the earth. Through the fixed

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point let a straight line be drawn parallel to the earth's polar axis; and let it be projected on the plane of (ξ, η): this line we shall take to be the ξ-axis, and reckon it positive in such a way that when the plane is horizontal that direction shall be southwards; and the positive direction of the η-axis we shall take to be such that that direction may be westwards when the plane is horizontal: hereby, if the plane of (ξ, η) is horizontal at the place of observation, we shall have the same arrangement and the same system of axes as in Fig. 63. Let the positive direction of $o\zeta$ be so taken as to be *away from* the earth's surface when the plane of (ξ, η) is horizontal; and let $\nu =$ the angle at which the ζ-axis is inclined to the earth's polar axis; so that when the ζ-axis is vertical, ν is the co-latitude of the place of observation.

Let ω, as heretofore, be the diurnal angular velocity of the earth. Then, taking account of its direction,

$$\omega_\xi = \omega \sin \nu, \quad \omega_\eta = 0, \quad \omega_\zeta = -\omega \cos \nu; \tag{152}$$

and let us suppose the dimensions of the body to be such that the force of gravity is the same for all parts of it: and let M (for the instant) be the mass of the body, then $\text{X}' - \text{X}_t$, $\text{Y}' - \text{Y}_t$, $\text{Z}' - \text{Z}_t$ are constant for all parts of the body; and we may abbreviate the expressions and denote them thus:

$$\text{X}' - \text{X}_t = \text{E}, \quad \text{Y}' - \text{Y}_t = \text{F}, \quad \text{Z}' - \text{Z}_t = \text{G}; \tag{153}$$

and if $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ is the place of the centre of gravity relatively to the moving axes, then

$$\Sigma . m\{(\text{X}' - \text{X}_t)d\xi + (\text{Y}' - \text{Y}_t)d\eta + (\text{Z}' - \text{Z}_t)d\zeta\} = \text{M}\{\text{E}d\bar{\xi} + \text{F}d\bar{\eta} + \text{G}d\bar{\zeta}\}; \tag{154}$$

which may be substituted in the right-hand member of (151); and this being integrated will give the value of the relative vis viva.

258.] We will take however a special case of motion of this kind; and consider that of the gyroscope of M. Foucault, of which a diagram has been given in Fig. 21, and of which the construction and arrangement have been described in Art. 95. The centre of gravity of the whole machine, which coincides with those of the several parts of it, is at the centre of the rotating disc; and this point remains at relative rest, whatever are the rotations of the disc and of the metallic circles. At this point therefore the systems of axes originate.

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Now M. Foucault contrived in some of his experiments that the axis of the disc should be constrained to move in a given plane, fixed relatively to the earth: we will in the first place consider this case, and investigate the phænomena which the machine presents to an observer moving with the earth. The problem in its dynamical form is this:

A heavy body of revolution rotates rapidly about its axis of figure; its centre of gravity is fixed relatively to the earth, and the rotation-axis of the body can move only in a plane, which is likewise fixed relatively to the earth: it is required to determine the motion of this moving axis when regard is had to the diurnal rotation of the earth.

We will take the fixed plane, in which only the rotation-axis of the body can move, to be the plane of (ξ, η); so that ν is the angle at which this plane is inclined to that of the terrestrial equator. As the rotating disc is a solid of revolution, $A = B$; and if the axis of the disc is that of ζ', C is the principal moment relative to it, and we will suppose it to be greater than A. Also, as the rotation-axis is in the plane of (ξ, η), $\theta = 90°$.

259.] We must in the first place investigate the values of L, M, N which occur in (147), and are the moments of the impressed couples acting on the body relative to its principal axes, and to which the absolute angular velocity-increments are due. Now these couples are the same whether the body is or is not at rest relatively to the earth. We may therefore investigate the values which they have when the body is at relative rest; that is, when its absolute angular velocities are only those which are due to its attachment to the revolving earth. Suppose therefore the body to be carried round with the earth, and its ζ'-axis to be inclined at an angle θ to the terrestrial axis, then θ is constant; let $\omega_1, \omega_2, \omega_3$ be the angular velocities of the body about its principal axes; then,

$$\left.\begin{aligned} \omega_3 &= \omega \cos\theta, \\ (\omega_1^2 + \omega_2^2)^{\frac{1}{2}} &= \omega \sin\theta. \end{aligned}\right\} \qquad (155)$$

Euler's equations, which connect these quantities with L, M and N, become in this case,

$$\left.\begin{aligned} A\frac{d\omega_1}{dt} + (C - A)\,\omega_2\,\omega\cos\theta &= L, \\ A\frac{d\omega_2}{dt} - (C - A)\,\omega_1\,\omega\cos\theta &= M, \\ 0 &= N. \end{aligned}\right\}$$

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Also we have

$$\omega_1 = \sin\phi \sin\theta \frac{d\psi}{dt}, \quad \omega_2 = \cos\phi \sin\theta \frac{d\psi}{dt}, \quad \omega_3 = \frac{d\phi}{dt} + \cos\theta \frac{d\psi}{dt}; \quad (156)$$

$$\therefore \quad \omega_1^2 + \omega_2^2 = (\sin\theta)^2 \left(\frac{d\psi}{dt}\right)^2;$$

and therefore, from the second of (155), $\frac{d\psi}{dt} = \omega$; consequently, from the last of (156),

$$\frac{d\phi}{dt} = 0;$$

$$\because \quad \phi = \text{a constant}$$
$$= a \text{ (say)};$$

$$\therefore \quad \omega_1 = \omega \sin a \sin\theta, \qquad \omega_2 = \omega \cos a \sin\theta;$$

$$\therefore \quad \left.\begin{aligned} \text{L} &= \omega^2 (\text{C} - \text{A}) \cos a \sin\theta \cos\theta, \\ \text{M} &= -\omega^2 (\text{C} - \text{A}) \sin a \sin\theta \cos\theta; \end{aligned}\right\}$$

and these quantities are the right hand members of the first two equations of (147). But ω^2 is a factor of both; and our approximations are carried to such an extent that the squares and higher powers of ω are omitted. Therefore $\text{L} = \text{M} = 0$; and $\text{N} = 0$. Thus (147) become

$$\left.\begin{aligned} \text{A} \frac{d\omega_1'}{dt} + (\text{C} - \text{A}) \omega_2' \omega_3' &= 0, \\ \text{A} \frac{d\omega_2'}{dt} - (\text{C} - \text{A}) \omega_3' \omega_1' &= 0, \\ \text{C} \frac{d\omega_3'}{dt} \qquad\qquad\qquad &= 0; \end{aligned}\right\} \quad (157)$$

from the last of these equations

$$\omega_3' = \text{a constant}$$
$$= n \text{ (say)}; \quad (158)$$

the value of n will be perceived hereafter. Hence the first two of (157) become

$$\left.\begin{aligned} \frac{d\omega_1'}{dt} + \frac{\text{C} - \text{A}}{\text{A}} n \omega_2' &= 0, \\ \frac{d\omega_2'}{dt} - \frac{\text{C} - \text{A}}{\text{A}} n \omega_1' &= 0. \end{aligned}\right\} \quad (159)$$

Let $$\frac{\text{C} - \text{A}}{\text{A}} n = \mu;$$

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then integrating (159), we have

$$\left.\begin{aligned}\omega_1' &= \kappa\cos(\mu t-\tau),\\ \omega_2' &= \kappa\sin(\mu t-\tau);\end{aligned}\right\} \tag{160}$$

where τ and κ are constants introduced in integration, depending on the initial circumstances; as the former is determined by the commencement of the time, and the latter is the greatest value of ω_1' and ω_2': these we shall also explain hereafter.

Since $\theta=90°$ and is constant, $\frac{d\theta}{dt}=0$; therefore, from (146), we have

$$\omega_1 = \sin\phi\frac{d\psi}{dt}, \qquad \omega_2 = \cos\phi\frac{d\psi}{dt}, \qquad \omega_3 = \frac{d\phi}{dt}; \tag{161}$$

and from (144) we have

$$\left.\begin{aligned} \alpha_1 &= \cos\phi\cos\psi, & \alpha_2 &= \cos\phi\sin\psi, & \alpha_3 &= \sin\phi,\\ \beta_1 &= -\sin\phi\cos\psi, & \beta_2 &= -\sin\phi\sin\psi, & \beta_3 &= \cos\phi,\\ \gamma_1 &= \sin\psi; & \gamma_2 &= -\cos\psi; & \gamma_3 &= 0.\end{aligned}\right\}$$

Thus, if we substitute for ω_ξ, ω_η, ω_ζ from (152), (145) become

$$\left.\begin{aligned}\omega_1 &= \kappa\cos(\mu t-\tau)-\omega\{\sin\nu\cos\phi\cos\psi-\cos\nu\sin\phi\},\\ \omega_2 &= \kappa\sin(\mu t-\tau)+\omega\{\sin\nu\sin\phi\cos\psi+\cos\nu\cos\phi\},\\ \omega_3 &= n-\omega\sin\nu\sin\psi;\end{aligned}\right\} \tag{162}$$

and replacing ω_1, ω_2, ω_3 by their values given in (161), we have three equations, by means of which θ, ϕ, and ψ may be expressed as functions of t.

The equation of vis viva however will give a condition more useful for our present purpose. Equating the values given in (151) and (154), we have

$$d.\{\text{A}\,\omega_1^2+\text{B}\,\omega_2^2+\text{C}\,\omega_3^2\} = 2\,\text{M}\,\{\text{E}\,d\bar{\xi}+\text{F}\,d\bar{\eta}+\text{G}\,d\bar{\zeta}\};$$

but the centre of gravity $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ is fixed at the origin; therefore

$$d\bar{\xi} = d\bar{\eta} = d\bar{\zeta} = 0;$$

and replacing ω_1, ω_2, ω_3 by their values given in (161), we have

$$\begin{aligned}\text{A}\left(\frac{d\psi}{dt}\right)^2+\text{C}\,\omega_3^2 &= \text{a constant}\\ &= k^2\,(\text{say}).\end{aligned} \tag{163}$$

Let us consider the meaning of the constants, which have been introduced; and suppose Ω to be the relative velocity communicated to the body, when $t=0$; so that $\omega_3=\Omega$, and $\omega_1=\omega_2=0$, when $t=0$; therefore at that time, from (161), $\frac{d\psi}{dt}=0$; as the position of the principal axes of ξ' and η' in the plane of

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(ξ', η') is arbitrary, let us take that of ξ' to be in the plane of (ξ, η), when $t = 0$; and at that time also let $\psi = \psi_0$; then the preceding equations give

$$\left.\begin{aligned} 0 &= \kappa \cos \tau - \omega \sin \nu \cos \psi_0, \\ 0 &= \kappa \sin \tau - \omega \cos \nu, \\ \Omega &= n - \omega \sin \nu \sin \psi_0, \\ \mathrm{C}\,\Omega^2 &= k^2: \end{aligned}\right\} \tag{164}$$

whence we have

$$\left.\begin{aligned} \kappa^2 &= \omega^2 \{1 - (\sin \nu)^2 (\sin \psi_0)^2\}, \\ \tan \tau &= \cot \nu \sec \psi_0, \\ n &= \Omega + \omega \sin \nu \sin \psi_0, \\ k &= \Omega\, \mathrm{C}^{\frac{1}{2}}; \end{aligned}\right\} \tag{165}$$

so that from the last of (162,) and from (163), we have

$$\omega_3 - \Omega = \omega \sin \nu \{\sin \psi_0 - \sin \psi\}; \tag{166}$$

$$\mathrm{A}\left(\frac{d\psi}{dt}\right)^2 = \mathrm{C}\,(\Omega^2 - \omega_3^2). \tag{167}$$

Since $\omega \sin \nu$ is the component along the axis of ξ of the angular velocity of the earth, $\omega \sin \nu \sin \psi$ is the component of that angular velocity along the axis of ζ'; and $\omega \cos \nu$, which is the other component of the earth's velocity, has no effect along that axis, because its axis is the axis of ζ: thus (166) shews that the sum of the apparent angular velocity of the body about its own axis, and of the component of the earth's angular velocity about the same line, is constant throughout the motion.

Equation (167) shews that the apparent vis viva of the body is also constant.

Eliminating ω_3 between (166) and (167) we have

$$\mathrm{A}\left(\frac{d\psi}{dt}\right)^2 = \mathrm{C}\omega \sin \nu \{\sin \psi - \sin \psi_0\} \{2\Omega - \omega(\sin \psi - \sin \psi_0)\}; \tag{168}$$

from which equation ψ is to be found in terms of t. The equation however does not admit of integration in its present complete form. But in all experiments with the gyroscope a very rapid rotation is given to the body, so that Ω is very great in comparison of ω: we may therefore omit the last part of the last factor in the right hand member of (168), and employ the approximate equation

$$\mathrm{A}\left(\frac{d\psi}{dt}\right)^2 = 2\,\mathrm{C}\,\Omega\,\omega \sin \nu (\sin \psi - \sin \psi_0). \tag{169}$$

It will be convenient to make a slight change in the form of

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this equation. ψ is the angle measured in the plane of (ξ, η) between the axis of ξ and the line of intersection of the planes of (ξ, η) and (ξ', η'). The axis of ζ' is also in the plane of (ξ, η) and is perpendicular to this line of intersection: if therefore v is the angle between the axes of ξ and ζ',

$$\psi = v + 90^\circ; \tag{170}$$

and substituting v in (169), we have

$$A\left(\frac{dv}{dt}\right)^2 = 2\,C\,\Omega\,\omega \sin v\,(\cos v - \cos v_0). \tag{171}$$

Now this equation is the same in form as (40), Art. 131, which expresses the motion of a pendulum under the action of a constant force whose line of action is parallel to that of the line of the rod of the pendulum when it is at rest. Let us then compare the motion of the rotation-axis of the body as expressed by (171) with that of the pendulum; and let us assume the length of the pendulum to be unity; so that the constant force, under the action of which the rotation-axis may be supposed to move, is $\frac{C\,\Omega\,\omega \sin v}{A}$, the line of which is the ξ-axis, and is the projection southwards on the plane of (ξ, η) of the earth's polar axis. Now the pendulum vibrates through small arcs to equal distances on either side of the vertical line; and so will the rotation-axis of the disc vibrate over equal small angles on either side of the ξ-axis. And as the pendulum remains always at rest in a vertical line, if it is ever at rest in it; so will the rotation-axis always be at relative rest along the ξ-axis, if it is ever at rest in it. If therefore the rotation-axis is on the ξ-axis when $\frac{d\psi}{dt} = 0$, it always remains on it, and has no oscillation.

Also, as the pendulum has two positions of rest, one of stable equilibrium, when it hangs vertically downwards from its point of suspension, and another when it is balanced on its point of suspension, the centre of gravity having its lowest and its highest position in the two cases respectively; so are there two positions of relative rest of the rotation-axis of the disc, one of which is of stable, and the other of unstable rest. Now if ω and Ω have the signs given to them in (171); that is, if the direction of Ω is contrary to that of the earth, then the rest of the rotation-axis will be stable or unstable, according as the axis coincides with the positive or negative direction of the ξ-axis. And the contrary

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will be the case, when the direction of Ω is the same as that of the earth.

If the rotation-axis is in its position of stable rest, and is slightly disturbed therefrom by an extraneous force, it oscillates in the plane of (ξ, η) over a small angle on either side of the line of rest; and if τ is the time of an oscillation,

$$\tau = \pi \left\{ \frac{A}{C\,\Omega\,\omega \sin \nu} \right\}^{\frac{1}{2}}. \tag{172}$$

If however the rotation-axis is in its position of unstable rest, and is slightly displaced therefrom, it goes farther from that position and does not return to it, until it has passed through 360° in its plane of motion.

260.] In particular cases these results take forms which are of considerable interest.

(1) Let the plane of (ξ, η), in which the rotation-axis of the body is constrained to move, be horizontal at the place of observation: and let the latitude of the place be λ; then $\sin \nu = \cos \lambda$; and from (172),

$$\tau = \pi \left\{ \frac{A}{C\,\Omega\,\omega \cos \lambda} \right\}^{\frac{1}{2}}. \tag{173}$$

In this case the ξ-axis is the meridian line, of which the positive direction is that towards the south. If the direction of the rotation of the body is the same as that of the earth, the position of rest of the rotation-axis will be of stable or unstable equilibrium according as it is drawn from the point towards the north or towards the south; and if the rotation is contrary to that of the earth, the rotation-axis will be in stable or unstable rest according as its direction is due south or due north.

(2) Let the plane of (ξ, η), in which the rotation-axis is constrained to move, be the meridian plane at the place of observation; then $\nu = 90°$; and the line of relative equilibrium of the rotation-axis is parallel to the earth's polar axis; and the equilibrium of the axis is stable or unstable according as the direction of rotation is contrary to, or is the same as, that of the earth. In this case

$$\tau = \pi \left\{ \frac{A}{C\,\Omega\,\omega} \right\}^{\frac{1}{2}}; \tag{174}$$

so that, cæteris paribus, the time of oscillation is less in this case than it is when the rotation-axis moves in the horizontal

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plane; and generally the oscillations in the meridian plane are quicker than in any other plane.

(3) These last results however are not limited to the meridian plane; for $\sin \nu = 1$ for all planes drawn through the place of observation and parallel to the earth's polar axis.

(4) If the plane of (ξ, η) is perpendicular to the earth's polar axis, $\nu = 0$; and $\tau = \infty$; so that the rotation-axis of the solid is at rest for all positions in that plane.

(5) If the number of oscillations of the rotation-axis in the meridian plane is determined by observation, τ is known; and consequently, from (174),

$$\omega = \frac{\pi^2 A}{C \Omega \tau^2}; \tag{175}$$

and thus the angular velocity of the earth may be determined.

(6) If τ and τ' are the times of oscillation of the rotation-axis in the horizontal and the meridian planes respectively at a given place, corresponding to the same value of Ω, then

$$\cos \lambda = \frac{\tau'^2}{\tau^2}.$$

From all these theorems we conclude, that if the phænomena of the gyroscope are observed with sufficient care, we can by them determine the meridian line and the altitude of the pole at the place; and consequently the latitude: we can determine also the direction of the diurnal rotation of the earth, and, from (175), the mean length of the sidereal day. All these results then are confirmations, if they are required, of the evidence of that motion of the earth which astronomical phænomena suggest to us. And although the proof of the diurnal rotation, thus acquired, may not be as palpable as that afforded by astronomical observation, yet it is not to be rejected as useless, nor is its investigation to be regarded as idle speculation; for evidence supporting theories of cosmical phænomena is cumulative; and the value of any addition to it increases in geometrical ratio.

261.] The gyroscope again may be so arranged that the rotation-axis of the disc shall be constrained to move in a right circular cone, whose axis passes through the fixed point which is the centre of gravity of the disc and rings. If we take this axis for the ζ-axis, θ = the semi-vertical angle of the cone $= \alpha$ (say); and thus the equations which determine the motion of the instrument become

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$$\omega_1 = \sin a \sin\phi \frac{d\psi}{dt}, \quad \omega_2 = \sin a \cos\phi \frac{d\psi}{dt}, \quad \omega_3 = \frac{d\phi}{dt} + \cos a \frac{d\psi}{dt}; \quad (176)$$

so that the equation of vis viva becomes

$$\text{A} (\sin a)^2 \left(\frac{d\psi}{dt}\right)^2 + \text{C} (\omega_3^2 - \Omega_3^2) = 0; \quad (177)$$

and since $\gamma_1 = \sin\psi \sin a$, $\gamma_2 = -\cos\psi \sin a$, $\gamma_3 = \cos a$, from the third of (145), we have

$$\omega_3 - \Omega + \omega \sin\nu \sin a (\sin\psi - \sin\psi_0) = 0; \quad (178)$$

from which, and from (177), results may be deduced similar to those of the last Article, those indeed being only the particular forms which these take, when $a = 90°$. Thus the time of small oscillation of the rotation-axis of the disc

$$= \pi \left\{ \frac{\text{A} \sin a}{\text{C}\, \Omega\, \omega \sin\nu} \right\}^{\frac{1}{2}}. \quad (179)$$

262.] Finally, let us take the most general case which the gyroscope presents to us, and suppose the axis of the disc to be free from all constraint. Then the problem is, to determine the phænomena which, by reason of the earth's diurnal rotation, the motion of this rotation-axis exhibits to an observer placed on the earth.

I shall suppose a very rapid angular velocity Ω to be given to the disc, and the axis to be placed at the beginning of the time in relative rest. Let the centre of gravity of the disc be the origin of the systems of moving axes and of the principal axes of the disc. Let the line drawn through that point, and parallel to the earth's polar axis, be the ζ-axis, the positive direction being that towards the north; so that in this case, $\nu = 0$; and

$$\omega_\xi = \omega_\eta = 0; \qquad \omega_\zeta = -\omega; \quad (180)$$

then the plane of (ξ, η) is parallel to that of the earth's equator. At the beginning of the time, when the relative angular velocity of the disc about its own axis is Ω, and the rotation-axis is at relative rest, let θ_0 be the angle at which the rotation-axis, or the ζ'-axis, is inclined to the ζ-axis; and let the line of intersection of the planes of (ξ, η) and of (ξ', η') be the ξ- and the ξ'-axes; so that $\phi = \psi = 0$, when $t = 0$; at which time also $\frac{d\theta}{dt} = \frac{d\psi}{dt} = 0$.

Now as equations which have already been found are sufficient for the solution of this problem, I shall only refer to them;

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and deduce from them the results which we require. The complete values of the nine direction-cosines given in (144) are true in this case: and (145) become

$$\left.\begin{aligned} \omega_1 &= \omega_1' + \omega \sin\theta \sin\phi, \\ \omega_2 &= \omega_2' + \omega \sin\theta \cos\phi, \\ \omega_3 &= \omega_3' + \omega \cos\theta; \end{aligned}\right\} \tag{181}$$

(146) remain in the general form which they have in that place.

From (147) we deduce the results (158) and (160); viz.

$$\left.\begin{aligned} \omega_1' &= \kappa \cos(\mu t - \tau), \\ \omega_2' &= \kappa \sin(\mu t - \tau), \\ \omega_3' &= n; \end{aligned}\right\} \tag{182}$$

since however when $t = 0$, $\omega_1 = \omega_2 = 0$; $\omega_3 = \Omega$, and $\theta = \theta_0$; therefore from these and from (181) we have

$$\left.\begin{aligned} \omega_1' &= \omega \sin\theta_0 \sin\mu t, \\ \omega_2' &= -\omega \sin\theta_0 \cos\mu t; \end{aligned}\right\} \tag{183}$$

and consequently,

$$\left.\begin{aligned} \omega_1 &= \omega \sin\theta_0 \sin\mu t + \omega \sin\theta \sin\phi, \\ \omega_2 &= -\omega \sin\theta_0 \cos\mu t + \omega \sin\theta \cos\phi, \\ \omega_3 &= \Omega + \omega(\cos\theta - \cos\theta_0). \end{aligned}\right\} \tag{184}$$

The equation of relative vis viva gives

$$\text{A}\left\{\left(\frac{d\theta}{dt}\right)^2 + (\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2\right\} + \text{C}(\omega_3^2 - \Omega^2) = 0. \tag{185}$$

Again from (184) we have

$$\begin{aligned} \omega^2(\sin\theta_0)^2 &= (\omega_1 - \omega\sin\theta\sin\phi)^2 + (\omega_2 - \omega\sin\theta\cos\phi)^2 \\ &= \omega_1^2 + \omega_2^2 - 2\omega\sin\theta(\omega_1\sin\phi + \omega_2\cos\phi) + \omega^2(\sin\theta)^2 \\ &= \omega_1^2 + \omega_2^2 - 2\omega(\sin\theta)^2\frac{d\psi}{dt} + \omega^2(\sin\theta)^2; \end{aligned}$$

therefore

$$\omega_1^2 + \omega_2^2 = \omega^2\{(\sin\theta_0)^2 - (\sin\theta)^2\} + 2\omega(\sin\theta)^2\frac{d\psi}{dt},$$

and $\quad \omega_3^2 = \{\Omega + \omega(\cos\theta - \cos\theta_0)\}^2;$

therefore

$$\begin{aligned} \text{A}(\omega_1^2 + \omega_2^2) + \text{C}\,\omega_3^2 &= \text{A}\,\omega^2\{(\sin\theta_0)^2 - (\sin\theta)^2\} + 2\text{A}\,\omega(\sin\theta)^2\frac{d\psi}{dt} \\ &\quad + \text{C}\,\Omega^2 + 2\text{C}\,\Omega\,\omega(\cos\theta - \cos\theta_0) + \text{C}\,\omega^2(\cos\theta - \cos\theta_0)^2; \end{aligned} \tag{186}$$

but the left-hand member of this equation is by the equation of vis viva equal to $\text{C}\,\Omega^2$: consequently

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$$\text{A}(\sin\theta)^2\frac{d\psi}{dt}$$

$$= \frac{\cos\theta_0-\cos\theta}{2}\{2\text{C}\Omega+\text{A}\omega(\cos\theta+\cos\theta_0)+\text{C}\omega(\cos\theta-\cos\theta_0)^2\}; \quad (187)$$

eliminating $\frac{d\psi}{dt}$ between this equation and (185), replacing ω_3 by its value given in (184), and omitting terms involving ω^2, we have

$$\text{A}^2(\sin\theta)^2\left(\frac{d\theta}{dt}\right)^2 = \text{C}\Omega(\cos\theta_0-\cos\theta)\{\text{A}\omega(\sin\theta)^2+\text{A}\omega(\sin\theta_0)^2$$
$$+\text{C}\Omega(\cos\theta-\cos\theta_0)+\text{C}\omega(\cos\theta-\cos\theta_0)^2\}; \quad (188)$$

which equation determines θ in terms of t; and when this value of θ is substituted in (187), that gives ψ in terms of t.

263.] If the gyroscope does not rotate when $t=0$, that is, if $\Omega=0$, then $\frac{d\theta}{dt}=0$; so that θ does not vary, and is constantly equal to θ_0: therefore from (187), $\frac{d\psi}{dt}=0$; hence also $\omega_3=0$: so that notwithstanding the diurnal rotation of the earth, the axis of the disc remains at relative rest.

If however Ω is very great, and this is the ordinary case in the gyroscope, the most important term in the last factor of the right-hand member of (188) is $\text{C}\Omega(\cos\theta-\cos\theta_0)$, as the other terms involve ω, which is a small quantity: but these must not be wholly neglected, because such an omission is equivalent to the omission of ω, which is the angular velocity of the earth; and because in such a case the only circumstances which would satisfy (188) are $\frac{d\theta}{dt}=0$, $\theta=\theta_0$; and the axis of the gyroscope would be at relative rest. Of the small terms, however, we may without sensible error neglect those which have periodical values, and retain those which are constant; for we shall hereby obtain the general effect. Thus (188) becomes

$$\text{A}^2(\sin\theta)^2\left(\frac{d\theta}{dt}\right)^2 = \text{C}\Omega(\cos\theta_0-\cos\theta)\{\text{C}\Omega(\cos\theta-\cos\theta_0)+\text{A}\omega(\sin\theta_0)^2+\text{C}\omega(\cos\theta_0)^2\}$$

$$= \text{C}^2\Omega^2(\cos\theta_0-\cos\theta)\left\{\cos\theta-\cos\theta_0+\frac{\text{A}(\sin\theta_0)^2+\text{C}(\cos\theta_0)^2}{\text{C}}\frac{\omega}{\Omega}\right\}. \quad (189)$$

Now let us suppose a to be an angle such that

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$$\cos a = \cos\theta_0 - \frac{A(\sin\theta_0)^2 + C(\cos\theta_0)^2}{C}\frac{\omega}{\Omega}; \qquad (190)$$

which equation is possible, if $\frac{\omega}{\Omega}$ is positive, for all values of θ_0 between 0 and a limit a little less than 180°; and if $\frac{\omega}{\Omega}$ is negative for all values of θ_0 between 180° and a limit a little greater than 0. Thus (189) becomes

$$A^2(\sin\theta)^2\left(\frac{d\theta}{dt}\right)^2 = C^2\Omega^2(\cos\theta_0 - \cos\theta)(\cos\theta - \cos a); \qquad (191)$$

so that θ always lies between θ_0 and a, which, as (190) shews, are two angles nearly equal: and consequently the inclination of the rotation-axis of the disc to the ζ-axis is not constant, but varies within limits very near to each other.

From (191) by integration we have

$$\cos\theta = \frac{\cos\theta_0 + \cos a}{2} + \frac{\cos\theta_0 - \cos a}{2}\cos\frac{C\Omega}{A}t; \qquad (192)$$

and replacing a by its value

$$\cos\theta = \cos\theta_0 - \frac{A(\sin\theta_0)^2 + C(\cos\theta_0)^2}{2C}\frac{\omega}{\Omega}\left\{1 - \cos\frac{C\Omega}{A}t\right\}. \qquad (193)$$

Let us introduce the preceding value of $\cos\theta$ into (187): then omitting terms involving the square and higher power of ω,

$$\frac{d\psi}{dt} = \frac{A(\sin\theta_0)^2 + C(\cos\theta_0)^2}{2A(\sin\theta_0)^2}\omega\left\{1 - \cos\frac{C\Omega}{A}t\right\},$$

$$\therefore\quad \psi = \frac{A(\sin\theta_0)^2 + C(\cos\theta_0)^2}{2A(\sin\theta_0)^2}\left\{\omega t - \frac{A\omega}{C\Omega}\sin\frac{C\Omega}{A}t\right\}; \qquad (194)$$

the limits of integration being such that $\psi = 0$ when $t = 0$.

If then we consider the first terms in (193) and (194), which are the principal terms, it appears that the rotation-axis of the disc revolves uniformly round an axis parallel to the earth's axis in a direction contrary to that of the earth, and that it is inclined to this axis at an angle almost constant: and besides this general precessional motion the axis has also motion of nutation both parallel to and perpendicular to the plane of the earth's equator; and that the periodic time of these nutations $= \frac{2\pi A}{C\Omega}$; so that the periodic time is shorter the greater is the initial angular velocity of the disc.

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264.] From this investigation then the following results follow:

(1) If the disc of the gyroscope has not any initial angular velocity, it remains at relative rest with the earth, whether the earth rotates or not.

(2) If the disc rotates with a very rapid angular velocity, and is placed in a position of relative equilibrium with the earth, then that equilibrium would continue if the earth did not rotate; but if the earth rotates, the axis of disc has a relative motion.

(3) And the direction in which this motion takes place does not depend on that of the angular velocity of the disc, but is always opposite to that of the angular velocity of the earth; and consequently, if it is observed, it indicates the direction in which the earth rotates; and thus its motion supplies evidence of the rotation of the earth.

(4) The angle of inclination of the rotation-axis of the disc to the axis of the earth is nearly constant throughout the motion of the disc: there are however small nutational variations of this angle as well as of the precessional velocity of the axis, the periodic time of which decreases according as the angular velocity of the disc increases.

Here I must conclude this subject; for it would be quite beyond the scope of the present work to enter further into the details of it. Let me however refer the student to the Memoir of M. Quet, already alluded to, which is contained in Vol. XVIII of Liouville's Journal, and to which I have been largely indebted for the preceding Articles. This memoir also contains an investigation of the motion of the several rings of the gyroscope; as well as a further inquiry into the interpretation of the several equations, when small terms, which we have omitted, are taken account of.

I may also refer the reader to another mathematical investigation by M. Yvon Villarceau, which will be found in Vol. XIV, p. 343, Nouvelles Annales des Mathématiques; Paris, 1855.

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CHAPTER IX.

ON THE THEORY OF MACHINES IN MOTION.

265.] In Section 4, Chapter III, of the present Volume, a combination of the principle of virtual velocities and of D'Alembert's principle led to the general equation of vis viva of a system of moving particles; of which the form is

$$\Sigma . m v^2 - \Sigma . m v_0^2 = 2 \Sigma . \int m (\mathrm{X}\, dx + \mathrm{Y}\, dy + \mathrm{Z}\, dz); \qquad (1)$$

the right-hand member being a definite integral, and the limits of integration being the values of the several quantities at the times t and t_0 respectively, v and v_0 being the corresponding values of v at those times. For the truth of this theorem it is necessary that neither the connections between the several particles of the system, nor the acting forces, which produce the impressed increments in the right-hand member of (1), should be explicit functions of the time t. It is also necessary that $\Sigma . m (\mathrm{X}\, dx + \mathrm{Y}\, dy + \mathrm{Z}\, dz)$ should be an exact differential; and in that section several cases are given in which these conditions are satisfied.

Now the geometrical relations of the several parts of a machine (I use this word in its ordinary meaning) and the forces which act on a machine satisfy all these conditions; so that the preceding equation is applicable to them. I propose to apply it to the theory of their motion; for this is our immediate subject of inquiry. We shall hereby ascertain not only general properties of them, but also measures of their effects and criteria of their goodness or efficiency. Mathematical and precise definitions will be given of these terms; and we shall thereby be able to reduce their values to measurement and to number; and this is in such cases the end of all exact scientific inquiry.

It will be convenient however to put (1) into a slightly different form. The right-hand member of (1) consists of a series of groups of terms which arise from an equal number of different acting forces. Let F = that *moving force* of which $m\mathrm{X}$, $m\mathrm{Y}$, $m\mathrm{Z}$

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are the axial components; and let ds be the projection on the line of action of F of the distance through which its point of application moves in the time dt; so that

$$m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz) = \text{F}\,ds. \tag{2}$$

Let this transformation be made for all the acting forces; then the equation of vis viva becomes

$$\Sigma.mv^2 - \Sigma.mv_0^2 = 2\Sigma.\int \text{F}\,ds; \tag{3}$$

the right-hand member being a definite integral with limits corresponding to the times t and t_0; and F being the *momentum* impressed on the machine. It is this equation which we shall apply to the theory of machines in motion.

266.] A machine is an instrument by which the momentum arising from certain acting forces, and applied at one or more definite points in given lines of action, is transmitted to other points, and is at them and along given lines communicated to other matter.

A machine generally consists of many pieces, which are connected by articulations of various kinds, by axles or shafting common to two or more wheels, by sliding and rolling contact, &c.; the connections being such, that when one piece moves by the action of a force, many or all the other pieces also move.

The point at which the momentum of a moving force, or *power* as it is popularly called, is applied, is called the *driving* point: and the point at which the transmitted momentum is applied is called the *working* point of the machine; and the series of pieces which connect these two points is called the *train*.

Let the object of a machine be clearly understood. It is to enable us conveniently to apply at a certain point in a definite line of action and in a certain way momentum which arises from a certain power or moving force applied elsewhere. Thus it transmits momentum. It neither generates it nor destroys it. The action of it is in complete accordance with the law of inertia. All the momentum which is communicated to it either has been or may be abstracted from it. Let us now consider the mode in which (3) enables us to trace the relation between the momentum communicated to a machine and that which it either gives or is capable of giving at its working point. The sign of summation in the left-hand member of (3) includes all particles to

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which velocity is communicated by the action of the moving forces; and thus includes not only the moving parts in the train of the machine, but also the fixed framework, the supports, the ground or base on which it rests, and even the particles of the surrounding air, if so be that velocity of vibration or any other velocity is communicated to all or any of them. And the left-hand member expresses the excess of the vis viva of all the moving particles at the time t over the vis viva at the time t_0; that is, it is the increment of the vis viva of the whole system in the time $t-t_0$.

The right-hand member, viz. $2\,\text{Z}.\int \text{F}\,ds$, expresses twice the sum of all the definite integrals of the products of each impressed momentum-increment (F), and the space (ds) through which its point of application moves along its own line of action in the time dt; the limits of integration being the values of these several quantities at the times t and t_0 respectively. This product $\text{F}\,ds$ is called the *element of work* due to the force F; and the definite integral $\int \text{F}\,ds$, which is the sum of these elements, is called the *whole work* of the force in the time $t-t_0$; and in reference to this impressed work the force is called *labouring* force.

267.] Of these terms and definitions I will take some simple instances, and shew how exactly they coincide with our ordinary notion of *work*, which involves resistance overcome and space described.

Let us suppose the force to be the earth's attraction, acting on a mass m, of which W is the weight, or the impressed moving force. Let us suppose this weight to move from rest in a vertical line towards the earth, and ds to be the element of its path in the time dt: then $\text{W}\,ds$ is the element of work in the time dt. Let t be the time during which the body falls, and let h and z be the vertical distances from the earth's surface when $t=0$ and $t=t$ respectively.

The work in the time $t = \int \text{W}\,ds$

$$= \text{W}(h-z); \tag{4}$$

and is equal to the product of the weight and the vertical distance through which it has fallen. This then is the work which has been impressed by the labouring force of the earth's attrac-

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tion in the time t; and hence an equal amount of labouring force must act, and an equal amount of *work* must be spent on w, so as to put it into its original place. This work in a given time is measured by the product of the weight moved and the vertical distance through which it is moved. Work is imparted to a heavy body as it is removed further from the earth's surface, and is taken from it as the body moves nearer to the earth. Consequently, a heavy body has a greater amount of work in it at a greater altitude than it has at a less. The absolute amount of work too remains the same: whatever change it undergoes in the change of place of the body, an equal change in an opposite direction takes place when the body returns to its original place. This result is evident from (4); for if h is less than z, the work is negative, so that work must be imparted to the body to raise it to a greater altitude.

Suppose again that we have a series of weights, w_1, w_2, ..., whose altitude above the earth's surface at the times t_0 and t are severally h_1, z_1, h_2, z_2, ...; then the work in the time $t-t_0$

$$= \Sigma . w (h-z)$$
$$= (\bar{h}-\bar{z}) \Sigma . w, \qquad (5)$$

if $\bar{h}$ and $\bar{z}$ are the vertical distances of the centre of gravity of the weights above the earth's surface at the time t_0 and t. So that the work depends on the vertical distances between the places of the centre of gravity of the system of weights at the times t_0 and t respectively. If $\bar{h}$ is greater than $\bar{z}$, so that the centre of gravity is lower, work is taken by the earth from the weights; if $\bar{h}$ is less than $\bar{z}$, work is given by the earth to the weights, and their work is increased: if $\bar{h} = \bar{z}$, the work of the weights is unaltered.

Equation (4), which gives a mathematical definition of work, enables us to determine an unit of work, and thereby to measure other work, although it is only founded on the application of the general principle of vis viva to the particular case of a heavy body moving near to the earth's surface. Work done in the time t = the product of the weight moved and the vertical distance over which it is moved. Consequently, if the weight moved is an unit-weight, and the vertical distance over which it is moved is an unit-distance, the product of these two quantities is the unit of work: and we have the following definition:

An unit of work = an unit-weight × an unit of vertical distance. (6)

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In Britain we express distances in terms of feet, and weight in terms of pounds avoirdupois; and consequently the British unit of work is one pound raised one vertical foot; this is called a foot-pound; and work, which is defined by the product of numbers expressing the number of pounds raised, and the number of feet through which they are raised, is said to consist of such and such a number of foot-pounds. Thus, if 100 pounds are lifted through 5 feet, the work done is 500 foot-pounds; and if 50 pounds are lowered through 10 feet, work to the amount of 500 foot-pounds has been taken from them and communicated to something else.

In France distance is expressed in mètres, and weight in kilogrammes; and the unit of work is called a kilogrammètre.

Now this mode of estimating work done is applicable to not only machines, but also to living agents, as a man or a horse. Thus, if a man lifts a weight w through a vertical height h in a given time, $w\,h$, expressed in foot-pounds, is the work done by him in that time. So it is found that a man working on a tread-mill will raise himself through 10,000 feet in a day of 8 hours; and, taking the weight of his body to be 150lbs, his work in the day is 1,500,000 foot-pounds.

I should observe also that there is a peculiar unit of work called a horse-power, in terms of which the work of a steam engine and of other machines is ordinarily estimated. In Britain a horse-power is 550 foot-pounds in a second of time; that is, is 33,000 foot-pounds in a minute. In France the term "force de cheval" means 4,500 kilogrammètres in a minute; and this is equivalent to 32,549 English foot-pounds.

268.] Having said thus much on work, and the mode of measuring work, I will return to the consideration of the equation of vis viva, given in (3), and introduce into it these definitions; in which case the theorem of vis viva may be enunciated as follows:

In the motion of a system of particles, subject to connections which are independent of the time, and under the action of forces which do not explicitly involve the time, the increment of the sum of the vires vivæ of all the particles in a given time is equal to twice the sum of the work communicated to the system during the same time by all the acting forces.

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Of the forces which act on the several parts of the machine and produce the work, there are two kinds, which must be distinguished; there are moving forces, and there are resisting or retarding forces. The former are called positive forces, or the moving powers acting on the machine, and the effect of them is the moving work of the machine, and is that which is impressed at the driving points. The latter are called negative forces, and correspond to the whole resisting work of the machine; that is, not only to the work done by the machine at its working points, but also that due to the friction of the pieces of the machine one against another, or against the fixed framework or the supports; that due to the stiffness of cords and connecting bands; that due to the vibrations of the several particles of the machine, to the vibrations of the supports, of the ground on which they rest, of the surrounding air. All these are causes of work, and the work due to them is resisting work, which acts in a contrary direction to the moving work of the machine. These forces are by Carnot and by other writers distinguished according as the angles between their lines of action and the path described by their points of application in the time dt are acute or obtuse.

Let w_m be the moving work of all the forces acting on the machine during the time $t-t_0$; and let w_r be the resisting work due to the resisting forces in the same time; so that

$$\int F\,ds = w_m - w_r;$$

then equation (3) takes the form

$$\Sigma.m\,v^2 - \Sigma.m\,v_0^2 = 2\,(w_m - w_r); \tag{7}$$

so that the increment of vis viva is equal to twice the excess of the moving work over the resisting work.

The resisting work w_r consists of two parts; firstly, the useful work, which we will call w_u, which acts at the working points, and the production of which is the object of the machine; and secondly, the lost work, which we will call w_l; which is spent on the friction of the pieces, the vibrations of the several particles of the framework, the ground, and the surrounding air, as we have just now explained; so that

$$w_r = w_u + w_l. \tag{8}$$

Equation (7) embodies the theory of all machines in motion; and consequently a careful consideration of it will indicate the

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conditions which good machines ought to satisfy, and the general principles of their construction.

As to the circumstances which cause the lost work (w_l), I would observe, that the work due to friction may be much lessened by means of a proper choice of materials, and by unguents, &c., and that in all cases the quantity of it can be calculated. The work spent on the vibrations of the several parts of the machine cannot be calculated in the present state of the science of molecular physics; but the vis viva of the several particles due to this work, which enters in the left-hand member of (7), is nearly, if not quite, equal in quantity; so that the "lost work" due to this cause will disappear in the application of the equation. The work which is spent on the motion communicated to the supports, and which is eventually conveyed to the earth and lost, varies much in different machines, but is sometimes very considerable. It cannot however generally be calculated. That which is lost in giving motion to the particles of the surrounding air is very small, and consequently may be neglected.

269.] Let us now trace the action of a machine in motion with reference to the equation (7), which connects the vis viva with the work done. When a machine begins to move, the element of moving work is greater than that of the resisting work which is brought into action in the same time, and vis viva increases; this increase of vis viva continues until the elements of moving work and resisting work due to the same element of time are equal; then there is equilibrium between these two work-elements, no further increase of vis viva takes place, and the vis viva becomes a maximum.

Now the subsequent motion of the machine may be either uniform or periodic: let us suppose it to be uniform; and let t_0 be the time at which this state is reached; and let $\Sigma . m v_0^2$ be the vis viva which the machine then has. As the velocity of every particle continues the same, whatever is the time expressed by $t - t_0$, $\Sigma . m v^2 = \Sigma . m v_0^2$; consequently, from the right-hand member of (7), we have $w_m = w_r$, and the whole moving work is equal to and becomes resisting work; and we have

$$w_m = w_u + w_l; \tag{9}$$

and the machine transmits the whole moving work to the points at which the resisting work, both useful and lost, is applied, without loss or modification.

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When the motion of a machine is not uniform, but periodic, as it is in an ordinary steam engine, there is a continual increase or decrease of vis viva; and the element of moving work is not equal to the element of resisting work for every time-element. If however we consider an interval of time, at the beginning and the end of which the velocities of the different elements of the machine are the same, the left-hand member of (7) vanishes, and consequently $w_m = w_r$; that is, the whole moving work applied during that interval is equal to the corresponding whole resisting work. And thus, although the elements of the moving and the resisting works for any time-element are not equal, yet the sums of these elements are equal, for the interval at the beginning and end of which the vis viva of the machine is the same. And as this result is true for any one such interval, so is it true for the time of many such intervals.

If then we consider a machine during the whole time that it moves, that is, from the instant at which it begins to move, to that at which it comes to rest, $\Sigma . m v^2 = \Sigma . m v_0^2 = 0$, and consequently, corresponding to that whole time, $w_m = w_r$; so that whatever is the motion of the machine, whether it is uniform or periodic, or of any other nature, the work due to all the moving forces for the time during which the motion continues is equal to the whole resisting work developed in the same time.

It also appears from equation (7), that if the vis viva $\Sigma . m v^2$ at the time t is greater than the vis viva $\Sigma . m v_0^2$ at the time t_0, w_m is greater than w_r; and that if $\Sigma . m v^2$ is less than $\Sigma . m v_0^2$, w_m is less than w_r. Thus the vis viva of the machine increases or decreases according as w_m is greater or less than w_r; and the vis viva remains the same, if $w_m = w_r$. If therefore the moving work is greater than the resisting work, the vis viva of the machine increases, because the excess of the moving over the resisting work is being stored as vis viva in the several parts of it. Whereas, if the resisting work is greater than the moving work, that excess is taken from the machine in the form of a loss of vis viva. Hereby we see the reason why a machine, whose motion is not uniform, will yield a resisting work equal to a moving work, if these are considered during the whole time of motion. As the machine starts from rest, w_m is greater than w_r, and work is being stored in the machine in the form of vis viva, until the machine reaches a state at which its vis viva is

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a maximum. And as the machine returns to its state of rest it loses vis viva, which becomes resisting work, and eventually gives out just as much as it received at first; so that if w_m and w_r are estimated through the whole motion, $w_m = w_r$. Thus it is that a machine does not create work; it receives it; it may store it for a time; but finally it yields exactly the same quantity as has been given to it.

270.] Whenever the differential of the vis viva vanishes, then, from (3), $\Sigma . F\,ds = 0$; that is, the vis viva of the machine is a maximum or a minimum, or does not vary with a small motion of the machine, when all the forces, moving and resisting, acting on it at the time are in equilibrium; in equilibrium, I say; because $\Sigma . F\,ds$ is the sum of the virtual moments of the forces; and when this sum $= 0$, the forces are in equilibrium by reason of the principle of virtual velocities. Now $\Sigma . m v^2$ is a maximum or a minimum according as $\Sigma . F\,ds$ changes sign from $+$ to $-$, or from $-$ to $+$, as it passes through zero. In the former case, the moving forces are greater than the resisting forces before equilibrium takes place, and after equilibrium the resisting forces become greater than the moving forces; these circumstances indicate a state of stable equilibrium. In the latter case, all these circumstances are reversed, and the state is that of unstable equilibrium. So that the vis viva of the machine is a maximum or a minimum according as the equilibrium of it at the instant is stable or unstable. If the forces are in equilibrium, and no change of sign takes place in $\Sigma . F\,ds$, the vis viva of the machine is neither a maximum nor a minimum; in this case the equilibrium cannot be said to be either stable or unstable. If an infinitesimal motion of the machine takes place, then $\Sigma . F\,ds$ will pass through 0 from $+$ to $+$, or from $-$ to $-$, according as one or other of the two directions of motion is taken; and thus the equilibrium is said to be neutral. If $\Sigma . F\,ds = 0$, because $\Sigma . \int F\,ds$ is a constant, then the vis viva is constant, and the equilibrium of the machine is continuous.

271.] The goodness or the efficiency of a machine depends on the amount of useful work yielded by it in comparison of the moving work; so that the efficiency is mathematically defined by the following equation:

$$\text{the efficiency of a machine} = \frac{w_u}{w_m}. \qquad (10)$$

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The superior limit of this ratio is unity, the ratio in all machines being less than 1; and the nearer the ratio approaches to 1, so much more efficient is the machine. Thus, a machine is perfectly efficient when $w_u = w_m$, that is, when the useful work is equal to the moving work. This however is a theoretical state, which is never found in practice. And why? Because in considering the work done by a machine during the interval of time at the beginning and the end of which the vis viva of the machine is the same, so that no increase of vis viva, and consequently no increase of work, is stored in the machine, $w_r = w_m$; and consequently,

$$w_m = w_u + w_l: \tag{11}$$

thus part of w_m goes to w_l, which is the work lost in the various ways which have been enumerated and explained in Art. 268 above; and w_l can never be made wholly to disappear. We must however diminish it as far as possible, so that w_u may be as nearly as possible equal to w_m. All frictions therefore of rolling and sliding, which are not absolutely necessary, must be avoided; and when they cannot be avoided, the spaces over which they take place must be, as far as possible, reduced, and their work must be diminished by means of hard and polished materials and of unguents. All vibratory motion is also to be avoided; for although, as we have before observed, vis viva may be produced in parts of the machine which will be equal or almost equal to the work spent on it, and which may be retransferred to work, yet oscillatory motion of the parts produces oscillatory motion in the molecules of the parts, which is propagated from molecule to molecule by the elasticity of the materials through the supports, the framework, and the surrounding air, and is finally lost in the mass of the earth without the production of any useful work. We must also avoid all sudden impulses and blows; for they will be the cause of not only intense vibrations in the molecules of the machine and of its supports, but of changes of form of the parts and supports; which, although perhaps not considerable in extent, may consume a large quantity of work, because the resistance to such changes is very great. In short the most efficient machines move without noise, without displacement of any support, and we may almost say without a suspicion on the part of the observer of any strain or effort in any of its parts, or of work done by it.

Notwithstanding all precautions, if w_u is the useful work to

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be done by a machine, w_m, the moving work, must be greater than it; for we can never reduce w_l to zero. We perceive then the great error of those who are in search of perpetual motion. They propose to invent a machine by means of which useful work may be done without any moving work; or at least at the expense of moving work less than the useful work. This is impossible; for it is inconsistent with the truth of the equation of vis viva; and, as all known mechanical forces satisfy that equation, they cannot effect it *.

272.] Another property of a good machine is uniformity of the velocity of its several parts; so that when the machine is in its full working state, the variation of the velocity of every part during the motion may be as small as possible. This condition is desired, not because the transmission of work is rendered more effective thereby, nor because the amount of useful work is increased; but because, from an industrial point of view, the quality of the useful work is better, being more regular. Now in many machines not only is the moving work communicated irregularly at the driving point, as in a steam engine; but also the useful work at the working points is irregularly applied; as in coining, punching, shearing, slotting, &c. machines; so that $w_m - w_r$ is a quantity varying within limits, which are considerably distant, and thus the velocities of the several parts of the machine are far from uniform. It may also be remarked, that irregular motions produce vibrations of the molecules, which cause loss of work. It is consequently important to devise means by which uniformity may be obtained as far as possible; and the equation of vis viva, as we proceed to shew, suggests a method.

In most machines, by means of the train, the velocities of the different parts have constant ratios which depend on their relative positions, if we omit the forced molecular vibrations; and thus the velocity of any molecule may be expressed in terms of that of another which is arbitrarily chosen. Let us, to fix our thoughts, take this last to be the driving point, that at which the moving work is applied. Let v and v_0 be the maximum and minimum velocities of it which succeed each other, t and t_0 being the times at which these respectively take place; and let

* See a curious paper on this subject by G. B. Airy, M. A., &c., the present Astronomer Royal, in the Cambridge Philosophical Transactions, Vol. III.

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w_m and w_r be the moving and resisting works of the machine in the time $t-t_0$. Let the velocities of any molecule m at these times be respectively av and av_0; and as this will be true for all particles, a varying from particle to particle, the equation of vis viva takes the form

$$(v^2-v_0^2)\,\Sigma.m\,a^2 = 2\,(w_m-w_r)\,; \tag{12}$$

so that

$$v-v_0 = \frac{2}{v+v_0}\,\frac{w_m-w_r}{\Sigma.m\,a^2}. \tag{13}$$

Our object is to make $v-v_0$ as small as possible; so that the variations of the velocity of the driving point, and consequently of all the particles of the machine, may be as small as possible. w_m-w_r is a known quantity, varying with the variations of the moving and resisting works; which however cannot be so arranged as to make $v-v_0$ small. But the difference $v-v_0$ will be less, the greater $\Sigma.m\,a^2$ is; this therefore suggests the addition to a machine of large masses moving with great velocities. These are generally introduced in the particular form of large and heavy wheels moving with high velocities, having the greater parts of their mass in a thick rim of a considerable radius; because thereby the vis viva of the wheel is increased; they are called flywheels; and are generally placed on an axle near to the moving force of the machine, when the variations of the moving work are great; and near to the working points when the variations of the work at their points are great. Being wheels moving on fixed axles, the centre of gravity of them remains fixed during the motion, so that no work is consumed on its motion; a small quantity of work is spent on the friction at the bearings, and on the vibrations which are communicated by their motion to the surrounding air.

Flywheels not only give steadiness to a machine, by preventing great fluctuations of velocity, but when the variations of the moving and the resisting work are not periodic, they are employed to prevent too great an accumulation or a diminution of vis viva; this they effect by bringing into action a piece of mechanism called a governor, or a regulator, whereby the supply of moving work can be varied. Such is the sluice or valve, which adjusts the opening through which water is supplied to a water-wheel; the throttle valve, which regulates the space in the steam pipe through which steam is supplied to the cylinder; the damper, which regulates the supply of air to a furnace. The

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governor which is most commonly in use is Watt's centrifugal governor, of which a drawing is given in fig. 62, and which is described in Art. 234.

Flywheels also serve a most useful purpose in those machines wherein a large amount of work is required at the working points, not continuously, but on a sudden, as for instance in punching machines; because they contain a large quantity of vis viva which has been communicated from the moving work, has been stored in the form of vis viva, and is ready to become useful work at the working points.

273.] The conclusions of the preceding Articles are drawn from the equations of vis viva, (3), (4), and (7), the members of which severally are vis viva, work, and a weight moved through a given vertical space. Now if an equation is intelligible and applicable to any useful purpose, its members must be homogeneous, that is, the quantities which the two members consist of must be of the same kind; if one member is linear space only, the other must equally be so; and consequently if an equation involves space, time, and mass, which is the case with most dynamical equations, the dimensions of these must be the same in both members. I propose to prove the homogeneity of the preceding equations; and in so doing certain general principles will be stated which are of general application; and I shall incidentally prove that other fundamental equations are equally homogeneous.

Certain units must be assumed, in terms of which, by means of number, quantities of the same kind may be expressed.

I assume an unit of space, and an unit of time. Then, from the ordinary definitions of dynamics, we have the two following consequences:

The unit of velocity is an unit of space passed through in an unit of time.

The unit of accelerating force is that which impresses an unit of velocity in an unit of time.

Since velocity $= \frac{ds}{dt}$, velocity is of (1) dimension in space, and of (-1) dimension in time.

Since accelerating force $= \frac{d^2s}{dt^2}$, accelerating force is of (1) dimension in space, and of (-2) dimensions in time.

We require the definition of another quantity, viz., that of

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mass. Now in these and other applications, the matter which moves is terrestrial matter, of which a property is, that it attracts directly as its mass, and inversely as the square of the distance. Let an unit of mass be placed at, say, A; and let m units be placed at, say, B; the distance between A and B being r: then, by the law of gravitation,

$$\frac{d^2r}{dt^2} = -\frac{m}{r^2};$$

$$\therefore \quad m = -r^2\frac{d^2r}{dt^2}. \tag{14}$$

and consequently an unit of mass is that which produces an unit of accelerating force at an unit of distance.

Thus mass is of (3) dimensions in space, and of (−2) dimensions in time.

Hence we have the following:

Density $= \dfrac{\text{mass}}{\text{volume}}$; now volume is evidently of (3) dimensions in space; consequently density is of (0) dimensions in space and of (−2) dimensions in time.

Since weight $= \text{M}g$; weight is of (4) dimensions in space, and of (−4) dimensions in time.

Since work $= \text{M}.\text{v}^2 = 2\text{w}h$; work is of (5) dimensions in space, and of (−4) dimensions in time.

Hence, if the units of space and time are changed, the members expressing the quantities above will have to be changed in the ratios just now assigned. Suppose for instance the units of space and time to be diminished in the ratio of 1 to 2; so that what was s becomes $2s$, and what was t becomes $2t$. Then the new density is expressed by a number which is one-fourth of that which expresses the former density; weight is not altered; work is expressed by a number which is twice that which expresses the former work.

On these principles all our mechanical equations are homogeneous. Thus, if w is a weight lowered through a vertical distance $= h$; and v is the velocity given to a mass $=$ M by the work thus obtained: from the preceding Articles we have

$$\text{the work} = 2\text{w}h = \text{M}.\text{v}^2 = \text{vis viva};$$

wh is of (5) dimensions in space, and of (−4) dimensions in time; and the dimensions of $\text{M}.\text{v}^2$ are the same.

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Consider again the equation $s = \frac{1}{2} g t^2$; s is of (1) dimension in space; g is of (1) dimension in space, and of (−2) dimensions in time: hence the equation is evidently homogeneous.

Consider also the equation $v^2 = 2gh$; each member of this is evidently of (2) dimensions in space, and of (−2) dimensions in time.

It is unnecessary to cite more examples. In conclusion, however, I would observe on the advantage of testing the homogeneity on these principles of all dynamical equations in their original forms; for that remains a quality of them whatever are the operations to which they are subjected; and consequently if they are also homogeneous in their ultimate state after a series of operations, it affords a presumption that the operations have been correctly performed.

These principles are also capable of a much wider application; work, in the meaning of the word here given, is produced by elastic action, by magnetism, by heat, &c. We are hereby enabled to reduce all these to a comparison with mechanical work. Thus, for instance, we have now a mechanical equivalent for heat; if the temperature of one pound of water be raised one degree Fahrenheit, it has been determined by Mr. Joule that the work thus produced is equivalent to 772 foot-pounds. The investigation of the equivalence of mechanical work to that done by the agency of heat, electricity, magnetism, &c., is a matter of extreme interest and great importance. It is however too large to be entered on in this volume, even if it were appropriate to do so; for our limits would not allow justice to be done to it; and we can only refer the reader to the various treatises and memoirs on these subjects.

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CHAPTER X.

THE MOTION OF ELASTIC BODIES.

274.] The principles and laws of motion have thus far been applied to rigid bodies, and to systems of rigid bodies, the constituent molecules of which have been assumed to be in a state of relative rest during the motion; and the equations of motion by which problems have heretofore been solved have been deduced from these principles thus restricted. Our purpose is to apply them more generally. Ere, however, we do so, there are two reasons why we should repeat as concisely as possible the modification of the equations which this assumption of the relative rest of the constituent molecules introduces. (1) Because we have come to the end of our investigations on that subject, and it is good once more prominently to restate the conspicuous principle of the process so frequently employed: and (2) because in the present chapter we shall investigate equations expressing the motion of a particle which is not at rest relatively to its neighbouring particles, all being constituent molecules of a body; and our research will include the varying form of flexible bodies, (as they are called,) the molecules of which move relatively to each other; and our conception of such motions will be more exact when they are contrasted with those of the molecules of a rigid body in their chief differences.

The equations of motion of a rigid body are found by the following process: Let dm be an element of the body, and let (x, y, z) be its place at the time t, relatively to a system of coordinate axes fixed in space. Now this particle is supposed to be under the action of certain external forces, whereby a certain velocity or velocity-increment is impressed on it. In consequence of this external force it would have a definite expressed velocity-increment if it were alone, and thus free from all constraint from its surrounding molecules. As it is not free, the constraints enter as other forces, which, affecting its motion, produce a change of its expressed velocity-increments: these constraints we consider as internal forces, which produce their own effects; and these effects modify those which would otherwise take place.

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And consequently, if X, Y, Z are the axial components of the velocity-increment impressed on dm by the external forces, and if I is the resultant of the velocity-increment due to all the internal forces or constraints, of which α, β, γ are the direction-angles; then the equations of motion of dm are

$$\left.\begin{aligned} dm\left\{\text{X}-\frac{d^2x}{dt^2}\right\}+\text{I}\cos\alpha &= 0,\\ dm\left\{\text{Y}-\frac{d^2y}{dt^2}\right\}+\text{I}\cos\beta &= 0,\\ dm\left\{\text{Z}-\frac{d^2z}{dt^2}\right\}+\text{I}\cos\gamma &= 0;\end{aligned}\right\} \qquad (1)$$

from which also arise three other equations, which express the rotation of dm; viz.

$$\left.\begin{aligned} dm\left\{y\left(\text{Z}-\frac{d^2z}{dt^2}\right)-z\left(\text{Y}-\frac{d^2y}{dt^2}\right)\right\}+\text{I}(y\cos\gamma-z\cos\beta) &= 0,\\ dm\left\{z\left(\text{X}-\frac{d^2x}{dt^2}\right)-x\left(\text{Z}-\frac{d^2z}{dt^2}\right)\right\}+\text{I}(z\cos\alpha-x\cos\gamma) &= 0,\\ dm\left\{x\left(\text{Y}-\frac{d^2y}{dt^2}\right)-y\left(\text{X}-\frac{d^2x}{dt^2}\right)\right\}+\text{I}(x\cos\beta-y\cos\alpha) &= 0.\end{aligned}\right\} (2)$$

Equations of the same form as those in (1) and (2) are true for every molecule of the body. Let these be formed; then we shall have a series of groups of equations expressing the motion of every molecule, the sum of which will express the motion of the whole body. And here enters the characteristic of the rigidity of the body: all the internal forces and their consequent velocity-increments enter in pairs, of which the directions are opposite to each other; every constraint, acting from (say) dm to dm', has an equal and opposite constraint acting from dm' to dm: the law of the equality of action and reaction is true in this case of every pair of molecules; so that

$$\Sigma.\text{I}\cos\alpha = \Sigma.\text{I}\cos\beta = \Sigma.\text{I}\cos\gamma = 0. \qquad (3)$$

$$\Sigma.\text{I}(y\cos\gamma-z\cos\beta)=\Sigma.\text{I}(z\cos\alpha-x\cos\gamma)=\Sigma.\text{I}(x\cos\beta-y\cos\alpha)=0. \quad (4)$$

And therefore, adding together (1) and all its similar groups, and (2) and all its similar groups, we obtain the equations of motion of a rigid body which are given in (37) and (38) Art. 48.

The same process of reasoning is applicable to the motion in space of a system of rigid bodies moving relatively to each other, if the internal action of one on another is always accompanied with an equal and opposite reaction; because these will disappear

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in the summation of the several equations, when that extends to and includes all the molecules of all the moving bodies.

275.] In the problem of the present chapter, however, the subject of motion is a body, the molecules of which move one relatively to another, and the bounding form of which hereby changes. A fine vibrating string, a thin vibrating membrane, a mass of quivering jelly or caoutchouc, are such bodies as we here contemplate. In these the form of the bounding surface will change from time to time; and so also will the relative arrangement of the constituent molecules. When the molecules move one relatively to another, internal forces are brought into action which affect their motion: these are generally called elastic forces, and are of the nature explained in Art. 153, Vol. III. These forces vary from molecule to molecule, and also from time to time; so that if the body is referred to a system of axes fixed in space, and (x, y, z) is the place of dm at the time t, the elastic forces acting on dm are functions of x, y, z, and t. In the most general case we suppose external forces to act on the several molecules of the body; so that dm is acted on by these as well as by the elastic forces, and both will enter into its equations of motion. Thus, if $\imath$ is the whole elastic force acting on dm at the time t, and α, β, γ are the direction-angles of its line of action, the equations of motion of dm are those given in (1) and (2). Similar equations will express the motion of every particle of the body. Now we cannot take the sum of all these, and thereby determine the motion of the whole body, as the process is in the case of a rigid body; (1) because our object is to determine the form of the body at any time, and to do this it is necessary to determine the place of every particle at that time; so that the set of equations corresponding to a given particle must be separately considered, and its place therefrom determined: and (2) because all the internal forces may not be in equilibrium amongst themselves; and consequently the conditions (3) and (4) may not be satisfied. These internal or elastic or molecular forces, as they are called, may enter in pairs of equal and opposite forces in the interior of the body, and thus far may disappear in the sum corresponding to the sum of all the particles; but at the bounding surface they may be counteracted by and thus be in equilibrium with certain external forces thereat acting; so that all will not disappear in the sum of the groups of the equations corresponding to all the particles of the body. Herein

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then is the difference of the mode of formation of the equations of motion of a rigid body and of a molecule of a flexible body.

276.] These internal forces, which enter into the equations of motion of each molecule, depend on the molecular constitution of the body, and are what are commonly called elastic or molecular forces.

There are two principles on which the required equations of motion may be formed. We might assume a particular theory of the molecular constitution of an elastic body, and deduce from it the intensity, mode of action, law and direction of the elastic force which affects a certain molecule in a given position, and which corresponds to a given displacement. This method has been adopted in Art. 152—161 of Vol. III, and has been therein applied to the formation of the conditions of statical rest of the molecules of an elastic body. And we might take the elastic forces of restitution or of further separation, corresponding to a given displacement, to be those which are therein determined. This method would have an obvious advantage. It would give us the expression of elastic action in a most general form; that, viz., which affects a molecule of an elastic substance of three dimensions in space: a particular form of this would be a membrane or plate of infinitesimal thickness; and a still more particular form would be a thin thread or string: and the equations which express the motion of the molecules of these reduced forms of elastic matter would be reduced forms of the general equations.

Our knowledge, however, of molecular physics is at present too imperfect that equations founded on any general theory of elastic action should be made the basis of equations which express the motion of fine elastic strings, and of thin elastic plates or membranes; especially too in a didactic treatise. I prefer to take laws which have been established by observation, such as Hooke's law; and to deduce from them, special though they be, the equations which express the particular motion in question. These particular laws are doubtless parts of the more general law; and if the latter is true, the former are included in it. And consequently, it is my intention to state and explain the equations which express the elastic action of the constituent molecules, as it has been given in Vol. III, and to deduce from it the particular equations of strings and membranes; but it will be beyond our purpose to apply them further in the present work.

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277.] In the first place let us form the equations of motion of a perfectly flexible fine thread or string, which in the general case we conceive to be extensible, and to be elastic; so that when it is stretched, elastic forces of restitution are brought into action. We suppose the string to have been displaced from its position of statical rest by the action of some external forces, and consider it in its motion at the time t. To take the most general case, we will suppose it to be a curve of double curvature; and we will refer it to a system of rectangular axes fixed in space.

Let dm be an element of its mass, whose place at the time t is (x, y, z). Let ds = the length of this element, and let ρ = its density; let ω = the area of a transverse section of the string: so that $dm = \rho\omega\, ds$. Let X, Y, Z be the axial components of the impressed velocity-increments on dm; and let $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ be the axial components of the expressed velocity-increments. Let T = the tension of the thread at the point (x, y, z), which I take to be the beginning of ds: then, as the tangent is the line of action of T, the axial components of T are $\text{T}\frac{dx}{ds}, \text{T}\frac{dy}{ds}, \text{T}\frac{dz}{ds}$; and the axial components of the tension at $(x+dx, y+dy, z+dz)$, which is the other end of ds, are

$$\text{T}\frac{dx}{ds} + d.\text{T}\frac{dx}{ds}, \quad \text{T}\frac{dy}{ds} + d.\text{T}\frac{dy}{ds}, \quad \text{T}\frac{dz}{ds} + d.\text{T}\frac{dz}{ds}:$$

thus, according to (1), the equations of the motion of dm are

$$\left.\begin{aligned} \rho\omega\, ds\left\{\text{X} - \frac{d^2x}{dt^2}\right\} + d.\text{T}\frac{dx}{ds} &= 0, \\ \rho\omega\, ds\left\{\text{Y} - \frac{d^2y}{dt^2}\right\} + d.\text{T}\frac{dy}{ds} &= 0, \\ \rho\omega\, ds\left\{\text{Z} - \frac{d^2z}{dt^2}\right\} + d.\text{T}\frac{dz}{ds} &= 0. \end{aligned}\right\} \qquad (5)$$

I may observe that these equations have been found before; viz. in Art. 52, where their determination has been given in illustration of the principle of virtual velocities. I have chosen however again to investigate them, in order that the meaning of all the symbols involved in them may be clearly understood.

278.] Of these I will first take a most simple case. Let the string be fastened at one end O to a fixed point, and let it pass over a small pulley A, where OA = a, and have a weight = W

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attached to its other end: so that the tension of the string throughout is equal to W; and let W be so great that the weight of the string may in comparison of it be neglected. Let us however suppose $X = Y = Z = 0$; then, in its position of equilibrium, the string lies along the straight line OA; let it be slightly displaced by means of an external force; the displacement being so small that the angle at which any element of it is inclined to the line OA is infinitesimal. Our object is to investigate the law of the displacement of any particle which follows on this initial displacement. Let OA be the axis of x; and let (x, y, z) be the place of any element $(=\rho\omega ds)$ at the time t. Then, as the angle of inclination of ds to the axis of x is infinitesimal, we have approximately $ds = dx$. So that neglecting infinitesimals of the second order, each element of the string moves in a plane perpendicular to the line OA, and consequently the point of the string at A does not move: thus there will be no motion along the line OA, and $\frac{d^2x}{dt^2} = 0$, for all elements of the string. Thus the first of (5) gives $T =$ a constant; and the tension of the string is constant throughout its length and throughout the motion. Introducing these results into the last two equations of (5), they become

$$\left.\begin{aligned} \frac{d^2y}{dt^2} - \frac{T}{\rho\omega}\frac{d^2y}{dx^2} &= 0, \\ \frac{d^2z}{dt^2} - \frac{T}{\rho\omega}\frac{d^2z}{dx^2} &= 0. \end{aligned}\right\} \tag{6}$$

If, for the sake of simplicity, we suppose the curve of the string in its initial displacement to be in one plane, we may take that to be the plane of (x, y); and then the equation which represents the subsequent motion of the particle is

$$\frac{d^2y}{dt^2} - \frac{T}{\rho\omega}\frac{d^2y}{dx^2} = 0; \tag{7}$$

which is a partial-differential equation of the second order. The integration and interpretation of it I shall defer to the following Articles; because we shall again meet with equations of the same form. The homogeneity of it on the principles explained in Art. 273 deserves notice.

279.] Next let us investigate the motion of the particles of a thin heavy elastic string, which is homogeneous and of the same thickness throughout its length, stretched between two given

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points O and A; see Fig. 64; where OA $= l$. Let T_0 be the tension of the string at rest: which we assume to be so great that the weight may be neglected without sensible error in comparison of it: thus the string lies in the straight line joining O and A, when it is in statical equilibrium.

Now let us suppose the string to be put into motion by some external force; as a piano-forte string by the blow of the hammer, or the string of a harp by the finger of the player: hereby the particles are displaced both relatively and in space; and elastic forces of tension are brought into action, tending to restore the string to its original condition; and although the force which produces the displacement ceases to act, yet the particles of the string continue to move, and the string vibrates about its rectilinear position. We will take the most simple form of the problem, and suppose no other force to act: so that in (5), $X = Y = Z = 0$.

Let us consider the string in its vibrating state at the time t, and refer it to three rectangular axes originating at O, of which the x-axis coincides with OA. Let us take a particle ($= dm$) whose distance from O in its position of rest $= x$; and let ρ be its density: then if $\omega =$ the area of a transverse section of the string, $dm = \rho\omega\, dx$: let the place of this particle at the time t be $(x+\xi, \eta, \zeta)$; and let ρ' be its density in its displaced state, and ω' the corresponding area of the transverse section of the string; and let ds be the length-element of the curve which dm occupies: then, as the mass of dm is unaltered,

$$dm = \rho\omega\, dx = \rho'\omega' ds. \tag{8}$$

As the displacement of the particle is very small, ξ, η, ζ are all small: they are functions of x and t, and are to be expressed in terms of these variables. It is however to be observed that x is not a function of t. Thus (5) become

$$\left.\begin{aligned} \rho\omega\, dx \frac{d^2\xi}{dt^2} - d.T\frac{d(x+\xi)}{ds} &= 0, \\ \rho\omega\, dx \frac{d^2\eta}{dt^2} - d.T\frac{d\eta}{ds} &= 0, \\ \rho\omega\, dx \frac{d^2\zeta}{dt^2} - d.T\frac{d\zeta}{ds} &= 0. \end{aligned}\right\} \tag{9}$$

Now the length of the element $= dx$, under the action of the tension T_0: and $= ds$, under that of the tension T: consequently,

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if E is the *modulus* of elasticity, by Hooke's law, as explained in Art. 149, Vol. III,

$$ds = dx\left\{1 + \frac{T - T_0}{E}\right\}; \tag{10}$$

where, it will be observed, E is a *weight* depending on the nature of the string.

Also $$ds^2 = (dx + d\xi)^2 + d\eta^2 + d\zeta^2;$$

and as the displacements of the molecules are small, $d\xi$, $d\eta$, $d\zeta$ are so small, that all powers of them higher than the first may be neglected. Consequently,

$$ds = dx + d\xi; \tag{11}$$

$$\therefore \quad T = T_0 + E\frac{d\xi}{dx}.$$

Let us substitute all these values in (9); and, omitting small terms, we have

$$\left.\begin{aligned} \rho\omega\frac{d^2\xi}{dt^2} &= E\frac{d^2\xi}{dx^2}, \\ \rho\omega\frac{d^2\eta}{dt^2} &= T_0\frac{d^2\eta}{dx^2}, \\ \rho\omega\frac{d^2\zeta}{dt^2} &= T_0\frac{d^2\zeta}{dx^2}. \end{aligned}\right\} \tag{12}$$

Let $$\frac{E}{\rho\omega} = a^2, \qquad \frac{T_0}{\rho\omega} = b^2;* \tag{13}$$

then the preceding equations become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} &= a^2\frac{d^2\xi}{dx^2}, \\ \frac{d^2\eta}{dt^2} &= b^2\frac{d^2\eta}{dx^2}, \\ \frac{d^2\zeta}{dt^2} &= b^2\frac{d^2\zeta}{dx^2}; \end{aligned}\right\} \tag{14}$$

which are three partial linear differential equations with constant coefficients of the second order. As the variables in them are separated, we conclude that the vibrations of the string parallel to the three axes of ξ, η, ζ are independent of each other, and coexist without interference. The first equation expresses the vibrations along the string; these are called longitudinal vibrations; and those which are expressed by the last two equations take place at right angles to the axis of x, and are called

* It may be noticed that a and b are of (1) dimension in space, and of (−1) dimension in time, and thus represent velocities.

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transversal or lateral vibrations. The form however of all the equations is the same; and we need only discuss one of them, say the second; for the result of that will also give the solution of the first, if we change b into a; and will thus assign the nature of the longitudinal motion.

280.] These equations may be integrated by the method given in Art. 365, Vol. II; and the integral of the second of (14) is

$$\eta = \text{F}(x+bt) + f(x-bt); \tag{15}$$

where F and f are symbols of arbitrary functions, as yet undetermined.

As to this process of integration, I may by the way observe, that if we change the variables in the second of (14) by putting

$$x+bt = \alpha, \qquad x-bt = \beta,$$

the equation becomes

$$\frac{d^2\eta}{d\alpha\, d\beta} = 0; \tag{16}$$

$$\begin{aligned} \therefore \quad \eta &= \text{F}(\alpha) + f(\beta) \\ &= \text{F}(x+bt) + f(x-bt). \end{aligned}$$

Now our object is to determine the form of these functions by means of the initial or other circumstances of the string; and of the fact of the two points at O and A being fixed.

Let us suppose the equation of the curve of the string, when $t = 0$, to be

$$\eta = \Phi(x); \tag{17}$$

where η is the initial displacement parallel to the y-axis of the molecule whose distance from $\text{O} = x$. Also let us suppose the velocity parallel to the y-axis, when $t = 0$, to be given by the equation

$$\frac{d\eta}{dt} = b\phi'(x); \tag{18}$$

where $\phi'(x)$ is the derived function of $\phi(x)$. Φ and ϕ are symbols of known functions, and will be treated as known for all values of x between 0 and l; they are also subject to the condition that both vanish when $x = 0$, and when $x = l$.

Hence, when $t = 0$, from (16), (17), and (18), we have

$$\begin{aligned} \text{F}(x) + f(x) &= \Phi(x), \\ \text{F}'(x) - f'(x) &= \phi'(x); \\ \therefore \quad \text{F}(x) - f(x) &= \phi(x); \\ \therefore \quad \text{F}(x) &= \frac{\{\Phi(x)+\phi(x)\}}{2}, \end{aligned} \tag{19}$$

$$f(x) = \frac{\{\Phi(x)-\phi(x)\}}{2}; \tag{20}$$

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and consequently $\text{F}(x)$ and $f(x)$ are known for all values of x for which $\Phi(x)$ and $\phi(x)$ are known; that is, for all values of x between $x = 0$, and $x = l$.

The subject-variables however of F and f, as they are given in (15), are not limited by these values. The subject-variable of F is $x + bt$, and if b is positive, as we may take it to be, this varies as t increases through all positive values between 0 and ∞. And the subject-variable of f is $x - bt$, which has all values between l and $-\infty$; so that the complete solution requires the values of the functions corresponding to these values to be known. As b is a velocity, bt denotes a line, and is consequently homogeneous with x, and may be taken in addition to or subtraction from x along the x-axis.

Since the points O and A are fixed throughout the motion, $\eta = 0$, when $x = 0$ and when $x = l$; consequently, from (15), we have

$$\text{F}(bt) + f(-bt) = 0, \tag{21}$$

$$\text{F}(l+bt) + f(l-bt) = 0. \tag{22}$$

It appears from (21) that $f(-bt)$ and $\text{F}(bt)$ are equal and of contrary signs; so that if $\text{F}(bt)$ is known for all values of t between 0 and ∞, $f(-bt)$ is also known between those same limits.

In (22) let bt be replaced by $l + bt$; then

$$\begin{aligned} \text{F}(2l+bt) &= -f(-bt) \\ &= \text{F}(bt); \end{aligned} \tag{23}$$

which shews that the value of $\text{F}(bt)$ remains the same, when its subject-variable is increased by $2l$; consequently it is the same when the subject-variable is increased by $4l$, or $6l, \ldots$, or $2nl$, where n is a whole number. And therefore if the value of $\text{F}(bt)$ is known from $bt = 0$ to $bt = 2l$, the value is known for all values between $t = 0$ and $t = \infty$.

Again, in (22) let bt be replaced by $l - bt$, so that bt is less than l; then

$$\text{F}(2l-bt) = -f(bt); \tag{24}$$

but $f(bt)$ is known for all values of bt between 0 and l; consequently the value of $\text{F}(bt)$ is known for all values of bt between $bt = l$ and $bt = 2l$.

Hence the value of $\text{F}(a)$ is known for all values of a, from $a = 0$ to $a = \infty$; and these are the required limits.

Thus much as to F. And we have shewn above that all values of $f(\beta)$ are known from $\beta = 0$ to $\beta = -\infty$; and the initial equa-

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tion (20) gives all values of $f(\beta)$ from $\beta = l$ to $\beta = 0$; so that all values of $f(\beta)$ are known within the required limits.

It is worth observing that the values of $\text{F}(\alpha)$ for negative values of α, and that of $f(\beta)$ for values of β greater than l, have not been found in the preceding explanations; and they are not required; as their subject-variables are not within the limits given by the problem.

The values of the functions which express the ζ displacement may be found in a manner precisely similar.

Thus the form of the string in its displaced state, and the velocities of its several molecules parallel to the y- and z-axes at the time t will be known; and the problem will be completely solved, so far as the transversal vibrations are concerned.

Also, all that has been said on transversal vibrations is applicable, if we replace b by a, to the longitudinal vibrations of the string. In this case the initial equations will assign the position and the velocity along the x-axis of every particle of the string, when $t = 0$, between the limits $x = 0$ and $x = l$.

Thus the problem is completely solved. I propose however to interpret the result graphically, for the general motion of the string will be rendered clearer by means of a diagram. The results which will be exhibited might be derived from the equations just now discussed; but it will be more convenient to take a less general form, which will be equally expressive and more easily constructed.

281.] For this purpose I will assume that the string, having been disturbed, takes the form given by a known equation, such as (17), when $t = 0$; and that all its particles are then at rest; so that $\frac{d\eta}{dt} = 0$, when $t = 0$; consequently since

$$\eta = \text{F}(x+bt) + f(x-bt),$$

$$\frac{d\eta}{dt} = b\,\text{F}'(x+bt) - bf'(x-bt);$$

and therefore if $\frac{d\eta}{dt} = 0$, when $t = 0$,

$$\text{F}'(x) = f'(x);$$

$$\therefore \quad \text{F}(x) = f(x);$$

and
$$\eta = f(x+bt) + f(x-bt); \tag{25}$$

therefore, when $t = 0$,

$$\eta = 2f(x); \tag{26}$$

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Suppose however the equation to the curve of the string in its displaced state at rest, when $t = 0$, to be

$$\eta = \text{F}(x);$$

then, from (26), $\quad \text{F}(x) = 2f(x);$

and thus (26) becomes

$$\eta = \frac{1}{2}\{\text{F}(x+bt)+\text{F}(x-bt)\}. \tag{27}$$

This then is the η-displacement of the particle which is at $(x, 0, 0)$, when the string is rectilineal.

The function whose symbol is F is subject to the following conditions, which are derived from equations (21) ..... (24) of the preceding Article:

$$\text{F}(x) = -\text{F}(-x), \tag{28}$$

$$\text{F}(l+x) = -\text{F}(l-x); \tag{29}$$

$$\text{F}(x) = \text{F}(2l+x) = \text{F}(4l+x) = \ldots = \text{F}(2nl+x), \tag{30}$$

$$\text{F}(x) = -\text{F}(2l-x) = -\text{F}(4l-x) = \ldots\ldots \tag{31}$$

These equations enable us to infer a correct notion of the motion of the several molecules of the string, from the form which it has in its initial displaced state. From (28) it appears, that the curve represented by $y = \text{F}(x)$ is continued in similar forms on each side of O; the curve being on one side above, and on the other below the axis of x; and (29) shews that the curve is continued in similar forms on each side of A, the curve being on one side above, and on the other below the axis of x. Consequently the curve about the points O and A, and between these points, is of a form similar to that drawn in fig. 65, the plane of the paper being that of (x, y).

Again, to develope other properties of it: along the axis of x take from O in both directions a series of lengths, each of which is equal to l; viz., $\text{OA} = \text{AO}' = \text{O}'\text{A}' = \text{A}'\text{O}'' = \ldots = l$. Then equation (30) shews that whatever is the form of the curve between O and O′, it is the same between O′ and O″. And (31) shews that the form of the curve between O and A is the same as that between O′ and A, except that in the latter case it is inverted and lies below the axis of x. Thus the whole curve consists of similar portions drawn as in fig. 65.

Now this curve enables us to trace the motion of any particle of the vibrating string; and consequently the motion of the whole string. For the place P at the time t of the particle, which is at M ($\text{OM} = x$) when the string is straight, may thus be

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found: along OA take, on both sides of M, $MN = MN' = bt$; so that $NQ = F(x+bt)$, $N'Q' = F(x-bt)$, then, from (27),

$$MP = \frac{1}{2}\{NQ + N'Q'\}; \tag{32}$$

and by a similar process may the place of every element be determined at any time t.

282.] Now this molecule, and similarly every molecule of the string, and consequently the whole string, will oscillate; that is, the string will occupy a certain series of positions in succession, and will then be found in its original state; afterwards it will go through the same series again, and then return to its original state; and so on continually.

For if we assume $bt = 2l$, or $= 4l$, or $= 6l, \ldots$, or $= 2nl$, at the times corresponding to these intervals, we have, by means of (30) and (31),

$$\eta = \frac{1}{2}\{F(x) + F(x)\};$$
$$= F(x);$$

which is the value of η, when $t = 0$; and as this result is true for every point of the string, the string comes back to its original position at the times corresponding to these values. The interval between two successive states of like position is $\frac{2l}{b}$; which is therefore the periodic time of vibration.

When we take $bt = l$, or $= 3l$, or $= 5l, \ldots$, or $= (2n+1)l$, at the times corresponding to these intervals, we have

$$\eta = \frac{1}{2}\{F(x+l) + F(x-l)\};$$
$$= F(x+l);$$

so that in the time $= \frac{l}{b}$, every molecule will have described one half of the whole course which it describes in going from a position to the same position again. Thus the form of the string at these times is exactly similar to what it was when $t = 0$, but in an inverted position; as the lower line in fig. 64; the greatest ordinate being now at the same distance from A as it was from O in the original form of the string. The time which is occupied in this change of figure is one half of that occupied in a complete vibration.

Similarly we may consider the positions of the curve when

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$t = \frac{l}{2b}$, or when $t = \frac{3l}{2b}$: in the former case, the length of time is one-fourth of that of a complete vibration; and in the latter case, is three-fourths of that of a complete vibration. Similar results are also true when each of these times is increased by $\frac{2l}{b}$. It is to be observed, that during this vibratory motion the string never becomes a straight line.

Similar results to these are also true for the ζ-displacement; for in the last equation of (14) the same constant coefficient enters as in that for the η-displacement. Thus the periodic time of the complete path will be the same in both displacements. The forms of the functions may be different in the two cases, although they are both subject to the conditions developed in Art. 280.

This vibratory motion would continue perpetually if there were no diminution of vis viva of the string. In the case however of pianoforte or harp strings vis viva is lost for two reasons. The points O and A, to which the ends of the string are fastened, are not points rigidly fixed; so that vibrations are communicated to them from the string, and continued by means of their supports or framework to the earth, and thereby lost to the string; the string also vibrates in air, to the particles of which vibrations are communicated, and thus vis viva is taken from the string. Owing to these two causes, the oscillations of the string become gradually feebler, and eventually cease. The periodic time of the vibrations however is not changed.

283.] If the original position of the string-curve had been that drawn in fig. 66, where B is the middle point of OA, and where the two branches of the curve are exactly equal and similar, though their positions are inverted, the string would vibrate so that the point B would always remain fixed; that is, each half will vibrate as if the string were fixed at O and B, and at B and A; and the periodic time of vibration will be only one-half of what it is in the case already discussed. So, again, if the original string-curve had consisted of three or more equal and similar portions, intersecting OA at points equally distant from one another and from O and A, and being alternately above and below the line OA, the string will vibrate as if it were three or more distinct strings, and the points of it at which it intersects OA will remain at rest during the motion.

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These points are called nodes or nodal points, and the curve between two consecutive nodes is called a ventral segment.

284.] If τ is the time in which one complete transversal vibration of the string takes place,

$$\tau = \frac{2l}{b} = 2l\left(\frac{\rho\omega}{T_0}\right)^{\frac{1}{2}}; \tag{33}$$

so that, for a string of constant thickness and density, the time of transversal vibration varies as the length of the string directly, and as the square root of the tension in its straight form inversely.

And if n is the number of transverse vibrations which take place in a second of time,

$$n = \frac{1}{\tau} = \frac{1}{2l}\left(\frac{T_0}{\rho\omega}\right)^{\frac{1}{2}}; \tag{34}$$

so that for a string of constant thickness and density n varies as the square root of the tension in its straight form directly, and as the length of the string inversely.

Again, let τ' be the periodic time of a complete longitudinal vibration; then

$$\tau' = \frac{2l}{a} = 2l\left(\frac{\rho\omega}{E}\right)^{\frac{1}{2}}; \tag{35}$$

and let n' be the number of longitudinal vibrations which take place in a second of time,

$$n' = \frac{1}{\tau'} = \frac{1}{2l}\left(\frac{E}{\rho\omega}\right)^{\frac{1}{2}}. \tag{36}$$

Now if the string suspended by one end is stretched by a weight at the other end, E is that weight by which the length of the string will be doubled; so that the number of longitudinal vibrations of the string in a second of time varies directly as the square root of this weight.

From the preceding equations we have

$$\frac{n}{n'} = \left(\frac{T_0}{E}\right)^{\frac{1}{2}}; \tag{37}$$

but in the ordinary pianoforte strings, E is evidently very much greater than T_0, so that the number of longitudinal vibrations in a second of time is very much greater than that of the transversal vibrations.

The ratio of n to n' may also be expressed in the following way: suppose l to be the length of a string in its natural state,

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and Δl to be the increase of length when it is stretched by T_0: then, by Hooke's law,

$$l + \Delta l = l\left\{1 + \frac{T_0}{E}\right\};$$

$$\therefore \quad \frac{T_0}{E} = \frac{\Delta l}{l};$$

and

$$\frac{n}{n'} = \left(\frac{\Delta l}{l}\right)^{\frac{1}{2}}. \tag{38}$$

285.] Let us briefly notice these results in reference to the theory of music. When a string vibrates, the vibrations are imparted to the molecules of the surrounding air, and are through the medium of the air communicated to the tympanum of the ear, the auditory nerves of which are excited, and the sensation of sound is created. The ear recognises three special properties of sound, (1) the pitch, (2) the intensity, (3) a peculiar quality which is in England called technically the "quality" of a note, and is in France called "le timbre." The origin of this last property, and indeed most of its affections, it is very difficult to account for; it probably arises from many causes, amongst which may possibly be the difference of periodic time in the longitudinal and the transversal vibrations. The quality of note given by a string of a violincello played with a bow is very different to that given by a pianoforte string struck by a hammer, when the note or pitch is the same in both. The second property depends on the amplitude of vibration, and varies as the vis viva of the particles put into motion. The first property is that which is called the musical *tone* or *note*, and depends on the number of vibrations made by the string in a second of time. And thus the number of vibrations is taken as the measure of the pitch or note. The note is higher the greater the number of vibrations made in a second of time. Thus as to the note due to the transversal vibrations, the note varies inversely as the length of the string, and directly as the square root of the tension. Thus if the note of a given pianoforte string is too low, the string must be wound up, whereby the tension is increased.

The note given by a string when its two extremities are fixed, and when all the other points of the string vibrate in the motion, is called the *fundamental note* of the string: thus the fundamental note of a string is higher by one half than that of a string of which the length is twice as great. And thus, too, if a string is

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struck so that it has one node, the note in that case is twice as high as the fundamental note; and if it has two nodes, the note is thrice as high as the fundamental note; and so on. Thus the notes of strings are compared by means of their lengths and the distances between their nodes.

When two strings have equal periodic times, and vibrate together, they are, in musical language, said to be in unison. If two strings vibrate simultaneously, the resulting sound is most agreeable to the ear when they are in unison. Next to an unison the most agreeable concord is the octave, in which one string vibrates twice as fast as the other; that is, in which the times of vibration are as 1 : 2; and the note produced by the former is said to be an octave above that of the latter. Thus if a given string vibrates so that its middle point is a node, it produces the octave to the fundamental note.

It is invariably found, for all ears, that when two notes not in unison are sounded together, the resulting sounds are most agreeable when the times of vibration of the individual notes are in some simple proportions; say 1 : 2; 1 : 3; 1 : 4; 2 : 3; and that the concord is more agreeable the less the difference between the terms of the ratio.

Thus the octave of the octave of a note, or the fifteenth, as it is called in music, is an agreeable concord; for it consists of vibrations of two strings whose periodic times are as the numbers 1 : 4. Thus also two strings whose periodic times are in the ratio of 1 : 8, or in the ratio of 1 : 16, and so on, produce pleasant concords, and seem to partake of the perfection of the octave.

The next most simple numerical ratio is that of 1 : 3, in which we have three vibrations of one string corresponding in time to one vibration of the other. This concord is called a twelfth. If we replace the string, which vibrates once, by its octave, which vibrates twice in the given time, the times of vibration of the two strings are as 2 : 3: this also forms an agreeable concord, and is called a fifth.

It is however beyond the object of this treatise to enter further into this subject: and for other details, and for an exposition of the theory of music, founded on the preceding and other similar equations, I must refer the reader to the Treatise on Sound by Sir John F. W. Herschel, which was originally contained in the Encyclopædia Metropolitana.

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286.] A few words must be said on another process of finding the solution of the general differential equations given in (14); and now, as heretofore, I will take the second of the three equations to be the type-equation. It is impossible not at once to see that the equation

$$\eta = \{\text{A} \cos m\,bt + \text{B} \sin m\,bt\} \sin (mx + a) \tag{39}$$

satisfies the differential equation; A, B, m, and a being undetermined constants. Since however $\eta = 0$, when $x = 0$, and when $x = l$, whatever is the value of t, we have

$$0 = (\text{A} \cos m\,bt + \text{B} \sin m\,bt) \sin a, \tag{40}$$

$$0 = (\text{A} \cos m\,bt + \text{B} \sin m\,bt) \sin (m\,l + a), \tag{41}$$

From (40) it appears that $a = n\pi$, where n is any whole number; which however it is convenient to take to be positive, as no limitation to generality will be introduced thereby. And from (41),

$$m = -\frac{n\pi}{l}; \tag{42}$$

consequently (39) becomes of the form,

$$\eta = \left\{\text{A}_n \cos \frac{n\pi bt}{l} + \text{B}_n \sin \frac{n\pi bt}{l}\right\} \sin \frac{n\pi x}{l}. \tag{43}$$

And this equation satisfies the differential equation, and also fulfils the conditions that $\eta = 0$, when $x = 0$ and when $x = l$, whatever is the value of t.

Also, by reason of the linearity of the differential equation, it will be satisfied by a sum of expressions of the same form as the right-hand member of (43), so long as n is a positive integer: and thus

$$\eta = \Sigma . \left\{\text{A}_n \cos \frac{n\pi bt}{l} + \text{B}_n \sin \frac{n\pi bt}{l}\right\} \sin \frac{n\pi x}{l}; \tag{44}$$

the sign of summation indicating the sum of all similar expressions corresponding to integer values of n.

The A's and the B's are still undetermined. They are however to be expressed in terms of initial circumstances. These we will take to be the same as those in Art. 280; so that $\eta = \phi(x)$, and $\frac{d\eta}{dt} = b\phi'(x)$, when $t = 0$.

Consequently from (44),

$$\phi(x) = \Sigma . \text{A}_n \sin \frac{n\pi x}{l}, \tag{45}$$

$$\phi'(x) = \Sigma . \frac{n\pi}{l} \text{B}_n \sin \frac{n\pi x}{l}. \tag{46}$$

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his point of the process this method fails, because these ıtions will not determine the values of the undetermined stants, viz. the A's and the B's, in terms of the given functions which express the initial circumstances of the string. The form of the given functions is, it is to be observed, arbitrary; and the analytical solution is not perfect unless the constants are expressed in terms of them.

The following equations however, at first due to Lagrange, and afterwards deduced from a theory on definite integrals, more fully developed by Fourier and Poisson, will complete the solution. They may be found in most treatises on definite integrals.

$$\phi(x) = \frac{2}{l}\,\Sigma.\int_0^l \sin\frac{n\pi\xi}{l}\,\phi(\xi)\,d\xi\,\sin\frac{n\pi x}{l}; \tag{47}$$

$$\phi'(x) = \frac{2}{l}\,\Sigma.\int_0^l \sin\frac{n\pi\xi}{l}\,\phi'(\xi)\,d\xi\,\sin\frac{n\pi x}{l}. \tag{48}$$

Let these be substituted in (44); then we have

$$\eta = \frac{2}{l}\,\Sigma.\int_0^l \sin\frac{n\pi\xi}{l}\,\phi(\xi)\,d\xi\,\sin\frac{n\pi x}{l}\cos\frac{n\pi bt}{l}$$
$$+\frac{2}{\pi}\,\Sigma.\int_0^l \sin\frac{n\pi\xi}{l}\,\phi'(\xi)\,d\xi\,\frac{1}{n}\sin\frac{n\pi x}{l}\sin\frac{n\pi bt}{l}; \tag{49}$$

wherein the summation denoted by $\Sigma.$ indicates the sum of all these similar quantities corresponding to positive integer values of n. As this equation satisfies all the conditions of the problem, it is the complete integral of the original equation; and all the properties of the motion of the string, which have been deduced in Art. 280 from the arbitrary functions F and f, may be just as conveniently inferred from this equation.

Thus, according to this method, the motion of the string is expressed by a series of terms, each of which might exist alone, and might be the complete solution of the equation, if it agreed with the initial state of the string. The most general motion of the string however results from the coexistence or superposition of an infinity of vibratory motions, and the resulting note from the coexistence of the several notes which are due to these several single motions.

287.] We proceed now to other particular cases of the motion of elastic bodies and of their constituent molecules. And I will first consider the longitudinal vibrations of the molecules of a fine elastic rod.

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I shall assume the rod to be homogeneous, and in its natural state to be prismatic or cylindrical; so that if $\omega =$ the area of a transverse section, ω is constant throughout the length, and is infinitesimal because the rod is thin. I shall take the line which contains the centres of gravity of all thin transverse slices to be the x-axis. Although the rod is thin, that is, although the lines of its transverse section are infinitesimal in comparison of the length of the rod, yet its thickness is such that the rod is not bent by the forces acting on it.

Now we suppose the particles of the rod to be displaced longitudinally by a force acting in the direction of the length of the rod, so that each particle is displaced through a small distance along the x-axis. Moreover, we suppose that every particle in a thin transverse section or slice perpendicular to the x-axis is displaced through an equal distance; so that all particles which were in a given transverse slice before the displacement are in a transverse slice after the displacement. We also suppose that by this displacement, due to an external force, certain elastic forces are brought into action whose lines of action are parallel to the x-axis, and that the molecules subsequently vibrate under the action of these forces. It is this subsequent motion, when all other forces have ceased to act, which we shall now investigate.

Let O and A be the ends of the bar in its original state of rest; and let OA, its length, $= l$; let $\omega =$ the area of a transverse section, $\rho =$ the density. Let us consider the motion of a thin slice, whose distance from O $= x$, when $t = 0$, and whose thickness $= dx$; so that its mass $= \rho\omega\, dx$. This mass is unchanged during the motion.

Let $x + \xi$ be the distance of this elemental slice from O at the time t: ξ being a small quantity and evidently a function of both x and t. Let T be at the time t the tension drawing the slice towards O, and T $+ d$T that drawing it towards A: then, as no other force acts, the equation of motion is

$$\rho\omega\, dx \frac{d^2(x+\xi)}{dt^2} = d\mathrm{T};$$

$$\therefore\quad \rho\omega\, dx \frac{d^2\xi}{dt^2} = d\mathrm{T}. \tag{50}$$

We suppose the extension of the bar to vary according to Hooke's law: so that if E is the modulus of elasticity of the bar, according to Art. 149, Vol. III,

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$$dx + d\xi = dx\left\{1 + \frac{d\text{T}}{\text{E}}\right\};$$

$$\therefore \quad d\text{T} = \text{E}\frac{d\xi}{dx};$$

and consequently (50) becomes

$$\frac{d^2\xi}{dt^2} = \frac{\text{E}}{\rho\omega}\frac{d^2\xi}{dx^2}. \tag{51}$$

Let $$\frac{\text{E}}{\rho\omega} = a^2; \qquad \therefore \quad \frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}; \tag{52}$$

in which expressions we must observe that

(1) $\dfrac{d\xi}{dt}$ = the velocity of the slice $\rho\omega\, dx$;

(2) $\dfrac{d\xi}{dx}$ = the linear dilatation of an unit-length of the bar.

Now (52) is of precisely the same form as the equation for the vibrations of an elastic string, and is to be integrated in the same manner; but the initial circumstances of the two problems may be very different: and thus also will be the given arbitrary functions. Now I will suppose that when $t = 0$, $\xi = \Phi(x)$ and $\dfrac{d\xi}{dt} = a\phi'(x)$, where Φ and ϕ' denote arbitrary functions which are given for all values of x from 0 to l, when $t = 0$.

If an end of the bar is fixed, $\xi = 0$ and $\dfrac{d\xi}{dt} = 0$ at that point during the whole motion; and T is the pressure which the point has to bear. For a free end, $\dfrac{d\xi}{dx} = 0$, for all values of t.

Thus if the two ends are fixed, $\xi = 0$, and $\dfrac{d\xi}{dt} = 0$, when $x = 0$, and when $x = l$: so that

$$\xi = \frac{2}{l}\Sigma.\int_0^l \sin\frac{n\pi\xi}{l}\,\Phi(\xi)\,d\xi\,\sin\frac{n\pi x}{l}\cos\frac{n\pi a t}{l}$$
$$+ \frac{2}{\pi}\Sigma.\int_0^l \sin\frac{n\pi\xi}{l}\,\phi'(\xi)\,d\xi\,\frac{1}{n}\sin\frac{n\pi x}{l}\sin\frac{n\pi a t}{l}. \tag{53}$$

Hence if T is the time of a whole vibration of the slice, and n is the number of vibrations made by it in a second of time,

$$\text{T} = \frac{2l}{a} = 2l\left(\frac{\rho\omega}{\text{E}}\right)^{\frac{1}{2}}, \tag{54}$$

$$n = \frac{1}{2l}\left(\frac{\text{E}}{\rho\omega}\right)^{\frac{1}{2}}; \tag{55}$$

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so that the *note* of the bar is the same as that of a vibrating string of the same elasticity, thickness, density, and length.

If the end O is fixed, and the end A is free, then $\xi = 0$ and $\frac{d\xi}{dt} = 0$ when $x = 0$; and $\frac{d\xi}{dx}$, that is $\frac{d\Phi(x)}{dx}$, $= 0$ when $x = l$. Hence

$$\xi = \frac{2}{l}\Sigma.\int_0^l \sin\frac{(2n-1)\pi\xi}{2l}\Phi(\xi)\,d\xi \sin\frac{(2n-1)\pi x}{2l}\cos\frac{(2n-1)\pi at}{2l}$$
$$+\frac{4}{\pi}\Sigma.\int_0^l \sin\frac{(2n-1)\pi\xi}{2l}\phi'(\xi)\,d\xi\frac{1}{2n-1}\sin\frac{(2n-1)\pi x}{2l}\sin\frac{(2n-1)\pi at}{2l}; \quad (56)$$

because this value satisfies all the required conditions.

Now the values of ξ become the same whenever at is increased by any multiple of $4l$: consequently if T is the time of a complete vibration of a slice of the bar, and if n is the number of vibrations completed in a second of time,

$$\mathrm{T} = \frac{4l}{a} = 4l\left(\frac{\rho\omega}{\mathrm{E}}\right)^{\frac{1}{2}}, \quad (57)$$

$$n = \frac{1}{4l}\left(\frac{\mathrm{E}}{\rho\omega}\right)^{\frac{1}{2}}. \quad (58)$$

Thus the time of vibration is twice as long in this case as it is in the former, in which both ends of the bar are assumed to be fixed. And thus also the note due to the longitudinal vibrations of an elastic bar fixed at both ends is an octave higher than that due to it fixed at one end and having the other end free.

288.] Let us next consider the motion of the molecules of a thin elastic lamina, which is fixed along one edge, and which is otherwise free from external constraint; such as that of which the conditions of statical rest have been investigated in Art. 162—166, Vol. III. I shall employ the same symbols and the same diagram as in those Articles.

The lamina in its natural state is supposed to be plane; and to have been bent, as is assumed in those Articles, by the action of one or more external forces, and then left to itself. It subsequently vibrates by virtue of the elastic forces which have been brought into action by the original displacement. The problem in Vol. III is the determination of the curve which the lamina takes when it is bent by certain given forces: the problem herein to be treated is the subsequent motion of the lamina by reason of its elastic forces, when the original bending forces

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have ceased to act. For although the lamina is assumed to be very thin, yet its thickness is such that elastic forces are brought into action upon its being bent.

Now take the Figure 73 in Vol. III to be an enlarged representation of the lamina at the time t, except that there are no forces X and Y: then the equation of motion of the lamina may be formed as follows. Take any transverse section of it, as P″P P′, the mean fibre of which intersects the plane of (x, y) in the point (x, y). Then, if we suppose the fibre to be at rest for an instant, and L to be the moment of the elastic forces which are due to the part P″AP′ of the lamina, and act on the section about an axis perpendicular to the plane of the paper; by (178) Art. 463, Vol. III,

$$\text{L} = \frac{2kb\tau^3}{3\text{R}}; \tag{59}$$

where R is the radius of curvature of the bent lamina at the point P. But the forces of which this is the moment produce the motion of the part P″AP′ relative to the mean fibre of the section P″P′. Now let us take a thin slice of the lamina at a point between P and A; and as the lamina is very thin, we may consider its mass, which is equal to $2\rho\tau b ds'$, (ρ being the density), to be condensed into a particle at (x', y'), the place of its mean fibre; so that the expressed momentum-increments of this slice parallel to the axes of x and y are respectively

$$2\rho\tau b ds' \frac{d^2x'}{dt^2} \quad \text{and} \quad 2\rho\tau ds' \frac{d^2y'}{dt^2}. \tag{60}$$

Let us moreover assume the displacement of all particles of the lamina to be so small, that all velocity-increments parallel to the x-axis may be neglected; and we shall also assume the inclination to the x-axis of all elements of its curved fibres which are straight in their natural state, to be so small, that powers of $\frac{dy}{dx}$ above the first may be neglected. Thus

$$\frac{1}{\text{R}} = \frac{d^2y}{dx^2}, \qquad ds' = dx'. \tag{61}$$

Hence also the moment of the expressed velocity-increments of the part P″AP′ about an axis perpendicular to the plane of the paper, and passing through the mean fibre of P″P′ is

$$-2\rho\tau b \int_x^a \frac{d^2y'}{dt^2}(x'-x)\,dx'; \tag{62}$$

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and this in equilibrium with L; so that we have

$$\frac{2\,k\,b\,\tau^3}{3}\,\frac{d^2y}{dx^2} = -2\,\rho\,\tau\,b\int_x^a \frac{d^2y'}{dt^2}\,(x'-x)\,dx'. \tag{63}$$

Let us take the x-differential of this equation; x being the inferior limit of the definite integral in the right-hand member, and the differential being taken in accordance with the principles explained in Art. 178, Vol. II; then

$$\frac{k\,\tau^2}{3\,\rho}\,\frac{d^3y}{dx^3} = \int_x^a \frac{d^2y'}{dt^2}\,dx';$$

and again taking the x-differential

$$\frac{k\,\tau^2}{3\,\rho}\,\frac{d^4y}{dx^4} = -\frac{d^2y}{dt^2}; \tag{64}$$

so that the equation to the vibrating lamina is of the form

$$\frac{d^2y}{dt^2} + b^2\,\frac{d^4y}{dx^4} = 0. \tag{65}$$

This equation is not capable of integration in finite terms. But it is evident that

$$y = \{\text{A}\cos m^2\,b\,t + \text{B}\sin m^2\,b\,t\}\sin(m\,x + a) \tag{66}$$

satisfies the preceding equation; A, B, m, and a being undetermined constants: and since $y = 0$, when $x = 0$, whatever is the value of t, $a = n\,\pi$; where n is any integer number; so that

$$y = \{\text{A}_n\cos m^2\,b\,t + \text{B}_n\sin m^2\,b\,t\}\sin(m\,x + n\,\pi). \tag{67}$$

And as n may be any whole number, the complete solution will be

$$y = \Sigma.\{\text{A}_n\cos m^2\,b\,t + \text{B}_n\sin m^2\,b\,t\}\sin(m\,x + n\,\pi); \tag{68}$$

wherein the sign of summation denotes the sum of a series of similar quantities given by the several values of n, which admits of all integers.

All the undetermined coefficients must be found by means of the initial circumstances of the lamina, in the same way as similar questions have been treated in the preceding Articles.

289.] I come now to the last portion of our inquiry; viz. a statement of the general equations which express the motion of a molecule of an elastic body, homogeneous and of constant elasticity, existing in three finite dimensions in space. For this purpose, to avoid repetition, I must again refer the reader to Section 2, Chap. V, of Vol. III, wherein the theory of such

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elastic bodies has been explained, and equations have been found which express the conditions of the relative rest of the constituent molecules. From these the corresponding equations of motion are easily deduced. Suppose a molecule of a homogeneous elastic body, whose elasticity is constant, to be originally at (x, y, z), and to be displaced by the action of some external force, so that elastic forces are brought into action; and suppose it to move under the action of these last forces; the original disturbing force, and all other forces having ceased to act; and suppose its place at the time t to be $(x+\xi, y+\eta, z+\zeta)$; then the equations which express its motion are, see (160), Art. 160, Vol. III, evidently

$$\left.\begin{aligned}
(\lambda+\mu)\frac{d\Omega}{dx}+\mu\left\{\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}+\frac{d^2\xi}{dz^2}\right\} &= \rho\frac{d^2\xi}{dt^2},\\
(\lambda+\mu)\frac{d\Omega}{dy}+\mu\left\{\frac{d^2\eta}{dx^2}+\frac{d^2\eta}{dy^2}+\frac{d^2\eta}{dz^2}\right\} &= \rho\frac{d^2\eta}{dt^2},\\
(\lambda+\mu)\frac{d\Omega}{dz}+\mu\left\{\frac{d^2\zeta}{dx^2}+\frac{d^2\zeta}{dy^2}+\frac{d^2\zeta}{dz^2}\right\} &= \rho\frac{d^2\zeta}{dt^2};
\end{aligned}\right\} \quad (69)$$

in which Ω is the dilatation of a cube whose volume is the unit-volume, ρ is the constant density, λ and μ are constants depending on the elastic action of the body, and all the differentials are partial.

These equations may be also put into another form; take the x-, y-, and z- partial differentials of these respectively, and add them; then, since

$$\Omega = \frac{d\xi}{dx}+\frac{d\eta}{dy}+\frac{d\zeta}{dz}, \quad (70)$$

the sum becomes

$$(\lambda+2\mu)\left\{\frac{d^2\Omega}{dx^2}+\frac{d^2\Omega}{dy^2}+\frac{d^2\Omega}{dz^2}\right\} = \rho\frac{d^2\Omega}{dt^2}; \quad (71)$$

by means of which equation Ω is to be determined as a function of x, y, z, t. If this value of Ω is substituted in the three equations of (69), ξ, η, ζ may be determined in terms of x, y, z, and t, and the problem will then be completely solved.

I may in passing observe, that these equations are homogeneous on the principles of Art. 273.

290.] It will be beyond the object of our work to enter into a discussion on the equations of the preceding Articles, as it would lead us into wave-motion, and other subjects of a similar

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kind. Two special cases however, which have already been treated otherwise, are so easily deduced from these general equations, that it is desirable to insert them.

Equations (69) express the motion of a constituent molecule of any elastic body under the action of elastic forces which are made active by reason of a displacement produced by some external force. And as no limitation is made as to the dimensions of the body, the equations are applicable to thin plates or membranes, and to thin strings. These two cases I will consider as briefly as possible.

Imagine a series of molecules through the body, in its natural state, forming a thin plate or membrane, the bounding surfaces of which are planes at an infinitesimal distance apart. Further, imagine this thin plane membrane to be separated from the body, so that no forces, elastic or other, act normally to its plane surfaces, and its constituent molecules are at rest by the action of internal forces only. If this is homogeneous, and of constant elasticity, like the body of which it is a part, so is it a thin elastic membrane in the ordinary meaning of the term, as the parchment of a tambourine or the parchment in a drum-head.

Now we suppose this thin elastic membrane to lie in a plane; that is, we imagine a plane section of it equally, and thus infinitesimally, distant from its two plane surfaces, to be a geometrical plane. Moreover, we suppose it to be stretched with equal tension along all its edges by a stretching force which we will call τ. On the membrane thus stretched and at rest, we suppose an external force to act, so that the molecules are displaced: hereby elastic forces are brought into action tending to restore the membrane to its original position. We propose to determine the equations which express the motion of a particle due to these elastic forces, the disturbing force having acted once for all.

Let the system of reference be as follows: take the plane, which is equally distant from the two plane forces of the membrane, to be that of (x, y), and any point in that plane for the origin. Let ρ be the density and τ the thickness of the membrane; let the place of the type-molecule, whose motion we will consider, be $(x, y, 0)$ when it is at rest; and let its place at the time t be $(x + \xi, y + \eta, 0 + \zeta)$, so that ξ, η, ζ are the axial components of its displacement at the time t. Then, replacing Ω by its value given in (70), we have from (69)

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$$\left.\begin{aligned}(\lambda+2\mu)\left(\frac{d^2\xi}{dx^2}+\frac{d^2\xi}{dy^2}\right)&=\rho\frac{d^2\xi}{dt^2},\\(\lambda+2\mu)\left(\frac{d^2\eta}{dx^2}+\frac{d^2\eta}{dy^2}\right)&=\rho\frac{d^2\eta}{dt^2},\\\mu\left(\frac{d^2\zeta}{dx^2}+\frac{d^2\zeta}{dy^2}\right)&=\rho\frac{d^2\zeta}{dt^2}.\end{aligned}\right\}\qquad(72)$$

These equations may be integrated by the process explained in Art. 286, and the integrals will involve a series and certain arbitrary functions depending on the initial values. These will express vibrations of the molecules; of which the periodic time will be equal for those parallel to the x- and y-axes; that corresponding to the z-vibration will be different. The sound yielded by it will be much more intense, and will be the *note* of the membrane. The coefficients λ and μ will depend on the stretching weight τ.

291.] Lastly, let us suppose the elastic body to be a string which in its natural state is straight. That is, let us imagine such a string to be taken out of a homogeneous elastic body of constant elasticity, and to be under the action of a stretching force τ along its length, and to be subject at its surface to no other forces. We imagine this string to be displaced by the action of an external force; hereby certain elastic forces are brought into action, which tend to restore the string to its original straight state. The problem is, the determination of the motion of the molecules of the string under the action of these elastic forces.

Let us suppose the string at rest to lie along the axis of x; and let us consider the motion of a molecule whose place at rest is $(x, 0, 0)$; and let us suppose its place at the time t to be $(x+\xi, 0+\eta, 0+\zeta)$, where ξ, η, ζ are the axial components of the displacement at that time. Then (69) become

$$\left.\begin{aligned}(\lambda+2\mu)\frac{d^2\xi}{dx^2}&=\rho\frac{d^2\xi}{dt^2},\\\mu\frac{d^2\eta}{dx^2}&=\rho\frac{d^2\eta}{dt^2},\\\mu\frac{d^2\zeta}{dx^2}&=\rho\frac{d^2\zeta}{dt^2};\end{aligned}\right\}\qquad(73)$$

as these equations have the same form as (14) in the present Chapter, they may be integrated by the processes already explained, and will lead to the same results.

And here I must conclude the investigations on the motion

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of elastic bodies and their molecules. The inquiry is, for the subject, very imperfect; the completion of it would lead me far into the theory of definite integrals, the properties of periodic series simple and double, into wave-motion, &c., all of which would require more space than we can afford, and which might be inappropriate to a treatise, in which I have confined myself to elementary subjects. I cannot however refrain from recommending to the student the study of Leçons sur l'Elasticité des corps solides, by M. Lamé, Paris, 1852; and chiefly to Leçons 8me, 9me, and 10me, wherein he will find the subjects of the last two Articles especially treated of.

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CHAPTER XI.

THEORETICAL DYNAMICS.

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[It will be observed that many terms and symbols employed in this Chapter differ from the corresponding ones of the previous parts of the Treatise: this arises in part from the fact that Professor Donkin had not seen the previous Chapters when this was written. In the unavoidable absence of Professor Donkin, it has not been thought desirable to change either the one or the other. The advanced student indeed, for whom especially this Chapter is intended, will not require any alteration: he will understand the different terms and symbols by means of either the context or the explanations which are given. Whatever has been added to Professor Donkin's work is enclosed in square brackets. It should also be noticed that his MS. bears date Sept. 6, 1860.]

292.] The object of this Chapter is to give some account of the recent progress of theoretical dynamics. But no attempt will be made to follow accurately a historical order, or to assign every step in detail to its proper author. Such a plan would be hardly in accordance with the design of this Treatise, and it is moreover rendered unnecessary by Mr. Cayley's "Report on Dynamics," lately published *.

The reader's attention is requested to the following explanations of notation:

Throughout this Chapter total differentiation with respect to t (the time) will be denoted by accents; and accents will be used for no other purpose. Thus, instead of $\frac{d^2u}{dt^2} + \frac{d^2v}{dt^2}$ we shall write either $u'' + v''$ or $(u + v)''$, ..., and if u be a function con-

* Report of the British Association for 1857.

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taining t explicitly, and also involving $x, y, \ldots$, which are functions of t, we shall have

$$u' = \frac{du}{dt} + \frac{du}{dx}x' + \frac{du}{dy}y' + \ldots;$$

where $\frac{du}{dt}$ signifies the partial differential coefficient of u with respect to t, taken so far as t appears explicitly in u.

But no other distinction will in general be made, by means of notation, between the various possible meanings of differential coefficients; the interpretation of the symbols, if not clear from the context, will be explained in each case.

Secondly, expressions of the form $\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$ are of such frequent occurrence, that it is desirable to have a recognised abbreviation for them. The following has been found convenient, and will be adopted, namely*,

$$\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx} = \frac{d(u, v)}{d(x, y)}.$$

293.] The theorem of D'Alembert reduces the mathematical statement of every dynamical problem to the expression of conditions of equilibrium; and when these conditions are put in the general form, assigned by the principle of virtual velocities, there results a single formula, which may be written thus:

$$\Sigma . m(x''\delta x + y''\delta y + z''\delta z) = \Sigma(\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z)\dagger: \qquad (1)$$

in which x, y, z are the coordinates of the mass m, referred to rectangular axes fixed in space, and $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ are the components of the force applied at the point x, y, z.

It is necessary to observe, that the force here meant is not the so-called "accelerating force", that is, the force which would act on a unit of mass; but the total force, of whatever kind, which is impressed at the point (x, y, z). Otherwise the

* More generally, the symbol $\frac{d(u, v, w, \ldots)}{d(x, y, z, \ldots)}$ stands for the "Jacobian" determinant, of which the constituents are

$$\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}, \ldots; \qquad \frac{dv}{dx}, \frac{dv}{dy}, \frac{dv}{dz}, \ldots; \qquad \frac{dw}{dx}, \frac{dw}{dy}, \frac{dw}{dz}, \ldots.$$

This notation was proposed some time ago by the writer of this chapter (see Phil. Trans. for 1854, p. 72), and has since received the sanction of Mr. Cayley (Report on Dynamics, p. 5).

† [See equation (40), Art. 50.]

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formula would not include the case in which all or any of the forces are to be considered as pressures acting merely on mathematical surfaces, lines, or points; and having no relation to the magnitude of the masses which they tend to move. It is true that such forces are only mathematical fictions; but so are the conditions of almost all mechanical problems, treated as we are at present obliged to treat them.

It would however be out of place to enter into the questions suggested by this remark, because, for the purposes of this chapter, we are not concerned with the nature of the problem which gives rise to the formula (1), further than is necessary for a clear understanding of the meaning of the symbols.

In the most general case which occurs in practice, the values of X, Y, Z, at the time t, may depend in a given manner upon the time, the positions of all the points of the system, and the velocities and directions of their movements at the instant considered. In other words, X, Y, Z may be given functions of t, of all the coordinates $x, y, z, \ldots$, and of their first differential coefficients $x', y', z', \ldots$. This most general case has not, except in special problems, as yet been treated successfully, and we shall find it necessary to limit the significations of X, Y, Z; but the problems excluded by the limitation are comparatively unimportant.

294.] The meaning of the symbol of variation δ may be explained as follows: If $x, y, z, \ldots$ be the values of the coordinates in the actual position of the system at the time t, then $x+\delta x$, $y+\delta y$, $z+\delta z, \ldots$ are the values belonging to any other position which the system might have had at that time without violating the conditions by which its possible displacements are limited; provided only that the two positions be infinitely near to one another, so that $\delta x, \delta y, \ldots$ are infinitesimal.

Whatever be the nature of the conditions just mentioned, by which the motion of the system is constrained, they may always be supposed to be expressed by means of a certain number of equations of condition

$$L_1 = 0, \quad L_2 = 0, \ldots L_m = 0; \tag{2}$$

in which $L_1, L_2, \ldots$ are given functions of any or all of the coordinates, and may also contain t explicitly. Suppose n to be the whole number of coordinates involved in the formula (1), and m the number of equations of condition, m will be in all cases less than n; otherwise those equations would imply either a de-

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terminate fixed position of the system, or a determinate motion independent of the forces.

The formula (1) must be satisfied for every set of values of $\delta x, y, \ldots$ which satisfy the m equations,

$$\left.\begin{aligned} \frac{d\mathrm{L}_1}{dx}\delta x + \frac{d\mathrm{L}_1}{dy}\delta y + \ldots &= 0, \\ \frac{d\mathrm{L}_2}{dx}\delta x + \frac{d\mathrm{L}_2}{dy}\delta y + \ldots &= 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ \frac{d\mathrm{L}_m}{dx}\delta x + \frac{d\mathrm{L}_m}{dy}\delta y + \ldots &= 0. \end{aligned}\right\} \tag{3}$$

By means of these last equations any m out of the n quantities $\delta x, \delta y, \ldots$ may be expressed in terms of the rest, and when their values so expressed are substituted in (1), that formula will involve only the remaining $n-m$ variations; and since the values of these may be taken arbitrarily, without violating the m equations of condition, the coefficient of each must be separately equated to zero; and thus we obtain $n-m$ simultaneous linear differential equations of the second order, that is, as many as there are independent coordinates.

This process has been described briefly, because the reader is supposed to be already familiar with it*, at least in principle, and because we shall not have to perform it actually.

295.] Instead however of eliminating the variations in the manner just explained, we are of course at liberty to make use of any equation that can be obtained from (1) by substituting for $\delta x, \delta y, \ldots$ any admissible set of values; that is, any set which satisfies the m equations (3).

Among the infinite number of ways in which admissible values may be chosen for the variations $\delta x, \delta y, \ldots$, there are two which require particular notice.

First, let us suppose (what is usually, but not necessarily, the case) that every position of the system which is possible at any one time, is possible at any other time; which is the same thing as supposing that none of the equations of condition involve t explicitly. In this case, it is evident, that among the positions which the system might have had at the time t is that which it actually has at the time $t+dt$; but at the time $t+dt$, the coordinates $x, y, \ldots$ have become $x+dx$, $y+dy, \ldots$, where $dx = x'dt$,

* [See Arts. 51 and 52.]

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$dy = y'dt, \ldots$; so that we are at liberty to take as an admissible set of variations the displacements which actually happen in the time dt. In fact, since we are now supposing that the equations $\text{L}_1 = 0, \ldots$ do not involve t, the equations $\text{L}'_1 = 0, \ldots$, are

$$\frac{d\text{L}_1}{dx}x' + \frac{d\text{L}_1}{dy}y' + \ldots = 0, \ldots;$$

and comparing these with (3), we see that $\delta x, \delta y, \ldots$, may be taken proportional to $x', y', \ldots$, that is, to $dx, dy, \ldots$. But if L_1 contained t explicitly we should have

$$\text{L}'_1 = \frac{d\text{L}_1}{dt} + \frac{d\text{L}_1}{dx}x' + \frac{d\text{L}_1}{dy}y' + \ldots = 0,$$

which cannot be made to coincide with the equation

$$\delta\text{L}_1 = \frac{d\text{L}_1}{dx}\delta x + \frac{d\text{L}_1}{dy}\delta y + \ldots = 0$$

by any values whatever of $\delta x, \delta y, \ldots$*.

In the case now supposed, however, when the values $x'dt$, $y'dt, \ldots$, or $dx, dy, \ldots$, are substituted for $\delta x, \delta y, \ldots$ in (1), that formula becomes

$$\Sigma.m(x'x'' + y'y'' + z'z'')dt = \Sigma(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz);$$

or, if we put T for the vis viva of the system, that is,

$$\text{T} = \frac{1}{2}\Sigma.m(x'^2 + y'^2 + z'^2) = \frac{1}{2}\Sigma.mv^2;$$

we obtain

$$d\text{T} = \Sigma(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz),$$

an equation which we shall meet with afterwards in a somewhat different form.

296.] The second way of choosing admissible values for δx, $\delta y, \ldots$, is applicable in all cases without exception, and may be explained as follows:

The n coordinates $x, y, \ldots$ being subject to m equations of condition, it follows that any $n-m$ of the coordinates may be considered as absolutely independent; that is, their values might at any time t be assumed arbitrarily without violating the laws of constraint expressed by the equations of condition; and since the same thing is true at the time $t+dt$, it follows, that not only the $n-m$ coordinates, but also their first differential coefficients $x', y', \ldots$ might be arbitrarily assumed at the time t; but the values of these $2(n-m)$ quantities being given, those of the remaining m coordinates and of their first differential coefficients

* [This circumstance has been already alluded to in Art. 63.]

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are determinate. In fact, if $\xi, \eta, \ldots$ be these remaining coordinates, the m equations of condition would suffice to express each of them in terms of $x, y, \ldots$, and t, so that we should have m equations, such as

$$\xi = f(x, y, \ldots t);$$

from which we should get by differentiation

$$\xi' = \frac{df}{dt} + \frac{df}{dx}x' + \frac{df}{dy}y' + \ldots;$$

where it is evident that the values of both ξ and ξ' are given at the time t, if those of $x, y, \ldots, x', y', \ldots$ are given.

Now the motion of the system, under the action of the given forces, is completely determined if the positions of all its points, and the velocities and directions of their movements, be given at a determinate time; that is, if the values of all the coordinates $x, y, \ldots$, and their first differential coefficients $x', y', \ldots$, be given at that time; and since it has just been seen that all these quantities are given if any $n-m$ of the coordinates, with their first differential coefficients, be given, we conclude that the whole motion is determined if the values of these $2(n-m)$ quantities be given at any one time; it is convenient to take the instant when $t = 0$ for the time in question, and we may call the values of any quantities at that time their initial values.

From these considerations, it is easy to conclude that the final integral equations of the problem must contain $2(n-m)$ arbitrary constants, and no more; that is, that the values of all the coordinates must be expressible in terms of $2(n-m)$ arbitrary constants and t; for otherwise, the number of coordinates and first differential coefficients which it would be possible to assume arbitrarily at a given time would be either more or less than $2(n-m)$. The same conclusion follows from the theory of differential equations.

Hence we are in all cases at liberty to suppose that the actual value of every one of the coordinates at the time t is expressible by an equation of the form

$$x = f(a, b, \ldots t);$$

where $a, b, \ldots$ are the arbitrary constants, of which the number is twice that of the independent coordinates.

These constants may be the initial values of some set of independent coordinates and of their first differential coefficients, and must be expressible as functions of such initial values.

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297.] Now if we suppose the initial positions and velocities to receive infinitesimal alterations, or if, which comes to the same thing, we suppose the constants $a, b, \ldots$ to be changed into $a+\delta a$, $b+\delta b, \ldots$, the values of $x, y, \ldots$, at the time t, will be changed into $x+\delta x$, $y+\delta y, \ldots$, where

$$\delta x = \frac{dx}{da}\delta a + \frac{dx}{db}\delta b + \ldots,$$

$$\delta y = \frac{dy}{da}\delta a + \frac{dy}{db}\delta b + \ldots;$$

the partial differential coefficients $\frac{dx}{da}, \ldots$ being taken on the hypothesis that $x, y, \ldots$ are expressed, as above supposed, in terms of $a, b, \ldots, t$.

The values of $\delta x, \delta y, \ldots$, thus formed, are distinguished from other admissible sets of values by important properties, which we proceed to point out.

First, if we suppose $a, b, \ldots$ to have the values belonging to the actual motion of the system, so that the point (x, y, z) actually describes the path defined by the equations $x = f(a, b, \ldots, t), \ldots$, then the values $a+\delta a$, $b+\delta b, \ldots$ correspond to a motion which does not actually take place, but which might take place under the action of the existing forces, and would take place if the initial circumstances were suitably altered; so that the substitution, at every instant, of $x+\delta x$ for $x, \ldots$ would change the actual paths and velocities of all the points of the system into others not merely consistent with the given equations of condition, but consistent also with the action of the forces. All such paths and velocities may be called "dynamically possible." But if the values of $\delta x, \delta y, \ldots$ were merely chosen so as to be consistent with the equations of condition, without any further limitation, then the substitution at every instant of $x+\delta x$ for x, $\ldots$, would change the actual paths and velocities into others, which, though not inconsistent with the given laws of constraint of the system, could not be produced by the action of the existing forces. Such paths and velocities may be called "geometrically possible*" though dynamically impossible.

298.] But there is another, and in some respects more important, distinction.

* The expressions "dynamically possible" and "geometrically possible" are Sir W. R. Hamilton's.

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The values of δa, δb, ... are arbitrary infinitesimal constants. Let Δa, Δb, ... be any other set of similar values; and let δu, Δu be the increments of any function u, corresponding to the two sets of increments of the constants; so that if u be expressed as a function of $a, b, c, \ldots$, with or without t, we shall have

$$\delta u = \frac{du}{da}\delta a + \frac{du}{db}\delta b + \ldots,$$

$$\Delta u = \frac{du}{da}\Delta a + \frac{du}{db}\Delta b + \ldots.$$

Now suppose that in the above value of δu we change $a, b, \ldots$ into $a+\Delta a$, $b+\Delta b, \ldots$, without altering the values of $\delta a, \delta b, \ldots$; the corresponding increment of δu will be

$$\Delta\delta u = \frac{d^2u}{da^2}\Delta a.\delta a + \frac{d^2u}{da\,db}(\delta a.\Delta b + \delta b.\Delta a) + \ldots;$$

but the same expression will be obtained for $\delta\Delta u$ from the second of the above equations; consequently,

$$\Delta\delta u = \delta\Delta u. \tag{4}$$

In this equation u may evidently be any function of the coordinates $x, y, z, \ldots$ and their differential coefficients of all orders; the property expressed by it distinguishes those variations of u which are due to variations of the constants $a, b, \ldots$ from those which, though otherwise admissible, arise in a different manner. In fact, it will be found in general, either that the symbols $\Delta\delta$, $\delta\Delta$ are unmeaning, or that the above equation is not true.

299.] Lastly, we must notice a property which belongs to the variations denoted by δ or Δ without any limitation, namely, that the operations δ, or Δ, and $\frac{d}{dt}$ are commutative; that is,

$$\delta(u') = (\delta u)'; \tag{5}$$

where u is any function of $x, y, \ldots$ (with their differential coefficients) and t, and δx, δy, ... may be considered as perfectly arbitrary functions of t, subject to the sole restriction of being infinitesimal. In fact, the meaning of $\delta(u')$ is $(u+\delta u)'-u'$, that is, is $(\delta u)'$.

300.] Now let $x, y, \ldots$ be any n variables, functions of t, and subject to m equations of condition. Also let ξ, η, ... be other variables, of which the number is not less than $n-m$, and so connected with the former set of variables by given equations, which may involve t explicitly but may not involve the differ-

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ential coefficients of either set, that any variable of one set may be expressed as a function of variables of the other set, with or without t, by equations such as

$$\left.\begin{aligned} x &= \phi(\xi, \eta, \ldots, t), \\ \xi &= \psi(x, y, \ldots, t). \end{aligned}\right\} \tag{6}$$

It is to be observed, that the expressions on the right of these equations are, to a certain extent, indeterminate in form; for any function of $x, y, \ldots$ and t may be variously transformed by means of the given equations of condition; and the same may be said of any function of $\xi, \eta, \ldots$, if the number of the latter variables be greater than $n-m$.

Suppose then that u is any function of $t, x, y, \ldots, x', y', \ldots, x'', y'', \ldots$; and let r be the order of the highest differential coefficient contained in u. By means of the equations, such as the first of (6), $x, y, \ldots$ can be expressed as functions of $t, \xi, \eta, \ldots$; $x', y', \ldots$ as functions of $t, \xi, \eta, \ldots, \xi', \eta', \ldots$; and so on; so that u can be transformed into a function of $t, \xi, \eta, \ldots, \xi', \eta', \ldots$, in which the highest differential coefficient will still be of the order r. Also, $\delta x, \delta y, \ldots$ can be transformed by means of the equations

$$\delta x = \frac{dx}{d\xi}\delta\xi + \frac{dx}{d\eta}\delta\eta + \ldots;$$

where $\frac{dx}{d\xi}, \frac{dx}{d\eta}, \ldots$ are given functions of $t, \xi, \eta, \ldots$.

Let $\mathrm{E}_x u$ be for a moment an abbreviation for

$$\frac{du}{dx} - \left(\frac{du}{dx'}\right)' + \left(\frac{du}{dx''}\right)'' - \ldots,$$

the series being continued until it terminates of itself. Then the expression

$$\mathrm{E}_x u \,.\, \delta x + \mathrm{E}_y u \,.\, \delta y + \mathrm{E}_z u \,.\, \delta z + \ldots$$

can be transformed into another, involving the variables $\xi, \eta, \ldots$, with their variations and differential coefficients, instead of $x, y, \ldots$.

It is a known theorem in the Calculus of Variations that the result of this transformation is an expression of the same form, namely,

$$\mathrm{E}_\xi u \,.\, \delta\xi + \mathrm{E}_\eta u \,.\, \delta\eta + \ldots;$$

where u only differs from the former u in being expressed in terms of the new variables.

The direct demonstration* of this theorem in its general form

* For an indirect demonstration, see Lagrange, Mécanique Analytique, 2de partie, 4me section, 6, or De Morgan's Diff. Calc., p. 519.

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is somewhat complicated, and need not be given here, because the only case with which we are concerned is that in which u contains no differential coefficient of a higher order than the first, or

$$u = \text{F}(t, x, y, \ldots x', y', \ldots).$$

The theorem may then be conveniently written thus:

$$\text{Σ}.\left\{\left(\frac{du}{dx'}\right)' - \frac{du}{dx}\right\}\delta x = \text{Σ}.\left\{\left(\frac{du}{d\xi'}\right)' - \frac{du}{d\xi}\right\}\delta\xi; \tag{7}$$

and this admits of the following simple demonstration, due in principle to Sir W. R. Hamilton.

301.] Since u, when expressed in terms of the new variables, $\xi, \eta, \ldots$, will contain $\xi', \eta', \ldots$, only because it originally contains $x', y', \ldots$, we shall have

$$\frac{du}{dx'} = \frac{du}{d\xi'}\frac{d\xi'}{dx'} + \frac{du}{d\eta'}\frac{d\eta'}{dx'} + \ldots;$$

where the differentiations in $\frac{d\xi'}{dx'}, \ldots$ are performed on the supposition that $\xi', \eta', \ldots$ are expressed in terms of $t, x, y, \ldots, x', y', \ldots$: now ξ being expressed in terms of $t, x, y, \ldots$, we have

$$\xi' = \frac{d\xi}{dt} + \frac{d\xi}{dx}x' + \frac{d\xi}{dy}y' + \ldots;$$

and since $\frac{d\xi}{dt}, \frac{d\xi}{dx}, \ldots$, do not contain $x', y', \ldots$, we obtain at once by differentiation

$$\frac{d\xi'}{dx'} = \frac{d\xi}{dx}, \quad \frac{d\xi'}{dy'} = \frac{d\xi}{dy}, \ldots; \tag{8}$$

in like manner we should find

$$\frac{d\eta'}{dx'} = \frac{d\eta}{dx}, \quad \frac{d\eta'}{dy'} = \frac{d\eta}{dy}, \ldots:$$

hence the above expression for $\frac{du}{dx'}$ becomes

$$\frac{du}{dx'} = \frac{du}{d\xi'}\frac{d\xi}{dx} + \frac{du}{d\eta'}\frac{d\eta}{dx} + \ldots:$$

similarly we should have

$$\frac{du}{dy'} = \frac{du}{d\xi'}\frac{d\xi}{dy} + \frac{du}{d\eta'}\frac{d\eta}{dy} + \ldots;$$

and so on: whence, multiplying the first of these equations by δx, the second by $\delta y, \ldots$, and observing that

$$\frac{d\xi}{dx}\delta x + \frac{d\xi}{dy}\delta y + \ldots = \delta\xi,$$

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we obtain by addition,

$$\frac{du}{dx'}\delta x + \frac{du}{dy'}\delta y + \ldots = \frac{du}{d\xi'}\delta\xi + \frac{du}{d\eta'}\delta\eta + \ldots;$$

and differentiating this last equation with respect to t, and observing (Art. 299) that $(\delta x)' = \delta x'$, ...,

$$\left(\frac{du}{dx'}\right)'\delta x + \frac{du}{dx'}\delta x' + \ldots = \left(\frac{du}{d\xi'}\right)'\delta\xi + \frac{du}{d\xi'}\delta\xi' + \ldots$$

Now δu may be expressed in either of the following ways:

$$\delta u = \frac{du}{dx}\delta x + \frac{du}{dx'}\delta x' + \frac{du}{dy}\delta y + \ldots,$$

$$\delta u = \frac{du}{d\xi}\delta\xi + \frac{du}{d\xi'}\delta\xi' + \frac{du}{d\eta}\delta\eta + \ldots;$$

and if the first of these values be subtracted from the left-hand member, and the second from the right-hand member of the above equation, the result is the equation (7), which was to be established.

302.] We now return to the dynamical formula (1), Art. 293. In that formula, the position of the system at the time t is assigned by means of the rectangular coordinates $x, y, z, \ldots$; but it is evident that any other set of variables, $\xi, \eta, \zeta, \ldots$, connected with $x, y, z, \ldots$, in the manner supposed in Art. 300, would answer the same purpose. We may extend the meaning of the word "coordinates" so as to include all sets of variables of which the values at the time t determine the position of the system at that time. For instance, the position of a rigid system which has one fixed point may be defined in several ways by means of three angles, which may be called the "coordinates" of the system.

Now the theorem (7), Art. 300, enables us to express the left-hand member of the formula (1) in a form adapted to any system of coordinates whatever, in the following manner:

Let T* denote as before the vis viva of the system; then, in terms of the original coordinates, we have

$$2\mathrm{T} = \Sigma . m(x'^2 + y'^2 + z'^2);$$

but when T is expressed in terms of any other coordinates, $\xi, \eta, \ldots$, it will become in general a function of $\xi, \eta, \ldots, \xi', \eta', \ldots$, with or without t, not containing any differential coefficients of

* [Vis viva, as used here and in Art. 295, is one-half of the quantity which has heretofore been called by that name.]

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a higher order than the first. Hence we may put T for u in the equation (7), and, observing that

$$\frac{d\mathrm{T}}{dx'} = \Sigma . m x', \qquad \frac{d\mathrm{T}}{dx} = 0,$$

we obtain

$$\Sigma . m x'' \delta x = \Sigma . \left\{ \left(\frac{d\mathrm{T}}{d\xi'} \right)' - \frac{d\mathrm{T}}{d\xi} \right\} \delta \xi ;$$

which is the required form.

The terms $\mathrm{X}\delta x + \mathrm{Y}\delta y + \ldots$, on the right of (1), when expressed in terms of the new coordinates, will take the form $\mathrm{P}\delta\xi + \mathrm{Q}\delta\eta + \ldots$, where $\mathrm{P}, \mathrm{Q}, \ldots$ are functions of $\xi, \eta, \ldots$ with or without t; so that the equation (1) is finally reducible to the form

$$\Sigma . \left\{ \left(\frac{d\mathrm{T}}{d\xi'} \right)' - \frac{d\mathrm{T}}{d\xi} \right\} \delta \xi = \Sigma . \mathrm{P} \delta \xi . \tag{9}$$

In the most usual and important problems, $\mathrm{X}, \mathrm{Y}, \mathrm{Z}, \ldots$ are the partial differential coefficients with respect to $x, y, z, \ldots$ of a function U, called the force-function, which may also contain t, but does not contain $x', y', \ldots$: in this case we have

$$\mathrm{X}\delta x + \mathrm{Y}\delta y + \ldots = \delta \mathrm{U} ;$$

and the right-hand member of (9) is obtained by deriving $\delta\mathrm{U}$ from U expressed in terms of the new variables: thus

$$\delta \mathrm{U} = \frac{d\mathrm{U}}{d\xi} \delta \xi + \frac{d\mathrm{U}}{d\eta} \delta \eta + \ldots ;$$

and the equation may then be written in the form

$$\Sigma . \left\{ \left(\frac{d\mathrm{T}}{d\xi'} \right)' - \frac{d\mathrm{T}}{d\xi} - \frac{d\mathrm{U}}{d\xi} \right\} \delta \xi = 0 : \tag{10}$$

this may be abridged by putting $\mathrm{T} + \mathrm{U} = \mathrm{W}$; for, since U does not contain $\xi', \ldots, \frac{d\mathrm{T}}{d\xi'} = \frac{d\mathrm{W}}{d\xi'}, \ldots$; so that the formula becomes

$$\Sigma . \left\{ \left(\frac{d\mathrm{W}}{d\xi'} \right)' - \frac{d\mathrm{W}}{d\xi} \right\} \delta \xi = 0. \tag{11}$$

If the coordinates $\xi, \eta, \ldots$ be independent, that is, subject to no equations of condition, the coefficients of $\delta\xi, \delta\eta, \ldots$ must separately vanish; so that (11) is equivalent to the system of equations

$$\left(\frac{d\mathrm{W}}{d\xi'} \right)' = \frac{d\mathrm{W}}{d\xi}, \qquad \left(\frac{d\mathrm{W}}{d\eta'} \right)' = \frac{d\mathrm{W}}{d\eta}, \ldots . \tag{12}$$

We shall refer to these as the "Lagrangian" equations; a name given to them by Mr. Cayley.

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303.] It will be desirable to illustrate the preceding formulæ by some examples before proceeding further. First then, let it be required to express the equations of motion of a single material point by means of polar coordinates.

Let m be the mass of the point, x, y, z its rectangular coordinates, X, Y, Z the components of the force acting on m, so that the equations of motion in their primitive form are included in the formula

$$m(x''\delta x + y''\delta y + z''\delta z) = \mathrm{X}\delta x + \mathrm{Y}\delta y + \mathrm{Z}\delta z. \qquad (13)$$

In polar coordinates we have, according to the usual notation,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta;$$

and we must express $2\mathrm{T} = m(x'^2 + y'^2 + z'^2)$ in terms of $r, \theta, \phi, r', \theta', \phi'$. Differentiating, we find

$$\begin{aligned} x' &= r'\sin\theta\cos\phi + \theta' r\cos\theta\cos\phi - \phi' r\sin\theta\sin\phi, \\ y' &= r'\sin\theta\sin\phi + \theta' r\cos\theta\sin\phi + \phi' r\sin\theta\cos\phi, \\ z' &= r'\cos\theta \qquad\quad - \theta' r\sin\theta; \end{aligned}$$

hence we easily obtain

$$2\mathrm{T} = m\{r'^2 + r^2\theta'^2 + r^2(\sin\theta)^2\phi'^2\};$$

and we know (Art. 302) that the left-hand member of (13) will become

$$\left\{\left(\frac{d\mathrm{T}}{dr'}\right)' - \frac{d\mathrm{T}}{dr}\right\}\delta r + \left\{\left(\frac{d\mathrm{T}}{d\theta'}\right)' - \frac{d\mathrm{T}}{d\theta}\right\}\delta\theta + \left\{\left(\frac{d\mathrm{T}}{d\phi'}\right)' - \frac{d\mathrm{T}}{d\phi}\right\}\delta\phi.$$

Now
$$\frac{d\mathrm{T}}{dr'} = mr', \qquad \frac{d\mathrm{T}}{dr} = mr\{\theta'^2 + (\sin\theta^2)\phi'^2\},$$
$$\frac{d\mathrm{T}}{d\theta'} = mr^2\theta', \qquad \frac{d\mathrm{T}}{d\theta} = mr^2\sin\theta\cos\theta\,\phi'^2,$$
$$\frac{d\mathrm{T}}{d\phi'} = mr^2(\sin\theta)^2\phi', \quad \frac{d\mathrm{T}}{d\phi} = 0;$$

and these values reduce the above expression to the following:

$$m\{r'' - r\theta'^2 - r(\sin\theta)^2\phi'^2\}\delta r$$
$$+ m\{(r^2\theta')' - r^2\sin\theta\cos\theta\,\phi'^2\}\delta\theta + m\{r^2(\sin\theta)^2\phi'\}'\delta\phi.$$

The right-hand member of (13) will always be reducible to the form $\mathrm{P}\delta r + \mathrm{Q}\delta\theta + \mathrm{R}\delta\phi$; and in the case in which X, Y, Z are of the form $\frac{d\mathrm{U}}{dx}, \frac{d\mathrm{U}}{dy}, \frac{d\mathrm{U}}{dz}$, then also P, Q, R will be of the form $\frac{d\mathrm{U}}{dr}, \frac{d\mathrm{U}}{d\theta}, \frac{d\mathrm{U}}{d\phi}$.

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If the motion of m be unconstrained, δr, $\delta\theta$, $\delta\phi$ are all arbitrary, and the formula (13) breaks up into three separate equations.

304.] As a second example, let us consider the transformation from fixed rectangular axes to moving rectangular axes with the same origin.

*Let x, y, z be the coordinates of the material point m, referred to the fixed axes; ξ, η, ζ the coordinates of the same point, referred to the moving axes; and let the position of the moving system at the time t be defined in the usual manner by the equations

$$x = a_1\xi + a_2\eta + a_3\zeta,\ y = b_1\xi + b_2\eta + b_3\zeta,\ z = c_1\xi + c_2\eta + c_3\zeta;$$

where the nine direction-cosines $a_1, a_2, \ldots$ are given functions of t. It will be convenient to introduce the usual symbols $\omega_1, \omega_2, \omega_3$ for the angular velocities of the moving system of axes, estimated about the axes of ξ, η, ζ respectively. Let a rotation about the axis of ζ be positive when its direction is such that the axis of ξ is following the axis of η; then, with similar conventions as to the other axes, we shall have

$$a_3 a_2' + b_3 b_2' + c_3 c_2' = -(a_2 a_3' + b_2 b_3' + c_2 c_3') = \omega_1,$$
$$a_1 a_3' + b_1 b_3' + c_1 c_3' = -(a_3 a_1' + b_3 b_1' + c_3 c_1') = \omega_2,$$
$$a_2 a_1' + b_2 b_1' + c_2 c_1' = -(a_1 a_2' + b_1 b_2' + c_1 c_2') = \omega_3.$$

Now differentiating the equations $x = a_1\xi + a_2\eta + a_3\zeta, \ldots$ we obtain

$$x' = a_1\xi' + a_2\eta' + a_3\zeta' + a_1'\xi + a_2'\eta + a_3'\zeta,$$
$$y' = b_1\xi' + b_2\eta' + b_3\zeta' + b_1'\xi + b_2'\eta + b_3'\zeta,$$
$$z' = c_1\xi' + c_2\eta' + c_3\zeta' + c_1'\xi + c_2'\eta + c_3'\zeta;$$

and hence, observing the above values of $\omega_1 \ldots$, and the known relations between the nine direction-cosines, including the equations $a_1 a_1' + b_1 b_1' + c_1 c_1' = 0, \ldots$,

$$a_1 x' + b_1 y' + c_1 z' = \xi' + \omega_2\zeta - \omega_3\eta,$$
$$a_2 x' + b_2 y' + c_2 z' = \eta' + \omega_3\xi - \omega_1\zeta,$$
$$a_3 x' + b_3 y' + c_3 z' = \zeta' + \omega_1\eta - \omega_2\xi;$$

and finally, by adding the squares of these expressions on each side

$$x'^2 + y'^2 + z'^2 = (\xi' + \omega_2\zeta - \omega_3\eta)^2 + (\eta' + \omega_3\xi - \omega_1\zeta)^2 + (\zeta' + \omega_1\eta - \omega_2\xi)^2.$$

* [This is the problem which has already been investigated in the first Section of Chapter VIII. The difference of notation will be observed.]

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The meaning of the terms in this expression is easily seen: thus, ξ' is the velocity of the point (ξ, η, ζ) relative to the moving axes, estimated parallel to the axis of ξ; and $\omega_2\zeta - \omega_3\eta$ is the velocity, relative to fixed space, and estimated in the same direction, which the same point would have if (ξ, η, ζ) were invariable; the sum is the total component of the velocity relative to fixed space. Hence the value of T, expressed in terms of the new coordinates, becomes

$$\mathrm{T} = \frac{1}{2}\Sigma . m\{(\xi' + \omega_2\zeta - \omega_3\eta)^2 + (\eta' + \omega_3\xi - \omega_1\zeta)^2 + (\zeta' + \omega_1\eta - \omega_2\xi)^2\}; \quad (14)$$

which is the expression to be used in forming the left-hand member of the equation (9), Art. 302.

The right-hand member of that equation is to be obtained by transforming the expression $\Sigma(\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z)$.

Now from the equations

$$x = a_1\xi + a_2\eta + a_3\zeta, \ldots,$$

we have $$\delta x = a_1\delta\xi + a_2\delta\eta + a_3\delta\zeta, \ldots$$

$a_1, a_2, \ldots$, being treated as invariable in the differentiation denoted by δ, because $\delta x, \ldots$ refer to displacements in space which might subsist at the time t. Hence

$$\mathrm{X}\,\delta x + \mathrm{Y}\,\delta y + \mathrm{Z}\,\delta z =$$
$$(a_1\mathrm{X} + b_1\mathrm{Y} + c_1\mathrm{Z})\delta\xi + (a_2\mathrm{X} + b_2\mathrm{Y} + c_2\mathrm{Z})\,\delta\eta + (a_3\mathrm{X} + b_3\mathrm{Y} + c_3\mathrm{Z})\delta\zeta;$$

now $a_1\mathrm{X} + b_1\mathrm{Y} + c_1\mathrm{Z}$ is the component of the force acting at (x, y, z), estimated in the direction of the axis of ξ; so that if we call this component Ξ, the term involving $\delta\xi$ becomes $\Xi\,\delta\xi$; and similar conclusions will result for the other terms. Thus the right-hand member of the equation (9) may be represented by $\Sigma . \Xi\,\delta\xi$; the summation referring to all the coordinates as well as to all the points, and the coefficient of each variation being the corresponding component of force.

305.] As an illustration of the general formulæ of the preceding Article, we may take the following problem:

To find the modification introduced into the treatment of dynamical problems, referring to motion near the earth's surface, when the earth's rotation is taken into account.

Neglecting the curvature of the path of the earth's centre, and assuming that the forces concerned in the problem are independent of the earth's position in its orbit, we may consider the centre as fixed.

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Let the primitive axes of coordinates then have their origin at the earth's centre, the positive axis of z being directed to the north pole, the axes of x and y being in the plane of the equator, but fixed in space; the positive axis of y being to the east of that of x.

We wish to take as new axes a system fixed relatively to the earth, and having its origin at a given point on the earth's surface.

As an intermediate step, let ξ, η, ζ refer to axes fixed in the earth and parallel to the required axes, but having their origin at the centre. If then we call ω the angular velocity of the earth's rotation, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ the direction-cosines of the polar axis referred to the axes of ξ, η, ζ, we shall have

$$\omega_1 = \omega\cos\alpha, \quad \omega_2 = \omega\cos\beta, \quad \omega_3 = \omega\cos\gamma;$$

and the expression (14) Art. 304, becomes

$$\text{T} = \frac{1}{2}\Sigma.m\left[\{\xi' + \omega(\zeta\cos\beta - \eta\cos\gamma)\}^2\right.$$
$$\left. + \{\eta' + \omega(\xi\cos\gamma - \zeta\cos\alpha)\}^2 + \{\zeta' + \omega(\eta\cos\alpha - \xi\cos\beta)\}^2\right];$$

and it only remains to remove the origin to the required point on the surface by writing $\xi+\xi_0$, $\eta+\eta_0$, $\zeta+\zeta_0$ instead of ξ, η, ζ, where ξ_0, η_0, ζ_0 are the coordinates of the point in question, referred to the axes of ξ, η, ζ with centre as origin. This being done, the following values will be found without difficulty:

$$\frac{d\text{T}}{d\xi'} = m\xi' + m\omega\{(\zeta+\zeta_0)\cos\beta - (\eta+\eta_0)\cos\gamma\},$$

$$\frac{d\text{T}}{d\xi} = m\omega(\eta'\cos\gamma - \zeta'\cos\beta) + m\omega^2(\xi+\xi_0)$$
$$- m\omega^2\cos\alpha\{(\xi+\xi_0)\cos\alpha + (\eta+\eta_0)\cos\beta + (\zeta+\zeta_0)\cos\gamma\};$$

consequently

$$\left(\frac{d\text{T}}{d\xi'}\right)' - \frac{d\text{T}}{d\xi} = m\xi'' + 2m\omega(\zeta'\cos\beta - \eta'\cos\gamma) - m\omega^2(\xi+\xi_0)$$
$$+ m\omega^2\cos\alpha\{(\xi+\xi_0)\cos\alpha + (\eta+\eta_0)\cos\beta + (\zeta+\zeta_0)\cos\gamma\};$$

from which the forms of the terms referring to η and ζ are obvious.

If we call l the latitude of the place at which the origin is fixed, and take the plane of (ξ, η) horizontal, the axis of ξ being directed to the south and that of η to the east, we shall have

$$\cos\alpha = -\cos l, \quad \cos\beta = 0, \quad \cos\gamma = \sin l;$$

also $\xi_0 = 0$, and η_0, ζ_0 are given quantities, the former being small, of which the values are easily assigned in terms of the earth's axes and of l.

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Thus we obtain *

$$\left(\frac{d\mathrm{T}}{d\xi'}\right)' - \frac{d\mathrm{T}}{d\xi} = m\xi'' - 2m\omega \sin l\,\eta' - m\omega^2 (\sin l)^2 \xi$$
$$- m\omega^2 \sin l \cos l\,(\zeta + \zeta_0),$$
$$\left(\frac{d\mathrm{T}}{d\eta'}\right)' - \frac{d\mathrm{T}}{d\eta} = m\eta'' + 2m\omega\,(\xi' \sin l + \zeta' \cos l) - m\omega^2 (\eta + \eta_0),$$
$$\left(\frac{d\mathrm{T}}{d\zeta'}\right)' - \frac{d\mathrm{T}}{d\zeta} = m\zeta'' - 2m\omega \cos l\,\eta' - m\omega^2 \sin l \cos l\,\xi$$
$$- m\omega^2 (\cos l)^2 (\zeta + \zeta_0);$$

so that, finally, the general equations of motion of any system under the circumstances supposed are comprised in the following formula:

$$\begin{aligned}
&\Sigma.m\,(\xi''\delta\xi + \eta''\delta\eta + \zeta''\delta\zeta) \\
&\quad + 2\omega \sin l\,\Sigma.m\,(\xi'\delta\eta - \eta'\delta\xi) + 2\omega \cos l\,\Sigma.m\,(\zeta'\delta\eta - \eta'\delta\zeta) \\
&\quad - \omega^2 (\sin l)^2\,\Sigma.m\,\xi\delta\xi - \omega^2 \cos^2 l\,\Sigma.m\,(\zeta + \zeta_0)\,\delta\zeta - \omega^2\,\Sigma.m\,(\eta + \eta_0)\,\delta\eta \\
&\qquad - \omega^2 \sin l \cos l\,\Sigma.m\,\{(\zeta + \zeta_0)\,\delta\xi + \xi\delta\zeta\} = \Sigma.m\,\Xi\,\delta\xi.
\end{aligned}$$

306.] It is not intended in this Chapter to discuss particular problems; and the examples given in the last three Articles have been inserted only because the Lagrangian formulæ, if left in their general shape without illustration, would probably fail to convey precise notions to the mind of a reader coming to them for the first time.

We proceed now to an important transformation of these formulæ, due † to Sir W. R. Hamilton, without stopping to introduce at this stage the consequences derived from them by Lagrange; because these, with many other results, are more easily obtained from the Hamiltonian form.

Conforming to the notation of recent writers, we will denote the "coordinates" in any dynamical problem by $q_1, q_2, \ldots$, so that the general formula (11) Art. 302, becomes

$$\Sigma.\left\{\left(\frac{d\mathrm{w}}{dq'}\right)' - \frac{d\mathrm{w}}{dq}\right\}\delta q = 0; \tag{15}$$

in which we shall suppose that w may be any function whatever of $q_1, q_2, \ldots, q_1', q_2', \ldots$, and t.

If now we put

$$\frac{d\mathrm{w}}{dq_1'} = p_1, \qquad \frac{d\mathrm{w}}{dq_2'} = p_2, \ldots, \tag{16}$$

* [The equations given in (55), Art. 237, are identical with these when the signs of ω and η are changed.]

† A first step towards this transformation was made by Poisson; but we have not space for details on this point.

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we may suppose q_1', q_2', ... to be expressed, by means of these equations, in terms of $p_1, p_2, \ldots, q_1, q_2, \ldots$, with or without t; and when these values of q_1', q_2', ... are introduced in the formula (15), that formula, together with (16), will give a set of equations involving the two sets of variables $q_1, q_2, \ldots, p_1, p_2, \ldots$, with their first differential coefficients, instead of the one set $q_1, q_2, \ldots$ with their first and second differential coefficients.

Thus, if the coordinates $q_1, q_2, \ldots, q_n$, be an independent set, instead of n differential equations of the second order we shall have $2n$ of the first order.

The general form of these $2n$ equations was first assigned by Sir W. R. Hamilton. His demonstration depends upon the particular character of the function T in most actual problems; and the following, which is slightly different and more general, is therefore substituted.

307.] The principle of the demonstration may be most clearly exhibited independently, in the form of the following theorem:

If P be any function of the n quantities $x_1, x_2, \ldots x_n$, and if n other quantities $y_1, y_2, \ldots y_n$ be defined by the equations

$$y_1 = \frac{d\mathrm{P}}{dx_1}, \quad y_2 = \frac{d\mathrm{P}}{dx_2}, \ldots, \quad y_n = \frac{d\mathrm{P}}{dx_n}; \tag{17}$$

then, if by means of these equations, $x_1, \ldots, x_n$ be expressed in terms of $y_1, y_2, \ldots, y_n$, their values will be of the form

$$x_1 = \frac{d\mathrm{Q}}{dy_1}, \quad x_2 = \frac{d\mathrm{Q}}{dy_2}, \ldots, \quad x_n = \frac{d\mathrm{Q}}{dy_n}; \tag{18}$$

where $$\mathrm{Q} = -\mathrm{P} + x_1 y_1 + x_2 y_2 + \ldots + x_n y_n: \tag{19}$$

in which $x_1, \ldots, x_n$ on the right are supposed to be expressed in terms of $y_1, \ldots, y_n$.

Also if P contain any other quantities, $\xi, \ldots$, besides $x_1, \ldots, x_n$, then

$$\frac{d\mathrm{P}}{d\xi} = -\frac{d\mathrm{Q}}{d\xi};$$

the differentiation with respect to ξ being in each case performed only so far as ξ appears explicitly.

To prove this we have, if the symbol d operate only on $x_1, \ldots, x_n, y_1, \ldots, y_n$,

$$d\mathrm{P} = y_1\,dx_1 + y_2\,dx_2 + \ldots + y_n\,dx_n, \quad \text{by (17)};$$

but

$$d(x_1 y_1 + \ldots + x_n y_n) = y_1\,dx_1 + \ldots + y_n\,dx_n + x_1\,dy_1 + \ldots + x_n\,dy_n;$$

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hence, by subtraction,

$$d(x_1 y_1 + \ldots + x_n y_n - \text{P}) = x_1 dy_1 + \ldots + x_n dy_n;$$

an equation which must be identical if both sides be expressed in the same way. If therefore we put, as above, $\text{Q} = x_1 y_1 + \ldots - \text{P}$, and suppose $x_1, \ldots$, on each side, expressed in terms of $y_1, \ldots$, since $d\text{Q} = \frac{d\text{Q}}{dy_1} dy_1 + \ldots$, we must have $x_1 = \frac{d\text{Q}}{dy_1}, \ldots$, which proves the first part of the theorem.

To prove the second part, we observe that the value (19) of Q will contain ξ explicitly, partly because it is contained explicitly in P, as originally expressed, and partly because the values of $x_1, \ldots$, in terms of $y_1, \ldots$, when substituted in P and the other terms, will introduce it again. Hence we shall have

$$\frac{d\text{Q}}{d\xi} = -\frac{d\text{P}}{d\xi} - \frac{d\text{P}}{dx_1}\frac{dx_1}{d\xi} - \ldots - \frac{d\text{P}}{dx^n}\frac{dx_n}{d\xi}$$
$$+ y_1 \frac{dx_1}{d\xi} + \ldots + y_n \frac{dx_n}{d\xi};$$

but since $y_1 = \frac{d\text{P}}{dx_1}, \ldots$, this equation becomes simply $\frac{d\text{Q}}{d\xi} = -\frac{d\text{P}}{d\xi}$, which was to be proved.

Let us now apply this theorem to transform the formulæ (15) and (16), Art. 306.

The equations (16) being exactly similar to (17) of this Article, it follows, that if we put

$$\text{H} = -\text{W} + p_1 q_1' + p_2 q_2' + \ldots + p_n q_n',$$

and express $q_1', \ldots q_n'$ on the right in terms of $p_1, \ldots, p_n, \ldots$, we shall have

$$q_1' = \frac{d\text{H}}{dp_1}, \ldots, \qquad q_n' = \frac{d\text{H}}{dp_n};$$

and moreover, since, besides $q_1', q_2', \ldots$, W contains also the quantities $q_1, q_2, \ldots$, analogous to $\xi, \ldots$, we shall have also

$$\frac{d\text{W}}{dq_1} = -\frac{d\text{H}}{dq_1}, \qquad \frac{d\text{W}}{dq_2} = -\frac{d\text{H}}{dq_2}, \ldots;$$

so that the formula (15) will become

$$\Sigma\left(p' + \frac{d\text{H}}{dq}\right)\delta q = 0.$$

These results may be summed up as follows:

If W be any function of $q_1, q_2, \ldots, q_1', q_2', \ldots, t$, the formula

$$\Sigma\left\{\left(\frac{d\text{W}}{dq'}\right)' - \frac{d\text{W}}{dq}\right\}\delta q = 0$$

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is transformed into the system

$$\left.\begin{aligned} \Sigma\left(p' + \frac{d\mathrm{H}}{dq}\right)\delta q = 0, \\ q_1' = \frac{d\mathrm{H}}{dp_1}, \quad q_2' = \frac{d\mathrm{H}}{dp_2}, \quad \ldots \end{aligned}\right\} \tag{20}$$

by the following substitutions:

$$\frac{d\mathrm{W}}{dq_1'} = p_1, \qquad \frac{d\mathrm{W}}{dq_2'} = p_2, \ldots,$$

$$\mathrm{H} = p_1 q_1' + p_2 q_2' + \ldots - \mathrm{W};$$

where, in forming the expression for H, we are to express q_1', q_2', ... in terms of $p_1, p_2, \ldots, q_1, q_3, \ldots$, so that H is in general a function of $p_1, p_2, \ldots, q_1, q_2, \ldots$, and t.

One case deserves particular notice, because it occurs in most actual dynamical problems. If W be of the form T + U, where T is homogeneous and of the second degree in $q_1', q_2', \ldots$, and U does not contain $q_1', q_2', \ldots$, then $p_1 = \frac{d\mathrm{T}}{dq_1'}, \ldots$, and therefore

$$p_1 q_1' + p_2 q_2' + \ldots = q_1' \frac{d\mathrm{T}}{dq_1'} + \ldots = 2\mathrm{T};$$

hence, in this case,

$$\mathrm{H} = 2\mathrm{T} - \mathrm{W} = \mathrm{T} - \mathrm{U},$$

where T is to be expressed in terms of $p_1, p_2, \ldots, q_1, q_2, \ldots$.

If $q_1, q_2, \ldots$ be a set of independent coordinates, say n in number, then the system (20) gives $2n$ separate equations, namely, those obtained by giving to i all integer values from 1 to n inclusive in the two following:

$$p_i' = -\frac{d\mathrm{H}}{dq_i}, \qquad q_i' = \frac{d\mathrm{H}}{dp_i}. \tag{21}$$

We shall call these, as Mr. Cayley has done, the "Hamiltonian" equations*. In treating of their general properties it is usually unnecessary to take any account of the nature of the problems

* The Lagrangian equations may be considered as a particular case of a more general form, upon which the solution of a class of problems in the Calculus of Variations depends; and it has been shewn by M. Ostrogradsky, that this more general form is susceptible of a transformation which includes that of sir W. R. Hamilton as a particular case. See "Mémoire sur les équations différentielles relatives au problème des isopérimètres, 1848." Mém. de l'Acad. Imper. des Sciences de St. Petersburg. Sciences Math. et Phys. t. iv, 1850.

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which give rise to such a system. H is to be considered merely as a given function of $p_1, p_2, \dots p_n, q_1, q_2, \dots, q_n$, and t.

308.] The complete solution of the $2n$ simultaneous differential equations of the first order, represented by the formula (21), would consist of $2n$ equations involving the variables $p_1, \dots, q_1, \dots$, and t, with $2n$ arbitrary constants. Any one such equation may be called an "integral equation;" but it is desirable to distinguish by a separate name that particular form of integral equation in which a function of variables only is equated to an arbitrary constant. We shall call such an equation an "integral." Thus the general form of an integral will be

$$c = f(p_1, \dots p_n, q_1, \dots q_n, t);$$

where the function on the right contains no arbitrary constant; and it is a convenient abbreviation to speak of such an integral as* "the integral c."

Thus a complete solution of the system (21) may be supposed to consist of $2n$ integrals. But in order that $2n$ integrals may constitute a complete solution, it is necesssry that they should be independent; that is, that no identical relations should subsist between the functions equated to the arbitrary constants. If such relations did subsist, the variables might be eliminated, and one or more equations be obtained involving the constants only, so that the constants would not be all arbitrary.

Hence the problem of integrating the system of equations (21) may be stated as follows:

"To find $2n$ independent functions of $p_1, p_2, \dots p_n, q_1, q_2, \dots q_n$, and t, each of which is constant by virtue of the differential equations (21)."

On the other hand, the same problem might be regarded as having for its object "to express each of the $2n$ variables, $p_1, \dots, q_1, \dots$, as a function of $2n$ arbitrary constants and t."

If a complete solution were obtained in either of these forms, it is evident that algebraical processes only would be required to deduce from it a solution in the other form, as well as an infinite variety of "integral equations."

* This expression however "the integral c" is, to avoid circumlocution, used not only to signify the equation $c = f(p_1, \dots)$, but also to denote either side of that equation separately, viz., either the constant c, or the function $f(p_1, \dots)$, which has that constant value. The last is the most usual meaning.

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The consideration of the two forms just mentioned is of the greatest theoretical importance, though neither of them is in general obtained as a direct result of existing methods of integration.

Inasmuch as all complete solutions of the same system of differential equations must be equivalent to one another, it follows that any arbitrary constant belonging to one solution must be capable of being expressed as a function of the arbitrary constants belonging to any other solution.

309.] Any $2n$ functions of the variables $p_1, \ldots, q_1, \ldots$, and t, may be called *elements*, provided that the equations by which they are defined are algebraically sufficient to determine conversely the $2n$ variables $p_1, \ldots, q_1, \ldots$, as functions of the elements and t. Thus, if the elements $u_1, u_n, \ldots u_{2n}$ be defined by $2n$ equations, such as

$$u_1 = f(p_1, \ldots, q_1, \ldots, t),$$

then it must not be possible to eliminate all the $2n$ variables, $p_1, \ldots, q_1, \ldots$, from these equations.

From the above definition, it is evident that a complete solution of the differential equations would be obtained if any set of elements were expressed in terms of arbitrary constants and t.

It is also evident, that the functions which are equated to arbitrary constants in any complete set of integrals are "elements." Thus elements may be either variable or constant.

310.] It may be useful to exhibit at this stage, for the sake of clearness, the equations of a simple dynamical problem in the Hamiltonian form. For this purpose we may take the case of motion of a single material point about a fixed centre of force. Let m be the mass of the moving point; then, taking the origin of the polar coordinates r, θ at the fixed centre, and the plane of the motion for the plane of the angle θ, we shall have

$$2\mathrm{T} = m(r'^2 + r^2\theta'^2);$$

and the force-function U will be a given function of r, say $\mathrm{U} = \phi(r)$. Then, writing q_1 instead of r, and q_2 instead of θ, we have

$$2\mathrm{T} = m(q_1'^2 + q_1^2 q_2'^2);$$

consequently,

$$p_1 = \frac{d\mathrm{T}}{dq_1'} = mq_1', \qquad p_2 = \frac{d\mathrm{T}}{dq_2'} = mq_1^2 q_2';$$

from which we have

$$q_1' = \frac{p_1}{m}, \qquad q_2' = \frac{p_2}{mq_1^2},$$

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and therefore H, which in this case is T − U expressed in terms of p_1, p_2, q_1, q_2, becomes, see equation (20),

$$\text{H} = \frac{1}{2m}\left(p_1^2 + \frac{p_2^2}{q_1^2}\right) - \phi(q_1);$$

and the four equations (21), Art. 307, become

$$p_1' = \frac{p_2^2}{m q_1^3} + \phi'(q_1), \qquad q_1' = \frac{p_1}{m};$$

$$p_2' = 0, \qquad q_2' = \frac{p_2}{m q_1^2}.$$

311.] If u, v be any functions whatever, containing the variables $p_1, \ldots, p_n, q_1, \ldots, q_n$, then it is convenient to employ the following symbol: let

$$(u, v) = \Sigma\left(\frac{du}{dp_i}\frac{dv}{dq_i} - \frac{du}{dq_i}\frac{dv}{dp_i}\right),$$

or in the notation of Art. 292,

$$(u, v) = \Sigma\frac{d(u, v)}{d(p_i, q_i)};$$

the summation extending to all values of i from 1 to n.

The reader will hardly require to be reminded that the symbols (u, v), (p_i, q_i) on the right of the last equation, have only an accidental resemblance to the (u, v) on the left, without any connexion of meaning.

For example, if u, v contain p_1, p_2, q_1, q_2 only, then

$$(u, v) = \frac{du}{dp_1}\frac{dv}{dq_1} - \frac{du}{dq_1}\frac{dv}{dp_1} + \frac{du}{dp_2}\frac{dv}{dq_2} - \frac{du}{dq_2}\frac{dv}{dp_2}.$$

From the above definition the following consequences are easily deduced by means of the elementary principles of differentiation:

$$(u, v) = -(v, u,) \qquad (u, u) = 0,$$

$$(p_i, q_i) = 1, \qquad (q_i, p_i) = -1,$$

and $(p_i, q_j) = 0$, if j be different from i.

Also, if a be any function of $v, w, \ldots$, then

$$(u, a) = \frac{da}{dv}(u, v) + \frac{da}{dw}(u, w) + \ldots.$$

Again, if v contain $p_1, q_1, \ldots$ explicitly, and also $\alpha, \beta, \ldots$ functions of $p_1, q_1, \ldots$, then

$$(u, v) = (u, \bar{v}) + \frac{dv}{d\alpha}(u, \alpha) + \frac{dv}{d\beta}(u, \beta) + \ldots,$$

where $(u, \bar{v})$ represents the expression formed by differentiating v only so far as it contains $p_1, q_1, \ldots$ explicitly.

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Lastly, if u, v contain explicitly any other quantity, say z, besides the variables $p_1, q_1, \ldots$, then the partial differential coefficient of (u, v), taken explicitly with respect to z, is

$$\frac{d}{dz}(u, v) = \left(\frac{du}{dz}, v\right) + \left(u, \frac{dv}{dz}\right).$$

312.] The following theorem will be of use afterwards:

Let u, v, w be any three functions whatever, containing $p_1, q_1, \ldots$, with or without other quantities, then

$$\{u, (v, w)\} + \{v, (w, u)\} + \{w, (u, v)\} = 0. \qquad (22)$$

For if this expression were developed, each term would, irrespective of sign, consist of the product of one second differential coefficient, and two first differential coefficients. Thus we should have terms in which u is twice differentiated, arising from $\{v, (\omega, u)\}$ in the three forms

$$\frac{dv}{dp_i}\frac{dw}{dp_j}\frac{d^2u}{dq_i dq_j}, \qquad \frac{dv}{dq_i}\frac{dw}{dq_j}\frac{d^2u}{dp_i dp_j}, \qquad \frac{dv}{dp_i}\frac{dw}{dq_j}\frac{d^2u}{dq_i dp_j},$$

including the case of $j = i$; but the same terms would arise from $\{w, (u, v)\}$ with the contrary signs, as the reader will easily verify. The same thing may be said of the terms in which v and w are twice differentiated. Hence the equation (22) is satisfied identically, as was to be shewn.

The properties established in this and the preceding Articles are independent of any suppositions as to the meanings of p_1, $q_1, \ldots$, and of the relations established by the differential equations (21), to the consideration of which we now return.

313.] Suppose a complete solution of the equations (21), namely,

$$p_i' = -\frac{d\mathrm{H}}{dq_i}, \qquad q_i' = \frac{d\mathrm{H}}{dp_i},$$

to have been obtained, so that each of the $2n$ variables $p_1, \ldots, p_n, q_1, \ldots, q_n$, is a given function of t, and of $2n$ arbitrary constants $c_1, c_2, \ldots, c_{2n}$;

Also let two independent sets of arbitrary infinitesimal variations be attributed to the constants, and denoted by the symbols δ, Δ, so that we should have

$$\delta p_i = \frac{dp_i}{dc_1}\delta c_1 + \frac{dp_i}{dc_2}\delta c_2 + \ldots,$$

$$\Delta p_i = \frac{dp_i}{dc_1}\Delta c_1 + \frac{dp_i}{dc_2}\Delta c_2 + \ldots;$$

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then the expression

$$\delta p_1 \Delta q_1 - \Delta p_1 \delta q_1 + \delta p_2 \Delta q_2 - \Delta p_2 \delta q_2 + \dots,$$

or

$$\Sigma (\delta p_i \Delta q_i - \Delta p_i \delta q_i) \tag{23}$$

is constant. That is, if the above values of $\delta p_i, \dots$, in terms of the constants, their variations, and t, be introduced, t will disappear from the result, and the expression (23) will become a function of the constants $c_1, c_2, \dots$ and their variations $\delta c_1, \Delta c_1, \dots$ only.

This remarkable theorem was discovered by Lagrange, employing his own form of the differential equations. The following simple demonstration of it is due to Professor Boole.

H is a given function of $p_1, \dots, p_n, q_1, \dots, q_n, t$; and we have

$$\delta \mathrm{H} = \frac{d\mathrm{H}}{dp_1} \delta p_1 + \frac{d\mathrm{H}}{dq_1} \delta q_1 + \dots;$$

so that by the equations (21)

$$\delta \mathrm{H} = q_1' \delta p_1 - p_1' \delta q_1 + \dots;$$

and consequently

$$\delta \mathrm{H} = \Sigma (q_i' \delta p_i - p_i' \delta q_i).$$

Now performing the operation Δ on each side of this equation, we have

$$\Delta \delta \mathrm{H} = \Sigma (\Delta q_i' \delta p_i - \Delta p_i \delta q_i + q_i' \Delta \delta p_i - p_i \Delta \delta q_i);$$

in like manner we should find

$$\delta \Delta \mathrm{H} = \Sigma (\delta q_i' \Delta p_i - \delta p_i \Delta q_i + q_i \delta \Delta p_i - p_i \delta \Delta q_i);$$

hence, subtracting and observing that $\Delta \delta = \delta \Delta$,

$$0 = \Sigma (\Delta q_i' \delta p_i + \Delta q_i \delta p_i' - \Delta p_i' \delta q - \Delta p_i \delta q_i');$$

now $\Delta q_i' = (\Delta q_i)', \dots$; and thus this equation is equivalent to

$$0 = \Sigma (\delta p_i \Delta q_i - \Delta p_i \delta q_i)';$$

that is, the total differential coefficient with respect to t of the expression (23) vanishes, and that expression is therefore constant; which was to be proved.

314.] Suppose now that the $2n$ constants $c_1, c_2, \dots$ are the initial values of the variables, which we will denote by $\lambda_1, \lambda_2, \dots \lambda_n, \mu_1, \mu_2, \dots, \mu_n$; where $\lambda_1, \lambda_2, \dots$ are the initial values of $p_1, p_2, \dots$, and $\mu_1, \mu_2, \dots$ of $q_1, q_2, \dots$.

Since the value of $\Sigma (\delta p_i \Delta q_i - \Delta p_i \delta q_i)$ is independent of t, it is not altered by supposing $t = 0$; but when $t = 0$, the values of $\delta p_i, \delta q_i, \dots$ are $\delta \lambda_i, \delta \mu_i, \dots$; consequently,

$$\Sigma (\delta p_i \Delta q_i - \Delta p_i \delta q_i) = \Sigma (\delta \lambda_i \Delta \mu_i - \Delta \lambda_i \delta \mu_i). \tag{24}$$

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By the help of this equation we can shew that the initial values $\lambda_1, \mu_1, \ldots$, considered as constant elements, possess certain remarkable properties.

Each of these elements may be supposed to be expressed as a function of the variables, $p_1, q_1, \ldots$, and t. Let this supposition be called Hypothesis I.

On the other hand each of the variables, $p_1, q_1, \ldots$, may be supposed to be expressed in terms of the elements $\lambda_1, \mu_1, \ldots$, and t. Let this supposition be called Hypothesis II.

Now in equation (24) let $\delta p_i, \delta q_i$, on the left-hand side be expressed in terms of $\delta\lambda_1, \delta\mu_1, \ldots$; thus,

$$\delta p_i = \frac{dp_i}{d\lambda_i}\delta\lambda_1 + \frac{dp_i}{d\mu_1}\delta\mu_1 + \ldots,$$

$$\delta q_i = \frac{dq_i}{d\lambda_1}\delta\lambda_1 + \frac{dq_i}{d\mu_1}\delta\mu_1 + \ldots;$$

and let $\Delta\lambda_i, \Delta\mu_i$, on the right, be expressed in terms $\Delta p_1, \Delta q_1, \ldots$; thus,

$$\Delta\lambda_i = \frac{d\lambda_i}{dp_1}\Delta p_1 + \frac{d\lambda_i}{dq_1}\Delta q_1 + \ldots,$$

$$\Delta\mu_i = \frac{d\mu_i}{dp_1}\Delta p_1 + \frac{d\mu_i}{dq_1}\Delta q_1;$$

so that on both sides of the equation all terms involving δ will be variations of constants, and those involving Δ will be variations of variables; and each of these sets of variations may have arbitrary values assigned to them; hence the coefficients of corresponding terms on the two sides of the equation must be equal. Thus we obtain by comparing the coefficients of

$$\left.\begin{array}{lll}
\delta\lambda_j\,\Delta q_i, \text{ the equation} & \dfrac{dp_i}{d\lambda_j} = & \dfrac{d\mu_j}{dq_i}, \\[2ex]
\delta\mu_j\,\Delta q_i, \quad - - - - & \dfrac{dp_i}{d\mu_j} = & -\dfrac{d\lambda_j}{dq_i}, \\[2ex]
\delta\lambda_j\,\Delta p_i, \quad - - - - & \dfrac{dq_i}{d\lambda_j} = & -\dfrac{d\mu_j}{dp_i}, \\[2ex]
\delta\mu_j\,\Delta p_i, \quad - - - - & \dfrac{dq_i}{d\mu_j} = & \dfrac{d\lambda_j}{dp_i};
\end{array}\right\} \qquad (25)$$

in which equations the differentiation refers to Hypothesis II on the left, and Hypothesis I on the right; and in each of them i may be equal to j.

Now suppose any one of the constants, say λ_j, to be expressed, according to Hypothesis I, in terms of the variables, thus

$$\lambda_j = f(p_1, \ldots p_n, q_1, \ldots q_n, t);$$

if on the right of this equation each of the variables were ex-

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pressed, according to Hypothesis II, in terms of the constants and t, the equation would become identical; that is, the right-hand side would become identically λ_j; hence, if we differentiate each side with respect to λ_j, on Hypothesis II, the result on the right must $= 1$; but if we differentiate with respect to any other of the constants, the result must $= 0$; thus,

$$\frac{d\lambda_j}{dp_1}\frac{dp_1}{d\lambda_j} + \frac{d\lambda_j}{dq_1}\frac{dq_1}{d\lambda_j} + \frac{d\lambda_j}{dp_2}\frac{dp_2}{d\lambda_j} + \frac{d\lambda_j}{dq_2}\frac{dq_2}{d\lambda_j} + \ldots = 1;$$

and if c be any one of the constants except λ_j,

$$\frac{d\lambda_j}{dp_1}\frac{dp_1}{dc} + \frac{d\lambda_j}{dq_1}\frac{dq_1}{dc} + \frac{d\lambda_j}{dp_2}\frac{dp_2}{dc} + \frac{d\lambda_j}{dq_2}\frac{dq_2}{dc} + \ldots = 0.$$

Now in the first of these equations let the values of $\frac{dp_1}{d\lambda_j}, \frac{dq_1}{d\lambda_j}, \ldots$ given by (25) be substituted; and it will be seen that the result is

$$(\lambda_j, \mu_j) = 1;$$

see Art. 311. But if in the second of the above equations we take for c either λ_i, where i is not $= j$, or μ_i, where i may be either equal to j or not, the result in the first case is

$$(\lambda_j, \mu_i) = 0;$$

and in the second it is

$$-(\lambda_j, \lambda_i) = 0.$$

By supposing μ_j expressed in terms of $p_1, q_2, \ldots$, and reasoning in the same way, we should obtain the equation

$$(\mu_j, \mu_i) = 0.$$

Thus we see that the elements $\lambda_1, \ldots \lambda_n, \mu_1, \ldots \mu_n$, possess the properties expressed by the equations

$$(\lambda_i, \mu_i) = 1, \quad (\lambda_i, \mu_j) = 0, \quad (\lambda_i, \lambda_j) = 0, \quad (\mu_i, \mu_j) = 0. \quad (26)$$

If we call any pair, such as λ_i, μ_i, conjugate elements, the above properties may be briefly stated by saying, that if f, g be any two of the elements, then $(f, g) = \pm 1$ if f, g be conjugate, and $= 0$ in every other case.

315.] If $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ be elements, such that, f g representing any pair, the value of (f, g) is ± 1 or 0, according as f, g are a conjugate pair (that is, a pair such as a_i, b_i) or not, then these elements are called *canonical elements*.

It has been shewn in the preceding Articles that when the $2n$ variables, $p_1, q_1, \ldots$, are determined as functions of t and of

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$2n$ arbitrary constants, by means of the Hamiltonian equations (21), there exists one set of arbitrary constants, namely, the initial values, $\lambda_1, \mu_1, \ldots$, of the variables, which form a system of canonical constant elements. We shall now prove that the number of such systems is infinite.

In fact if $a_1, \ldots, a_n, b_1, \ldots, b_n$ be determined as functions of $\lambda_1, \mu_1, \ldots$ by the $2n$ equations

$$\frac{d\mathrm{A}}{da_i} = b_i, \qquad \frac{d\mathrm{A}}{d\mu_i} = \lambda_i, \tag{27}$$

where A is any arbitrary function of $a_1, \ldots a_n, \mu_1, \ldots \mu_n$, then $a_1, b_1, \ldots$ will be canonical elements. For we have

$$\delta\mathrm{A} = \frac{d\mathrm{A}}{da_1}\delta a_1 + \frac{d\mathrm{A}}{d\mu_1}\delta\mu_1 + \ldots,$$

$$= \Sigma\left(\frac{d\mathrm{A}}{da_i}\delta a_i + \frac{d\mathrm{A}}{d\mu_i}\delta\mu_i\right);$$

and consequently, by (27),

$$\delta\mathrm{A} = \Sigma(b_i\delta a_i + \lambda_i\delta\mu_i);$$

and performing the operation Δ on each side of this equation,

$$\Delta\delta\mathrm{A} = \Sigma(\Delta b_i\delta a_i + \Delta\lambda_i\delta\mu_i) + \Sigma(b_i\Delta\delta a_i + \lambda_i\Delta\delta\mu_i):$$

similarly we should find

$$\delta\Delta\mathrm{A} = \Sigma(\delta b_i\Delta a_i + \delta\lambda_i\Delta\mu_i) + \Sigma(b_i\delta\Delta a_i + \lambda_i\Delta\delta\mu_i);$$

whence, subtracting and observing that $\delta\Delta = \Delta\delta$,

$$\Sigma(\delta a_i\Delta b_i - \Delta a_i\delta b_i) = \Sigma(\delta\lambda_i\Delta\mu_i - \Delta\lambda_i\delta\mu_i).$$

But it has already been proved, see Art. 314, that the right-hand member of this equation is $= \Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i)$, so that we have

$$\Sigma(\delta a_i\Delta b_i - \Delta a_i\delta b_i) = \Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i);$$

from which it follows, that all the consequences deduced in Art. 314 from equation (24) will be true, if we substitute $a_1, b_1, \ldots$ for $\lambda_1, \beta_1, \ldots$, and in particular that the conditions

$$(a_i, b_i) = 1, \qquad (a_i, a_j) = (a_i, b_j) = (b_i, b_j) = 0,$$

will subsist.

316.] The equations last written are particular cases of the following general theorem, discovered by Poisson.

If f, g be any two integrals whatever of the Hamiltonian equations, then (f, g) is constant. Poisson's demonstration was obtained by means of the Lagrangian form of the equations.

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The following, founded on the Hamiltonian form, is much simpler:

If u be any function whatever of $p_1, q_1, \ldots$, and t, we have

$$u' = \frac{du}{dt} + \frac{du}{dp_1} p_1' + \frac{du}{dq_1} q_1' + \ldots;$$

but as $p_1, q_1, \ldots$ are supposed to satisfy the differential equations (21), we have $p_1' = -\frac{d\text{H}}{dq_1}$, $q_1' = \frac{d\text{H}}{dp_1}$, ..., so that

$$u' = \frac{du}{dt} + \Sigma\left(\frac{d\text{H}}{dp_i}\frac{du}{dq_i} - \frac{d\text{H}}{dq_i}\frac{du}{dp_i}\right);$$

$$= \frac{du}{dt} + (\text{H}, u); \quad \text{(see Art. 311).}$$

Now if we take $u = (f, g)$, we have, (Art. 311),

$$\frac{du}{dt} = \left(\frac{df}{dt}, g\right) + \left(f, \frac{dg}{dt}\right),$$

and therefore

$$(f, g)' = \left(\frac{df}{dt}, g\right) + \left(f, \frac{dg}{dt}\right) + \{\text{H}, (f, g)\};$$

now since f and g are integrals, $f' = 0$, and $g' = 0$; that is,

$$\frac{df}{dt} + (\text{H}, f) = 0, \qquad \frac{dg}{dt} + (\text{H}, g) = 0:$$

these equations are identically true; so that we may substitute $-(\text{H}, f)$, $-(\text{H}, g)$, for $\frac{df}{dt}$, $\frac{dg}{dt}$ respectively in the above expression for $(f, g)'$, and the result may be written thus,

$$(f, g)' = \{g, (\text{H}, f)\} + \{f, (g, \text{H})\} + \{\text{H}, (f, g)\};$$

but by the theorem proved in Art. 312, the expression on the right of this equation vanishes identically, and therefore $(f, g)' = 0$; or (f, g), is constant, which is the theorem to be demonstrated.

Here it is to be observed, that f and g represent given functions of the variables $p_1, q_1, \ldots$, and t, which are constant by virtue of the differential equations. But the constancy of the expression (f, g) may subsist in two different ways:

First, (f, g) may be identically constant, that is, a determinate numerical constant, or zero: this always happens when f and g belong to a set of canonical elements.

Secondly, (f, g) may be constant, not identically, but by virtue of the differential equations; and in this case

$$c = (f, g)$$

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will be an integral of the equations; but here again there are two cases, for c may be either an independent arbitrary constant or a function of f and g; in the latter case, the integral c is a combination of the integrals f, g, but in the former case, it is a distinct independent integral. Thus it may happen that the theorem will lead to the discovery of a new integral when two are known. For example, the problem of motion about a fixed centre of force, leads, as will be seen afterwards, to three integrals of the forms

$$\begin{aligned} h &= p_1^2+p_2^2+p_3^2-\phi(q_1^2+q_2^2+q_3^2), \\ e &= q_2p_3-q_3p_2, \\ f &= q_3p_1-q_1p_3; \end{aligned}$$

and it will be found on trial that $(h, e) = 0$, $(h, f) = 0$; but that $(e, f) = q_2p_1-p_2q_1$, which is neither identically constant nor expressible as a function of the other integrals: hence we may affirm that

$$g = p_2q_1-q_2p_1$$

is a new integral. But if we attempt to discover more integrals by the same method, we shall fail; for it will be found that

$$(h, g) = 0, \qquad (e, g) = f, \qquad (f, g) = -e.$$

It is to be observed also that the integral g is as easily discoverable by ordinary methods as the integrals e and f; so that in this case, and probably in general, the theorem is of no practical use as a means of obtaining new integrals, though very important in other points of view.

317.] We now come to a most important discovery, due to Sir W. R. Hamilton.

Suppose the solution of the system of equations (21), namely,

$$p_i' + \frac{d\text{H}}{dq_i} = 0, \qquad q_i' - \frac{d\text{H}}{dp_i} = 0,$$

to be given in the form of $2n$ integral equations involving the initial values, $\lambda_1, \mu_1, \ldots$, of the variables, as arbitrary constants. By means of these $2n$ equations each of the $2n$ variables could be expressed in terms of the $2n$ constants and t; and therefore the differential coefficients of the variables with respect to t could be expressed in the same way. Consider then the expression

$$p_1q_1' + p_2q_2' + \ldots + p_nq_n' - \text{H};$$

this being a given function of the variables, their first differen-

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tial coefficients, and t, might be expressed as above supposed, and would become a function of the $2n$ constants, $\lambda_1, \mu_1, \ldots$, and t. Suppose this function to be integrated with respect to t from $t = 0$, and let the result be called s; so that

$$\mathrm{s} = \int_0^t \{\Sigma (p_i q_i') - \mathrm{H}\}\, dt. \tag{28}$$

The value of s, obtained in this way, would be also a function of $\lambda_1, \mu_1, \ldots$, and t. But by means of the $2n$ integral equations we might express the $2n$ quantities, $\lambda_1, \lambda_2, \ldots \lambda_n, p_1, p_2, \ldots p_n$, in terms of the $2n$ quantities $\mu_1, \mu_2, \ldots \mu_n, q_1, q_2, \ldots q_n$, and t; and if the values of $\lambda_1, \ldots \lambda_n$, thus expressed, were substituted in the above value of s, the result would be of the form

$$\mathrm{s} = \mathrm{F}(q_1, q_2, \ldots q_n, \mu_1, \mu_2, \ldots \mu_n, t). \tag{29}$$

Now, taking the form (28), and using the symbol δ in the same sense as before, so that $\delta t = 0$, we have, by the rules of the calculus of variations,

$$\delta \mathrm{s} = \int_0^t [\Sigma \{p_i (\delta q_i)'\} + \Sigma (q_i' \delta p_i) - \delta \mathrm{H}]\, dt:$$

let H in this equation be supposed to be expressed in its original form as a function of $p_1, \ldots p_n, q_1, \ldots q_n, t$; then

$$\delta \mathrm{H} = \Sigma \left(\frac{d\mathrm{H}}{dp_i} \delta p_i + \frac{d\mathrm{H}}{dq_i} \delta q_i\right);$$

but by the differential equations (21),

$$\frac{d\mathrm{H}}{dp_i} = q_i', \qquad \frac{d\mathrm{H}}{dq_i} = -p_i',$$

hence $$\delta \mathrm{H} = \Sigma (q_i' \delta p_i - p_i' \delta q_i);$$

and if this be substituted in the above value of $\delta \mathrm{s}$, the result is

$$\delta \mathrm{s} = \int_0^t [\Sigma \{p_i (\delta q_i)'\} + \Sigma (p_i' \delta q_i)]\, dt$$
$$= \int_0^t \Sigma (p_i \delta q_i)' dt.$$

Thus $\delta \mathrm{s}$ turns out to be expressible as the integral of a perfect differential with respect to t. Performing the integration from $t = 0$, and observing that when $t = 0$ the values of p_i, δq_i are λ_i, $\delta \mu_i$, we obtain

$$\delta \mathrm{s} = \Sigma (p_i \delta q_i) - \Sigma (\lambda_i \delta \mu_i).$$

But if we suppose s to be expressed as in equation (29), we have

$$\delta \mathrm{s} = \Sigma \left(\frac{d\mathrm{s}}{dq_i} \delta q_i\right) + \Sigma \left(\frac{d\mathrm{s}}{d\mu_i} \delta \mu_i\right).$$

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Now these two values of δs involve the same set of $2n$ variations, $\delta q_1, \ldots \delta q_n, \delta\mu_1, \ldots \delta\mu_n$, which may all be considered as arbitrary and independent, because the $2n$ variables and $2n$ constants are only subject to $2n$ equations; so that the values of any set of $2n$ out of the $4n$ quantities could be assumed arbitrarily without contradicting the equations; consequently the coefficients of like variations must be equal, that is, the $2n$ equations,

$$\frac{ds}{dq_i} = p_i, \qquad \frac{ds}{d\mu_i} = -\lambda_i, \tag{30}$$

must be true; but these equations are obviously not true identically, and they contain the $2n$ arbitrary constants, $\lambda_1, \mu_1, \ldots$. Hence they can only be a particular form of the integral equations of the problem.

318.] It appears from the equations (30), just established, that if the single function s, expressed in the form (29), were known, a complete set of integral equations could be deduced from it by mere differentiation. Thus the complete integration of the system of differential equations (21) is made to depend upon finding the form of a single function. This is the most essential part of Sir W. R. Hamilton's discovery; but it must not be supposed that the above brief account of it represents the original form and manner of the author's investigation, much less that it gives any notion at all of the general contents of his two elaborate memoirs "On a General Method in Dynamics," contained in the Philosophical Transactions for 1834 and 1835.

319.] We proceed to examine the function s more closely. Recollecting that s is supposed to be expressed in terms of $q_1, \ldots q_n, \mu_1, \ldots \mu_n, t$, we have

$$\begin{aligned} s' &= \frac{ds}{dt} + \frac{ds}{dq_1} q_1' + \ldots \\ &= \frac{ds}{dt} + \Sigma(p_i q_i'), \text{ by (30)}; \end{aligned}$$

on the other hand, equation (28) gives by differentiation with respect to t, $\qquad s' = \Sigma(p_i q_i') - H$;

comparing these two values of s', we obtain

$$\frac{ds}{dt} + H = 0. \tag{31}$$

Now H is given as a function of $p_1, \ldots, q_1, \ldots, t$; say

$$H = f(p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n, t);$$

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also, by (30), $p_i = \frac{ds}{dq_i}$; hence (31) may be written in the form

$$\frac{ds}{dt} + f\left(\frac{ds}{dq_1}, \frac{ds}{dq_2}, \dots \frac{ds}{dq}, q_1, q_2, \dots q_n, t\right) = 0. \qquad (32)$$

But this equation contains, besides t, only the n variables $q_1, \dots q_n$, and the n constants $\mu_1, \dots \mu_n$; it cannot therefore be any combination of the integral equations (30), because those $2n$ equations would in general be insufficient to eliminate the $2n$ quantities $p_1, \dots p_n, \lambda_1, \dots \lambda_n$; hence it must be satisfied identically; in other words, it is a partial differential equation of which s is a solution.

320.] The above result is due to Sir W. R. Hamilton, and so is also the following: Differentiating H with respect to t, we have

$$\text{H}' = \frac{d\text{H}}{dt} + \Sigma\left(\frac{d\text{H}}{dp_i} p_i'\right) + \Sigma\left(\frac{d\text{H}}{dq_i} q_i'\right);$$

but, by the differential equations,

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i};$$

so that the above equation becomes simply

$$\text{H}' = \frac{d\text{H}}{dt}.$$

Now if H does not contain t explicitly, which is the case in ordinary* dynamical problems, then $\frac{d\text{H}}{dt} = 0$; therefore $\text{H}' = 0$,

or, $$\text{H} = \text{constant};$$

and this is an integral of the problem, called the "integral of vis viva." In this case the equation (31) is of the form

$$\frac{ds}{dt} + f(p_1, p_2, \dots p_n, q_1, q_2, \dots q_n) = 0;$$

and from this we may deduce an equation like (32) as before;

* Ordinary dynamical problems may be defined as those in which the law of "action and reaction" is maintained. This law is violated when, for instance, in the planetary theory, the disturbing planet is considered as being itself undisturbed; or when, in investigating the effect of the earth's rotation upon the motion of a pendulum, the pendulum is supposed not to affect the motion of the earth; or, speaking generally, whenever any part of a system is supposed to be subject to an "obligatory motion." In all these cases H will contain t explicitly.

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but since H is now constant, its value will not be altered by substituting for $p_1, q_1, \ldots$ their initial values; hence we may write

$$\frac{ds}{dt} + f(\lambda_1, \lambda_2, \ldots, \lambda_n, \mu_1, \mu_2, \ldots, \mu_n) = 0;$$

or, by (30),

$$\frac{ds}{dt} + f\left(-\frac{ds}{d\mu_1}, -\frac{ds}{d\mu_2}, \ldots, -\frac{ds}{d\mu_n}, \mu_1, \mu_2, \ldots, \mu_n\right) = 0; \quad (33)$$

which is a second partial differential equation satisfied by s in the case supposed.

321.] We must now notice an important extension of the preceding theory, discovered by Jacobi, and contained in the following theorem:

If H be $f(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t)$, and if s be any complete solution of the partial differential equation,

$$\frac{ds}{dt} + f\left(\frac{ds}{dq_1}, \frac{ds}{dq_2}, \ldots, \frac{ds}{dq_n}, q_1, q_2, \ldots, q_n, t\right) = 0, \quad (34)$$

containing the n arbitrary constants $a_1, a_2, \ldots, a_n$, then the $2n$ equations

$$\frac{ds}{dq_i} = p_i, \qquad \frac{ds}{da_i} = b_i$$

are a complete set of integral equations of the system

$$p_i' = -\frac{dH}{dq_i}, \quad q_i' = -\frac{dH}{dp_i}. \quad (35)$$

Here $b_1, b_2, \ldots, b_n$ are new arbitrary constants; and by a complete solution of the equation (34) is meant a solution containing, besides any constant merely added to s, n arbitrary constants $a_1, a_2, \ldots, a_n$, in such a manner that they cannot be all eliminated so as to produce a partial differential equation of the first order, without employing all the $n+1$ differential coefficients

$$\frac{ds}{dq_1}, \ldots, \frac{ds}{dq_n}, \frac{ds}{dt}.$$

Before proving the theorem we must investigate a criterion by which it may be known whether any solution of such an equation as (34) be complete or not.

Suppose then $s = \psi(a_1, \ldots, a_n, q_1, \ldots, q_n, t)$ to be a solution; and this to give

$$\frac{ds}{dq_1} = \phi_1(a_1, \ldots, a_n, q_1, \ldots, q_n, t),$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{ds}{dq_n} = \phi_n(a_1, \ldots, a_n, q_1, \ldots, q_n, t):$$

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now if it be impossible to eliminate the n quantities $a_1, \ldots, a_n$, between these equations, the solution is evidently complete, because the additional equation obtained by differentiating with respect to t will then be required.

But it is known* that the supposed elimination is always impossible unless the determinant formed with the n^2 constituents

$$\begin{matrix} \dfrac{d\phi_1}{da_1}, & \dfrac{d\phi_1}{da_2}, \ldots, & \dfrac{d\phi_1}{da_n} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \dfrac{d\phi_n}{da_1}, & \dfrac{d\phi_n}{da_2}, \ldots, & -\dfrac{d\phi_n}{da_n} \end{matrix}$$

vanishes identically.

We may therefore express the required criterion as follows:

In order that the solution s may be complete, it is only necessary that the determinant formed with the n^2 constituents

$$\begin{matrix} \dfrac{d^2 s}{da_1\, dq_1}, & \dfrac{d^2 s}{da_2\, dq_1}, \ldots, & \dfrac{d^2 s}{da_n\, dq_1} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ \dfrac{d^2 s}{da_1\, dq_n}, & \dfrac{d^2 s}{da_2\, dq_n}, \ldots, & \dfrac{d^2 s}{da_n\, dq_n} \end{matrix}$$

shall not vanish identically. In the abridged notation, explained in the note of Art. 292, this determinant would be written

$$\frac{d\left(\dfrac{ds}{dq_1}, \dfrac{ds}{dq_2}, \ldots, \dfrac{ds}{dq_n}\right)}{d\,(a_1, a_2, \ldots, a_n)}.$$

322.] This being premised, let

$$s = \psi\,(a_1, a_2, \ldots, a_n, q_1, q_2, \ldots, q_n, t)$$

be a complete solution of the equation (34). We have to prove that if $p_1, p_2, \ldots, p_n$ be defined by the n equations $\dfrac{ds}{dq_i} = p_i$,

* If the elimination be possible, it will lead to an identical relation between $\phi_1, \ldots, \phi_n$, considered as functions of $a_1, \ldots, a_n$, say $F(\phi_1, \phi_2, \ldots, \phi_n) = 0$; differentiating this equation with respect to a_i we obtain

$$\frac{dF}{d\phi_1}\frac{d\phi_1}{da_i} + \ldots + \frac{dF}{d\phi_n}\frac{d\phi_n}{da_i} = 0,$$

and if from the n equations given by this formula, on putting $i = 1, \ldots, i = n$, we eliminate the n terms $\dfrac{dF}{d\phi_1}, \ldots, \dfrac{dF}{d\phi_n}$, the result is $\Delta = 0$, Δ being the determinant mentioned in the text.

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and if the $2n$ variables $p_1, \dots, p_n, q_1, \dots, q_n$, be determined as functions of t, and constants, by those equations joined with the n further equations $\frac{ds}{da_i} = b_i$, then p_i, q_i will satisfy the differential equations (35).

Putting p_i for $\frac{ds}{dq_i}$ in (34), that equation becomes

$$\frac{ds}{dt} + f(p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n, t) = 0, \quad (36)$$

which is identical on the supposition that $p_1, \dots, p_n$ are expressed in the form $\frac{ds}{dq_1}, \dots, \frac{ds}{dq_n}$. Differentiating the equation with respect to a_i on this supposition, and observing that

$$\frac{dp_1}{da_i} = \frac{d^2s}{da_i\, dq_1}, \dots,$$

we have

$$\frac{d^2s}{da_i\, dt} + \frac{df}{dp_1}\frac{d^2s}{da_i\, dq_1} + \frac{df}{dp_2}\frac{d^2s}{da_i\, dq_2} + \dots + \frac{df}{dp_n}\frac{d^2s}{da_i\, dq_n} = 0.$$

Again, differentiating the equation $\frac{ds}{da_i} = b_i$ totally, with respect to t, we find

$$\frac{d^2s}{dt\, da_i} + \frac{d^2s}{dq_1\, da_i} q_1' + \frac{d^2s}{dq_2\, da_i} q_2' + \dots + \frac{d^2s}{dq_n\, da_i} q_n' = 0;$$

and subtracting the former equation from the latter,

$$\frac{d^2s}{da_i dq_1}\left(q_1' - \frac{df}{dp_1}\right) + \frac{d^2s}{da_i dq_2}\left(q_2' - \frac{df}{dp_2}\right) + \dots + \frac{d^2s}{da_i dq_n}\left(q_n' - \frac{df}{dp_n}\right) = 0.$$

Suppose the n equations obtained from this by giving i all its values $1, 2, \dots, n$, to be written down; then it is evident, from the theory of algebraic linear equations, that one of two things must be true; either the determinant formed with the n^2 coefficients $\frac{d^2s}{da_i\, dq_j}$ must vanish, or else each of the n terms $q_i' - \frac{df}{dp_i} = 0$. But it was shewn in the last Article that the former supposition would imply that s was not a complete solution of (34). Hence the latter alternative is alone admissible; and since $f(p_1, \dots, p_n, q_1, \dots, q_n, t)$ is the same as H, we have $q_i' = \frac{dH}{dp_i}$; which is one part of the conclusion to be proved.

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To establish the other part, differentiate the equation (36) on the same supposition as before, with respect to q_i, observing that $p_i = \frac{d\text{S}}{dq_i}$, and $\frac{dp_i}{dq_j} = \frac{dp_j}{dq_i}$; the result may then be written,

$$\frac{dp_i}{dt} + \frac{df}{dq_i} + \frac{df}{dp_1}\frac{dp_1}{dq_1} + \frac{df}{dp_2}\frac{dp_i}{dq_2} + \dots + \frac{df}{dp_n}\frac{dp_i}{dq_n} = 0;$$

on the other hand, differentiating the equation $p_i = \frac{d\text{S}}{dq_i}$ totally with respect to t, we obtain

$$p_i' = \frac{dp_i}{dt} + \frac{dp_i}{dq_1}q_1' + \frac{dp_i}{dq_2}q_2' + \dots + \frac{dp_i}{dq_n}q_n';$$

and combining these two equations,

$$p_i' + \frac{df}{dq_i} = \frac{dp_i}{dq_1}\left(q_1' - \frac{df}{dp_1}\right) + \dots + \frac{dp_i}{dq_n}\left(q_n' - \frac{df}{dp_n}\right):$$

but it has just been shewn that every term on the right of this equation vanishes; hence $p_i' + \frac{df}{dq_i} = 0$, or $p_i' = -\frac{d\text{H}}{dq_i}$, which completes the demonstration.

323.] The equations $\frac{d\text{S}}{dq_i} = p_i$, $\frac{d\text{S}}{da_i} = b_i$, give

$$\delta\text{S} = \Sigma(p_i\,\delta q_i) + \Sigma(b_i\,\delta a_i);$$

and if from this we form the expression for $\Delta\delta\text{S}$, and subtract from it the analogous expression for $\delta\Delta\text{S}$, equating the result to zero, see the process of Art. 315, we obtain the equation

$$\Sigma(\delta p_i\,\Delta q_i - \Delta p_i\,\delta q_i) = \Sigma(\delta a_i\,\Delta b_i - \Delta a_i\,\delta b_i);$$

and this being similar to equation (24), Art. 314, it follows, as in Art. 315, that $(a_i, b_i) = 1$, $(a_i, a_j) = (a_i, b_j) = (b_i, b_j) = 0$; so that $a_1, \dots, a_n, b_1, \dots, b_n$, are canonical elements.

324.] The formulæ established in Arts. 321, 322 may be somewhat modified in the case in which H does not contain t explicitly. We have seen, Art. 320, that in this case $h = \text{H}$ is one of the integrals of the problem; h being an arbitrary constant, called the "constant of vis viva."

Suppose then that

$$\text{H} = f(p_1, \dots, p_n, q_1, \dots, q_n);$$

and let V be a complete solution of the equation

$$f\left(\frac{d\text{V}}{dq_1}, \dots, \frac{d\text{V}}{dq_n}, q_1, \dots, q_n\right) = h; \tag{37}$$

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that is, a solution containing, besides h, $n-1$ arbitrary constants $a_1, a_2, \ldots, a_{n-1}$, in such a manner that these $n-1$ constants cannot be all eliminated without employing all the n differential coefficients $\frac{dv}{dq_1}, \ldots, \frac{dv}{dq_n}$. Then, if we take

$$s = -ht + v,$$

it is evident that s will be a solution of the equation

$$\frac{ds}{dt} + f\left(\frac{ds}{dq_1}, \ldots, \frac{ds}{dq_n}, q_1, \ldots, q_n\right) = 0;$$

for we have $\frac{ds}{dt} = -h$; and since $\frac{ds}{dq_i} = \frac{dv}{dq_i}$, the second term in the equation just written $= h$, by (37), so that the two terms destroy one another. Also s is a complete solution, for it is evident that h cannot be eliminated in addition to $a_1, \ldots a_{n-1}$, without employing the differential coefficient $\frac{ds}{dt}$.

If then the function v is known, the integral equations of the problem will be expressible as follows: since $\frac{ds}{dq_i} = \frac{dv}{dq_i}$, we shall have n equations $\frac{dv}{dq_i} = p_i$. Also, since s contains $a_1, \ldots, a_{n-1}$ only as they are contained in v, $\frac{ds}{da_i} = \frac{dv}{da_i}$, so that there will be $n-1$ equations,

$$\frac{dv}{da_i} = b_i. \tag{38}$$

The remaining integral equation is $\frac{ds}{dh} =$ constant. But

$$\frac{ds}{dh} = -t + \frac{dv}{dh},$$

and therefore, putting τ for the constant, we may write the equation

$$\frac{dv}{dh} = t + \tau. \tag{39}$$

In this way we should obtain a solution involving the canonical elements

$$\left.\begin{array}{l} h, a_1, a_2, \ldots a_{n-1}, \\ \tau, b_1, b_2, \ldots b_{n-1}. \end{array}\right\} \tag{40}$$

It is to be observed, that the only one of the integral equations containing t is the equation (39). And it is evident, that if the $2n$ integral equations were solved algebraically, so as to express each of the $2n$ elements (40) in terms of the variables, that is,

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so as to obtain $2n$ "integrals" properly so called, only one of these integrals would contain t, and would be of the form

$$\tau = -t + \phi(p_1, \dots p_n, q_1, \dots q_n).$$

Lastly, it is to be noticed, that whenever the constant of vis viva is one of a set of canonical elements, the element conjugate to it is the constant τ, which is added to t.

325.] Although the preceding theory assigns in a very remarkable manner certain forms in which the integral equations of the system (21) are capable of being expressed, yet it gives no assistance whatever towards the actual integration of that system. For the discovery of the function s, according to Sir W. R. Hamilton's definition of it, would require a knowledge of a complete set of integrals, and according to Jacobi's definition, would depend on the solution of a partial differential equation (34), which is a problem as difficult as the integration of the system (21) itself. In fact, the most hopeful way of attempting the integration of (34) would in general be to make it depend upon that of the system (21). See Boole's Differential Equations, Chapter XIV, Art. 14, and also page 475.

It will be seen however, from a theorem now to be demonstrated, that a knowledge of half the integrals of the system (21) will, when certain conditions are fulfilled, lead to the discovery of the function s, and therefore enable us to complete the integration.

326.] *Theorem**. Suppose $a_1, a_2, \dots, a_n$ to be n integrals of the system

$$p_i' = -\frac{d\mathrm{H}}{dq_i}, \qquad q_i' = \frac{d\mathrm{H}}{dp_i}, \qquad (i = 1 \text{ to } i = n),$$

where H is a given function of $p_1, \dots, p_n, q_1, \dots, q_n, t$; then, if the $\frac{n(n-1)}{2}$ conditions $(a_i, a_j) = 0$ subsist, where i, j is any pair of the n indices, the remaining integrals may be found as

* When this theorem was given by the writer as new, in the Phil. Trans. for 1854, p. 85, he had no means of knowing that it had been previously discovered by M. Liouville, and communicated to the Bureau des Longitudes in 1853; for no accessible notice of this communication appears to have been printed before 1855 (in Liouville's Journal). As M. Liouville has referred to the question of priority, it seemed necessary to mention the subject here. See note to p. 31 of Mr. Cayley's Report, in which for "second part" read "first part."

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follows: By means of the given n integrals, let $p_1, p_2, \dots p_n$ be expressed in terms of $q_1, \dots q_n, a_1, \dots a_n, t$; and let these values be introduced in H, so that H will be expressed in the same way. Then the values of $p_1, p_2, \dots, p_n, -\text{H}$, will be the partial differential coefficients with respect to $q_1, q_2 \dots q_n, t$, of a certain function of $a_1, \dots a_n, q_1 \dots q_n, t$; and calling this function s, we may find it by integrating the expression

$$d\text{s} = p_1 dq_1 + p_2 dq_2 + \dots - \text{H}\, dt; \tag{41}$$

of which the right-hand member is a perfect differential. And the remaining integral equations will be

$$\frac{d\text{s}}{da_1} = b_1, \dots, \qquad \frac{d\text{s}}{da_n} = b_n; \tag{42}$$

where $b_1, \dots, b_n$ are n new arbitrary constants.

327.] The proof of this theorem consists of two parts. First, we have to shew that the above expression for $d\text{s}$ is a perfect differential. Putting e, f for any two of the integrals $a_1, \dots, a_n$, suppose

$$e = \phi(p_1, p_2, \dots p_n, q_1, q_2, \dots q_n, t).$$

If in this equation the values of $p_1, \dots p_n$ were expressed, as above supposed, in terms of $q_1, \dots, a_1 \dots$, it would become identical. Differentiating it therefore on this hypothesis with respect to q_i, we have

$$0 = \frac{de}{dq_i} + \frac{de}{dp_1}\frac{dp_1}{dq_i} + \dots + \frac{de}{dp_n}\frac{dp_n}{dq_i};$$

and in like manner

$$0 = \frac{df}{dq_i} + \frac{df}{dp_1}\frac{dp_1}{dq_i} + \dots + \frac{df}{dp_n}\frac{dp_n}{dq_i}:$$

and if we multiply the first of these equations by $\frac{df}{dp_i}$, and the second by $\frac{de}{dp_i}$, and subtract, there results an equation which may be written thus:

$$\frac{de}{dp_i}\frac{df}{dq_i} - \frac{de}{dq_i}\frac{df}{dp_i} = \Sigma_j \left\{ \frac{dp_j}{dq_i}\left(\frac{de}{dp_j}\frac{df}{dp_i} - \frac{de}{dp_i}\frac{df}{dp_j}\right)\right\};$$

or, writing a_α, a_β instead of e, f, and employing the notation of Art. 292,

$$\frac{d(a_\alpha, a_\beta)}{d(p_i, q_i)} = \Sigma_j \left\{ \frac{dp_j}{dq_i}\frac{d(a_\alpha, a_\beta)}{d(p_j, p_i)}\right\}.$$

If now the terms on each side be summed with respect to i, the

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result on the left is (a_α, a_β); and observing that on the right the term multiplied by $\frac{dp_i}{dq_j}$ will only differ in sign from that multiplied by $\frac{dp_j}{dq_i}$, we obtain

$$(a_\alpha, a_\beta) = \Sigma \left\{ \left(\frac{dp_j}{dq_i} - \frac{dp_i}{dq_j} \right) \frac{d(a_\alpha, a_\beta)}{d(p_j, p_i)} \right\};$$

the summation referring both to i and j, and extending to every binary combination.

From this equation it appears that in order that the condition $\frac{dp_j}{dq_i} = \frac{dp_i}{dq_j}$ may subsist for every pair of indices i, j, it is necessary that the condition $(a_\alpha, a_\beta) = 0$ shall be satisfied for every pair of the n integrals $a_1, \ldots, a_n$. We ought in strictness to prove that no other conditions are necessary. As however the rigorous* proof of this involves certain preliminary theorems, which have not been given in this Chapter, it is here omitted; but it will be found in the Philosophical Transactions for 1854, pp. 84, 85.

Assuming then that $\frac{dp_j}{dq_i} = \frac{dp_i}{dq_j}$, in order to prove that the expression for ds, (41), is a perfect differential, it is necessary also to shew that $\frac{dp_i}{dt} = -\frac{d\text{H}}{dq_i}$. Here H is supposed to be expressed, like p_i, in terms of $a_1, \ldots, a_n, q_1, \ldots, q_n, t$. Putting for a moment (H) to denote H in its original form, namely, a function of $p_1, \ldots, p_n, q_1, \ldots, q_n, t$, we have

$$\frac{d\text{H}}{dq_i} = \frac{d(\text{H})}{dq_i} + \frac{d(\text{H})}{dp_1}\frac{dp_1}{dq_i} + \ldots + \frac{d(\text{H})}{dp_n}\frac{dp_n}{dq_i}:$$

now $\quad \frac{d(\text{H})}{dq_i} = -p_i', \quad \frac{d(\text{H})}{dp_1} = q_1' \ldots, \quad \frac{dp_1}{dq_i} = \frac{dp_i}{dq_1}, \ldots;$

hence this equation becomes

$$\frac{d\text{H}}{dq_i} = -p_i' + q_1'\frac{dp_i}{dq_1} + \ldots + q_n'\frac{dp_i}{dq_n};$$

* It might seem sufficient to say that the number of equations $\frac{dp_j}{dq_i} = \frac{dp_i}{dq_j}$ is the same as that of the equations $(a_i, a_j) = 0$, viz. $\frac{1}{2}n(n-1)$, and that therefore the former set cannot imply any conditions not involved in the latter; but this reasoning is not absolutely conclusive.

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but p_i being expressed as above supposed, we have

$$p_i' = \frac{dp_i}{dt} + \frac{dp_i}{dq_1} q_1' + \ldots + \frac{dp_i}{dq_n} q_n';$$

whence the above value of $\frac{d\text{H}}{dq_i}$ becomes simply

$$\frac{d\text{H}}{dq_i} = -\frac{dp_i}{dt};$$

which is the condition that was to be established, and completes the proof that $p_1 dq_1 + \ldots - \text{H} dt$ is a perfect differential.

Secondly, we have to shew that $\frac{d\text{S}}{da_i}$ is constant by virtue of the differential equations, or that $\left(\frac{d\text{S}}{da_i}\right)' = 0$.

Now*

$$\left(\frac{d\text{S}}{da_i}\right)' = \frac{d^2\text{S}}{dt\, da_i} + \frac{d^2\text{S}}{dq_1\, da_i} q_1' + \ldots + \frac{d^2\text{S}}{dq_n\, da_i} q_n';$$

and $\frac{d\text{S}}{dt} = -\text{H}$, whence $\frac{d^2\text{S}}{dt\, da_i} = -\frac{d\text{H}}{da_i}$; also $\frac{d\text{S}}{dq_1} = p_1$, whence $\frac{d^2\text{S}}{dq_1\, da_i} = \frac{dp_1}{da_i}$; and $q_1' = \frac{d(\text{H})}{dp_1}, \ldots$; so that we obtain

$$\left(\frac{d\text{S}}{da_i}\right)' = -\frac{d\text{H}}{da_i} + \frac{d(\text{H})}{dp_1}\frac{dp_1}{da_i} + \ldots + \frac{d(\text{H})}{dp_n}\frac{dp_n}{da_i};$$

but from the relation between H and (H), we have

$$\frac{d\text{H}}{da_i} = \frac{d(\text{H})}{dp_1}\frac{dp_1}{da_i} + \ldots + \frac{d(\text{H})}{dp_n}\frac{dp_n}{da_i},$$

for H only contains a_i because it is introduced in the values of $p_1, p_2, \ldots, p_n$. Hence the above equation becomes $\left(\frac{d\text{S}}{da_i}\right)' = 0$, or $\frac{d\text{S}}{da_i}$ = constant; which was to be proved.

328.] The formulæ (41), (42), Art. 326, admit of modification in the case in which the so-called "principle of vis viva" holds good; namely, that in which H, in its original form, does not contain t explicitly, so that $\text{H} = h$ is an integral.

In this case, suppose $h, a_1, a_2, \ldots, a_{n-1}$ to be n integrals, satis-

* The reader will not understand this reasoning unless he be careful to recollect the way in which H, (H), S, $p_1, \ldots, p_n$, are supposed to be expressed.

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fying the conditions $(h, a_i) = 0$, $(a_i, a_j) = 0$ for every pair. It is evident that when the n equations, expressing the values of h, $a_1, \ldots$, in terms of the variables, are solved for $p_1, \ldots, p_n$, and the values of these latter quantities substituted in H, the result will be identically the constant h, so that the value of $d\mathrm{s}$ (41) will become

$$d\mathrm{s} = -h\,dt + p_1\,dq_1 + \ldots + p_n\,dq_n,$$

where $p_1, \ldots, p_n$ cannot contain t, otherwise this expression would not be a perfect differential. Hence we shall have

$$\mathrm{s} = -ht + \mathrm{v};$$

where v is a function of $q_1, \ldots, q_n, h, a_1, \ldots, a_{n-1}$, not containing t, and given by the equation

$$d\mathrm{v} = p_1\,dq_1 + \ldots + p_n\,dq_n; \tag{43}$$

of which the right-hand member is a perfect differential.

We shall now have

$$\frac{d\mathrm{s}}{dh} = -t + \frac{d\mathrm{v}}{dh}, \qquad \frac{d\mathrm{s}}{da_1} = \frac{d\mathrm{v}}{da_1}, \ldots;$$

so that if we put τ for the constant conjugate to h, the equations corresponding to (42) may be written

$$\frac{d\mathrm{v}}{dh} = t + \tau, \qquad \frac{d\mathrm{v}}{da_1} = b_1, \ldots, \qquad \frac{d\mathrm{v}}{da_{n-1}} = b_{n-1}. \tag{44}$$

329.] It is useful to observe, that if c be any integral whatever, not containing t explicitly, and h the integral of vis viva, the condition $(h, c) = 0$ necessarily subsists. For we have, see the general expression for u', Art. 316,

$$0 = c' = \frac{dc}{dt} + (\mathrm{H}, c);$$

but $\frac{dc}{dt} = 0$ by hypothesis, and (H, c) is the same as (h, c); consequently $(h, c) = 0$.

From this proposition, combined with the results of the last Article, it is evident, that when the independent coordinates in any dynamical problem in which the principle of vis viva subsists are only two in number, if, besides the integral of vis viva, one other integral, not containing t explicitly, be given, the discovery of the two remaining integrals is reducible to quadratures. For v is given by the equation $d\mathrm{v} = p_1\,dq_1 + p_2\,dq_2$, of which the right-hand member is necessarily a perfect differential. The reader will find it a good exercise to treat the problem of the

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simple conical pendulum in this manner, omitting the effect of the earth's rotation.

330.] In the practical use of the theorem explained in Arts. 326—328, the following circumstances should be attended to. Instead of first finding the function s by integrating the expression $p_1 dq_1 + p_2 dq_2 + \ldots - \text{H}\, dt$, and then forming the integral equations $\frac{d\text{s}}{da_1} = b_1, \ldots$, it is generally more convenient to perform the differentiations with respect to $a_1, \ldots$, first, and the integrations afterwards. Thus we should have

$$b_i = \int \left\{ \frac{dp_1}{da_i} dq_1 + \frac{dp_2}{da_i} dq_2 + \ldots - \frac{d\text{H}}{da_i} dt \right\};$$

but the solution thus obtained will not be canonical, unless care be taken so to assume the inferior limits of the integrals that the functions equated to $b_1, b_2, \ldots$, shall be, as they ought to be, the differential coefficients of one and the same function of $a_1, \ldots, a_n, q_1, q_2, \ldots$.

To shew that this condition will not necessarily subsist, we may take the case which most frequently occurs, namely, that in which the value of p_i contains none of the variables except q_i, so that each term in the value of $d\text{s}$ will be, omitting indices, of the form $\phi(q, a_1, a_2, \ldots, a_n)\, dq$, and so far as one such term is concerned, we should have

$$\text{s} = \int_{\text{A}}^{q} \phi(q, a_1, a_2, \ldots, a_n)\, dq,$$

where A is an arbitrary function of $a_1, \ldots, a_n$. Differentiating this with respect to a_i, we obtain

$$\frac{d\text{s}}{da_i} = \int_{\text{A}}^{q} \frac{d\phi}{da_i} dq - \phi(\text{A}, a_1, \ldots, a_n) \frac{d\text{A}}{da_i};$$

that is,

$$\int_{\text{A}}^{q} \frac{d\phi}{da_i} dq = \frac{d}{da_i} \int_{\text{A}}^{q} \phi . dq + \phi(\text{A}, a_1, \ldots, a_n) \frac{d\text{A}}{da_i}:$$

from which it is evident that differentiating first and integrating afterwards will not in general give the same result as the converse process.

We see however that the values of

$$\int_{\text{A}}^{q} \frac{d\phi}{da_1} dq, \qquad \int_{\text{A}}^{q} \frac{d\phi}{da_2} dq, \ldots$$

will be the differential coefficients with respect to $a_1, a_2, \ldots$ of

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$\int_{\text{A}}^{q} \phi . dq$, provided that A, which must be the same in each of these integrals, be so assumed, that

$$\phi(\text{A}, a_1, \ldots, a_n) \frac{d\text{A}}{da_i} = 0$$

for every value of i. This condition may be secured in either of two ways: first, by taking $\text{A} = 0$, or $\text{A} =$ any determinate constant; that is, not involving $a_1, \ldots$; or secondly, by taking for A any root of the equation

$$\phi(x, a_1, \ldots, a_n) = 0;$$

or, which is the same thing, a value of q which would make p vanish.

331.] As an illustration of the principal formulæ hitherto established, we will take the important problem of the motion of a single particle about a fixed centre of force, in treating which our object will be to obtain a set of canonical elements. The advantage of this form of solution will appear afterwards.

Let m be the mass of the particle, x, y, z its rectangular coordinates at the time t, referred to fixed axes having their origin at the centre of force.

Then if r, θ, λ be polar coordinates, such that

$$x = r \cos\lambda \cos\theta, \qquad y = r \cos\lambda \sin\theta, \qquad z = r \sin\lambda,$$

r is the radius vector; and we may call θ the longitude, reckoned from the axis of x, in the plane of (x, y); and λ the latitude, or the angle between r and the plane of (x, y).

In rectangular coordinates $2\text{T} = m(x'^2 + y'^2 + z'^2)$; but if x', y', z' be expressed in terms of $r, r', \ldots$, it will be found that

$$2\text{T} = m\{r'^2 + r^2(\cos\lambda)^2\theta'^2 + r^2\lambda'^2\}.$$

We shall take r, θ, λ as the coordinates of the problem, corresponding to q_1, q_2, q_3, in the general formulæ. Let the variables conjugate to them, corresponding to p_1, p_2, p_3, be denoted, for greater clearness, by $r_{,}, \theta_{,}, \lambda_{,}$; so that we shall have

$$r_{,} = \frac{d\text{T}}{dr'} = mr', \quad \theta_{,} = \frac{d\text{T}}{d\theta'} = mr^2(\cos\lambda)^2\theta', \quad \lambda_{,} = \frac{d\text{T}}{d\lambda'} = mr^2\lambda';$$

and if the values of r', θ', λ' in terms of $r_{,}, \theta_{,}, \lambda_{,}$, given by these equations, be substituted in T, the result is

$$\text{T} = \frac{1}{2m}\left\{r_{,}^2 + \frac{(\sec\lambda)^2}{r^2}\theta_{,}^2 + \frac{\lambda_{,}^2}{r^2}\right\}. \tag{45}$$

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Since the force is central, the force-function U is a function of r, say $U = \phi(r)$; so that we shall have

$$H = T - \phi(r);$$

in which T stands for the expression (45) just given. See Art. 307. The six differential equations of the problem in the Hamiltonian form would then be

$$r_{,}' = -\frac{dH}{dr}, \qquad \theta_{,}' = -\frac{dH}{d\theta}, \qquad \lambda_{,}' = -\frac{dH}{d\lambda},$$

$$r' = \frac{dH}{dr_{,}}, \qquad \theta' = \frac{dH}{d\theta_{,}}, \qquad \lambda' = \frac{dH}{d\lambda_{,}};$$

but they are not required for our present purpose.

332.] Since H does not contain t explicitly, the integral of vis viva subsists; namely,

$$h = T - \phi(r).$$

We have now to find two other integrals, say c and k, such that $(h, c) = 0$, $(h, k) = 0$, $(c, k) = 0$. We know that the first two of these conditions will be satisfied if c, k be any integrals not containing t explicitly. Let us take then for c one of the three integrals expressing the conservation of areas; namely, $c = m(xy' - x'y)$; and for k, the equation obtained from the three by squaring and adding, namely,

$$k = m\{(yz' - zy')^2 + (zx' - xz')^2 + (xy' - yx')^2\}^{\frac{1}{2}}:$$

these however must be expressed in terms of the new variables; thus,

$$c = mr^2(\cos\lambda)^2\theta' = \theta_{,},$$

$$k = m\{r^2(x'^2 + y'^2 + z'^2) - r^2r'^2\}^{\frac{1}{2}} = \{\theta_{,}^2(\sec\lambda)^2 + \lambda_{,}^2\}^{\frac{1}{2}}.$$

Thus we have the three integrals

$$h = \frac{1}{2m}\left\{r_{,}^2 + \frac{(\sec\lambda)^2}{r^2}\theta_{,}^2 + \frac{1}{r^2}\lambda_{,}^2\right\} - \phi(r), \tag{i}$$

$$c = \theta_{,}, \tag{ii}$$

$$k = \{(\sec\lambda)^2\theta_{,}^2 + \lambda_{,}^2\}^{\frac{1}{2}}; \tag{iii}$$

and if it be recollected that the meaning of (u, v) is now

$$\frac{d(u, v)}{d(r_{,}, r)} + \frac{d(u, v)}{d(\theta_{,}, \theta)} + \frac{d(u, v)}{d(\lambda_{,}, \lambda)},$$

it will be found on trial, not only that $(h, c) = 0$, $(h, k) = 0$, as we know a priori, but also that $(c, k) = 0$.

Hence we may apply the method of Art. 328 to the equations (i), (ii), (iii); that is, we may find V from the equation

$$dV = r_{,}dr + \theta_{,}d\theta + \lambda_{,}d\lambda,$$

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in which $r_{,}, \theta_{,}, \lambda_{,}$ are to be expressed in terms of $r, \theta, \lambda, h, c, k$, by means of (i), (ii), (iii).

It will be found without difficulty that

$$r_{,} = \left[2m\{h+\phi(r)\} - \frac{k^2}{r^2}\right]^{\frac{1}{2}},$$

$$\theta_{,} = c,$$

$$\lambda_{,} = \{k^2 - c^2(\sec\lambda)^2\}^{\frac{1}{2}};$$

and it is obvious that $r_{,}dr + \theta_{,}d\theta + \lambda_{,}d\lambda$ is, as it ought to be, a perfect differential.

Supposing v to be found by integration, the three remaining integral equations would be, Art. 328,

$$\frac{d\mathrm{v}}{dh} = t + \tau, \qquad \frac{d\mathrm{v}}{dc} = \alpha, \qquad \frac{d\mathrm{v}}{dk} = \beta;$$

where τ, α, β are three new constants.

The conclusions of Art. 330, however, shew, that we may perform the differentiations before the integrations, if the inferior limits of the latter be properly chosen; and in accordance with the rule there laid down we may take 0 as the limit for θ and λ, and a root of the equation $2m\{h+\phi(r)\} - \frac{k^2}{r^2} = 0$, say ρ, as the limit for r. This being understood, we should have

$$\mathrm{v} = \int_\rho^r \left\{2m(h+\phi(r)) - \frac{k^2}{r^2}\right\}^{\frac{1}{2}} dr + c\theta + \int_0^\lambda \{k^2 - c^2(\sec\lambda)^2\}^{\frac{1}{2}} d\lambda;$$

and the three new integral equations will be obtained by differentiating under the signs of integration, and integrating afterwards. The integrations with respect to r cannot be performed till $\phi(r)$ is assigned, and in fact will not be actually needed; those with respect to λ are left to the reader. The results are

$$m\int_\rho^r r\{2mr^2(h+\phi(r)) - k^2\}^{-\frac{1}{2}} dr = t + \tau, \tag{iv}$$

$$\theta - \sin^{-1}\frac{c\tan\lambda}{(k^2-c^2)^{\frac{1}{2}}} = \alpha, \tag{v}$$

$$-k\int_\rho^r \frac{1}{r}\{2mr^2(h+\phi(r)) - k^2\}^{-\frac{1}{2}} dr + \sin^{-1}\frac{k\sin\lambda}{(k^2-c^2)^{\frac{1}{2}}} = \beta. \tag{vi}$$

The equations (i), (ii), (iii), (iv), (v), (vi) comprise a canonical solution of the problem.

333.] It is easy to interpret the constants τ, α, β.

The equation (iv) shews that $t + \tau = 0$ when $r = \rho$, so that

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$-\tau$ is the time at which $r = \rho$. Now if we differentiate the equation with respect to t, we obtain

$$mr' = \left\{2m(h+\phi(r)) - \frac{k^2}{r^2}\right\}^{\frac{1}{2}};$$

and we took for ρ a value of r, which makes this expression vanish; hence, when $r = \rho$, $r' = 0$, and it is evident therefore that we may suppose ρ to be the minimum value of r. Thus $-\tau$ will be the time at which m is at the least distance from the centre of force.

Again, if we put ι for the inclination of the plane of the motion to the plane of $(x, y,)$ we have, by well-known principles, $c = k\cos\iota$; hence

$$\frac{c}{(k^2-c^2)^{\frac{1}{2}}} = \cot\iota, \qquad \frac{k}{(k^2-c^2)^{\frac{1}{2}}} = \frac{1}{\sin\iota};$$

equation (v) is therefore equivalent to

$$\sin(\theta - a) = \tan\lambda\cot\iota. \tag{46}$$

Next let ϑ be the "argument of latitude;" that is, the angle between r and the ascending node, or line joining m with the centre at the instant when m passes through the plane of (x, y) from the negative to the positive side; considering then the right-angled spherical triangle, formed by the intersections with a sphere of the plane of (x, y), and the planes of the angles ϑ and λ, in which ϑ is the hypothenuse and ι the angle opposite to λ, we see, from (46), that $\theta - a$ is the side which is in the plane of (x, y), and therefore a is evidently the longitude of the ascending node.

Lastly, equation (vi) shews that when $r = \rho$, $\sin\lambda = \sin\iota.\sin\beta$; but from the triangle above mentioned we have also $\sin\lambda = \sin\iota.\sin\vartheta$; hence we see that when $r = \rho$, $\vartheta = \beta$, from which it is evident that β is the angle between the least distance and the node.

334.] We shall briefly indicate the application of these results to the case of the undisturbed motion of a planet about the sun. Let μ = mass of sun + mass of planet; then, m being the mass of the planet, we shall have for the force-function $\phi(r) = \dfrac{m\mu}{r}$; also, from the ordinary theory of elliptic motion, we have

$$h = -\frac{m\mu}{a}, \qquad k = m\{\mu a(1-e^2)\}^{\frac{1}{2}}, \qquad c = m\{\mu a(1-e^2)\}^{\frac{1}{2}}\cos\iota.$$

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Hence we have the following set of canonical elements, arranged in conjugate pairs. The signs of the first pair have been changed, which is obviously allowable, and p is put for $a(1-e^2)$, that is, for the semi-latus-rectum;

$\frac{m\mu}{a}$, time of perihelion passage;

$m(\mu p)^{\frac{1}{2}}\cos\iota$, longitude of node;

$m(\mu p)^{\frac{1}{2}}$, distance of perihelion from node.

These elements were first given by Jacobi.

335.] Variation of elements. Returning to the general form of the differential equations,

$$p_i = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}, \qquad (i = 1, \text{ to } i = n),$$

let us suppose that a complete set of integrals $c_1, c_2, \ldots, c_{2n}$ has been obtained, where $c_1, c_2, \ldots$ represent certain functions of the variables $p_1, q_1, \ldots$, and t, which, by virtue of the differential equations, have constant values; and conversely, $p_1, p_2, \ldots$ are given functions of $c_1, c_2, \ldots$, and t. Now let $c_1, \ldots, c_{2n}$ continue to represent the same functions of the variables and t, but let the variables be determined as functions of t by a different set of differential equations, say,

$$p_i' = -\frac{d\text{Q}}{dq_i}, \qquad q_i' = \frac{d\text{Q}}{dp_i}; \tag{47}$$

where Q is, like H, a given function of $p_1, \ldots, p_n, q_1, \ldots, q_n, t$, but not the same function as H. Then the functions $c_1, \ldots, c_{2n}$ will no longer have constant values. In other words, the elements $c_1, c_2, \ldots$, which were constant in the former problem, are variable in the latter problem.

The theory of the variation of elements has for its principal object to transform the equations (47) by taking as new variables the functions $c_1, \ldots, c_{2n}$, instead of the original variables $p_1, \ldots, p_n, q_1, \ldots, q_n$.

The integration of the equations after this transformation would give $c_1, \ldots, c_{2n}$ as functions of t and $2n$ constants; and the variables $p_1, p_2, \ldots$ would then be known as functions of t, because they are the same functions of the elements $c_1, c_2, \ldots$ and t, as they were in the original problem.

In practice, it is usually convenient to put $\text{Q} = \text{H} + \Omega$, where H

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is the same as before, and Ω is a given function of $p_1, \ldots, p_n$, $q_1, \ldots, q_n$, t, and is called technically the "disturbing function." Thus the equations (47) become

$$p_i' = -\frac{d\text{H}}{dq_i} - \frac{d\Omega}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i} + \frac{d\Omega}{dp_i};$$

and these are transformed as follows: we have

$$c_1' = \frac{dc_1}{dt} + \Sigma\left(\frac{dc_1}{dp_i}p_i' + \frac{dc_1}{dq_i}q_i'\right);$$

that is, introducing the above values of p_i', q_i',

$$c_1' = \frac{dc_1}{dt} + (\text{H}, c_1) + (\Omega, c_1):$$

now c_1 is by hypothesis a function of $p_1, \ldots$ and t, which in the "undisturbed" problem, that is, when $\Omega = 0$, is constant; and this constancy would be expressed by the equation

$$\frac{dc_1}{dt} + (\text{H}, c_1) = 0,$$

which is identically true, all its terms being functions of $p_1, \ldots, q_1, \ldots, t$. But since c_1 and H are the same functions of $p_1, \ldots, q_1, \ldots, t$, in the disturbed, as they were in the undisturbed problem, the above equation is still identically true, and the expression for c_1' therefore becomes simply

$$c_1' = (\Omega, c_1):$$

here Ω is expressed in its original form as a function of $p_1, \ldots, q_1, \ldots, t$; but if we suppose $p_1, \ldots, q_1, \ldots$ to be expressed in terms of $c_1, \ldots, c_{2n}, t$, Ω will become a function of the latter quantities, and denoting, for a moment, the new form of Ω by $\bar{\Omega}$, we shall have, by the elementary properties of the symbol (u, v), Art. 311,

$$c_1' = (\Omega, c_1) = \frac{d\bar{\Omega}}{dc_1}(c_1, c_1) + \frac{d\bar{\Omega}}{dc_2}(c_2, c_1) + \ldots + \frac{d\bar{\Omega}}{dc_{2n}}(c_{2n}, c_1);$$

in which the first term vanishes, since $(c_1, c_1) = 0$.

The general form is evidently, omitting the line over Ω,

$$c_i' = \frac{d\Omega}{dc_1}(c_1, c_i) + \frac{d\Omega}{dc_2}(c_2, c_i) + \ldots + \frac{d\Omega}{dc_{2n}}(c_{2n}, c_i), \qquad (48)$$

involving $2n-1$ terms on the right-hand side. And by giving to i the values $1, 2, \ldots, 2n$, we obtain a set of $2n$ differential equations, which will involve only the new variables $c_1, c_2, \ldots$

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instead of $p_1, p_2, \ldots$, when the terms $(c_1, c_i)\ldots$, which are given functions of the old variables, are expressed in terms of the new. It follows, from what was proved in Art. 316, that when so expressed they will not contain t.

336.] But the formula (48) undergoes a remarkable simplification when $c_1, \ldots c_{2n}$, are a set of canonical elements; for in that case all the terms (c_1, c_i), (c_2, c_i), ... vanish except one, namely, that in which c_i is combined with its conjugate element, and then the term is either $+1$ or -1.

Suppose then that $a_1, \ldots a_n, b_1, \ldots b_n$ are canonical elements, so that $(a_i, b_i) = 1$, $(b_i, a_i) = -1$, and all other combinations give zero. The formula (48) will be seen at once to give

$$a_i' = -\frac{d\Omega}{db_i}, \qquad b_i' = \frac{d\Omega}{da_i}, \tag{49}$$

which are the transformed equations in this case, and are of the same general form as the original equations.

These formulæ were discovered in a particular case by Lagrange; namely, that in which $a_1, b_1, \ldots$ are the initial values of $p_1, q_1, \ldots$. The extension to canonical elements in general is due partly to Sir W. R. Hamilton and partly to Jacobi.

There are many points of interest and importance connected with the further development and application of the theory of the variation of elements, but we cannot afford space for them here.

337.] We shall conclude this Chapter by a brief notice of an addition to the general theory of the Hamiltonian equations made by M. Bour, and improved in accordance with a suggestion of M. Liouville*.

Let $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$, be, as before, a set of canonical integrals of the equations

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}. \tag{50}$$

* See Liouville's Journal, t. xx. pp. 185–200, 201, 202. The writer is not aware whether M. Bour's investigations have yet been published complete in the Memoirs of the Academy. He believes that the process and results in the text must be substantially what those of M. Bour become when H contains t, which is the generalization proposed by M. Liouville; but he can only speak from conjecture on this point.

[The Memoir of M. Bour, to which the note refers, is published in Tome XIV of the Mémoires des Savants Etrangers, p. 792; having been presented to the Academy of Sciences at Paris on March 5th, 1855.]

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If we represent any integral whatever by ζ, the equation $\zeta' = 0$ gives $\frac{d\zeta}{dt} + (\text{H}, \zeta) = 0$; which, written in full, would be

$$\frac{d\zeta}{dt} + \frac{d\text{H}}{dp_1}\frac{d\zeta}{dq_1} - \frac{d\text{H}}{dq_1}\frac{d\zeta}{dp_1} + \dots + \frac{d\text{H}}{dp_n}\frac{d\zeta}{dq_n} - \frac{d\text{H}}{dq_n}\frac{d\zeta}{dp_n} = 0. \quad (51)$$

In this equation $\frac{d\text{H}}{dp_1}, \frac{d\text{H}}{dq_1}, \dots$ are given functions of $p_1, \dots p_n, q_1, \dots q_n, t$; so that with reference to ζ it is a linear partial differential equation of the first order, of which any solution whatever is an integral of the equations (50), and of which the general solution might be put in the form

$$\zeta = f(a_1, \dots, a_n, b_1, \dots, b_n),$$

where f is an arbitrary function.

Thus the complete integration of the equations (50) would be effected if we could find the general solution of (51). But this transformation of the problem is practically useless, since the only known general method of treating the equation (51) requires the integration of the system (50), as the reader will see at once on applying Lagrange's method to the former.

The proposition we are about to demonstrate is the following:

Suppose m of the n integrals $a_1, \dots a_n$, to be known; say $a_1, a_2, \dots, a_m$, where m is of course less than n; and the condition $(a_i, a_j) = 0$ to subsist for every pair; then by means of these m integrals, m of the variables, say $p_1, p_2, \dots, p_m$, might be expressed in terms of

$$a_1, a_2, \dots, a_m, p_{m+1}, \dots p_n, q_1, \dots, q_n, t; \quad (52)$$

and by the substitution of these values of $p_1, \dots p_m$, H and ζ might be expressed in the same way: we shall prove that after this substitution the equation (51) would still subsist, the differentiations being all performed only so far as the variables would appear explicitly. But since H and ζ would no longer contain $p_1, \dots, p_m$ explicitly, all the terms in (51) which involve differentiation with respect to these m variables would disappear; and the equation would become

$$\frac{d\zeta}{dt} + \frac{d\text{H}}{dp_{m+1}}\frac{d\zeta}{dq_{m+1}} - \frac{d\text{H}}{dq_{m+1}}\frac{d\zeta}{dp_{m+1}} + \dots + \frac{d\text{H}}{dp_n}\frac{d\zeta}{dq_n} - \frac{d\text{H}}{dq_n}\frac{d\zeta}{dp_n} = 0; \quad (53)$$

in which the number of terms is diminished by $2m$.

In this equation H is a given function of the quantities (52); but as it does not involve $\frac{d\zeta}{dq_1}, \dots, \frac{d\zeta}{dq_m}$, the variables $q_1, \dots,$

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would be treated as constants in integration. Any solution of it will be a function of the same quantities (52), and will be an integral of (50); but the number of distinct solutions will be less by $2m$ than those of (51). In fact (53) will be satisfied by $a_{m+1}, \dots a_n, b_{m+1}, \dots b_n$, but not by $a_1, \dots a_m, b_1, \dots b_m$; and it follows that any solution c of (53) will satisfy the conditions $(c, a_1) = 0$, $(c, a_m) = 0$, but any two solutions, c, e, will not necessarily satisfy the condition $(c, e) = 0$.

338.] We proceed to demonstrate what has been stated in the last Article. Let H, when transformed by substituting for $p_1, \dots p_m$ the values obtained from the given integrals $a_1, \dots a_m$ in terms of the quantities (52), be denoted by $\bar{\text{H}}$. The equation (51) is

$$\frac{d\zeta}{dt} + (\text{H}, \zeta) = 0;$$

but by the elementary property of the symbol (u, v), before referred to, we have

$$(\text{H}, \zeta) = (\bar{\text{H}}, \zeta) + \frac{d\bar{\text{H}}}{da_1}(a_1, \zeta) + \dots + \frac{d\bar{\text{H}}}{da_m}(a_m, \zeta);$$

in which the expression $(\bar{\text{H}}, \zeta)$ is to be formed by differentiating $\bar{\text{H}}$ with respect to the variables only so far as they appear explicitly. Thus the equation (51) becomes

$$\frac{d\zeta}{dt} + (\bar{\text{H}}, \zeta) + \frac{d\bar{\text{H}}}{da_1}(a_1, \zeta) + \dots + \frac{d\bar{\text{H}}}{da_m}(a_m, \zeta) = 0;$$

and this would still be satisfied by putting $\zeta =$ any integral of (50). But if ζ be any one of the integrals $a_1, \dots a_n, b_{m+1}, \dots b_n$, we shall have $(a_1, \zeta) = 0$, $(a_2, \zeta) = 0, \dots (a_m, \zeta) = 0$, whereas one of these terms would be unity if ζ were one of the integrals $b_1, \dots b_m$. Consequently the equation

$$\frac{d\zeta}{dt} + (\bar{\text{H}}, \zeta) = 0 \tag{54}$$

will be satisfied by $a_1, \dots, a_n, b_{m+1}, \dots b_n$, but not by $b_1, \dots b_m$.

In this equation however ζ is still supposed to be expressed in terms of all the variables $p_1, \dots, q_1, \dots$, and t. But if we suppose ζ to be transformed in the same way as H, and then to be denoted by $\bar{\zeta}$, we shall have

$$\frac{d\zeta}{dt} = \frac{d\bar{\zeta}}{dt} + \frac{d\bar{\zeta}}{da_1}\frac{da_1}{dt} + \dots + \frac{d\bar{\zeta}}{da_m}\frac{da_m}{dt},$$

$$(\bar{\text{H}}, \zeta) = (\bar{\text{H}}, \bar{\zeta}) + \frac{d\bar{\zeta}}{da_1}(\bar{\text{H}}, a_1) + \dots + \frac{d\bar{\zeta}}{da_m}(\bar{\text{H}}, a_m);$$

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so that the equation (54) becomes

$$\frac{d\bar{\zeta}}{dt} + (\bar{\mathrm{H}}, \bar{\zeta}) + \frac{d\bar{\zeta}}{da_1}\left\{\frac{da_1}{dt} + (\bar{\mathrm{H}}, a_1)\right\} + \ldots + \frac{d\bar{\zeta}}{da_m}\left\{\frac{da_m}{dt} + (\bar{\mathrm{H}}, a_m)\right\} = 0,$$

in which all the terms after the first two vanish; for, since a_1 is an integral, we have $a_1' = 0$, that is, $\frac{da_1}{dt} + (\mathrm{H}, a_1) = 0$; but

$$(\mathrm{H}, a_1) = (\bar{\mathrm{H}}, a_1) + \frac{d\bar{\mathrm{H}}}{da_1}(a_1, a_1) + \frac{d\bar{\mathrm{H}}}{da_2}(a_2, a_1) + \ldots + \frac{d\bar{\mathrm{H}}}{da_m}(a_m, a_1),$$

and $(a_1, a_1) = 0$, $(a_2, a_1) = 0$, ... $(a_m, a_1) = 0$, so that $(\mathrm{H}, a_1) = (\bar{\mathrm{H}}, a_1)$; and therefore $\frac{da_1}{dt} + (\bar{\mathrm{H}}, a_1) = 0$; and the same reasoning applies to the other terms; finally, therefore, the equation becomes

$$\frac{d\bar{\zeta}}{dt} + (\bar{\mathrm{H}}, \bar{\zeta}) = 0;$$

and this will be satisfied by the same integrals as (54) with the following exceptions. Suppose the integral a_i to be

$$a_i = \phi(p_1, \ldots, p_n, q_1, \ldots, q_n, t); \tag{56}$$

then (54) is satisfied by putting for ζ the function ϕ on the right of this equation. And if this function be transformed by putting for $p_1, \ldots p_m$ their values in terms of the quantities (52), a_i will in general be expressed as a function of the same quantities, and (55) will be satisfied by taking this function for $\bar{\zeta}$; but if i be one of the indices $1, 2, \ldots, m$, it is evident that the transformation in question will reduce the right-hand member of (56) identically to a_i, so that ϕ will cease to contain any of the variables; and (55) will then only be satisfied by $\bar{\zeta} = a_i$, in the sense that all the differential coefficients of $\bar{\zeta}$ vanish separately; that is, in the sense that the equation is satisfied by putting $\bar{\zeta} =$ any constant.

It follows therefore that, in the ordinary sense, (55) is satisfied by $a_{m+1}, \ldots a_n, b_{m+1}, \ldots b_n$, expressed in terms of the quantities (52), but not by the remaining integrals of the canonical set.

If the distinctive notation of (55) be omitted, and the equation be written at length, the result is the equation (53) of the last Article.

These formulæ require some alteration in the case in which

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the principle of vis viva subsists; but we have not space to enter further into the subject.

339.] In taking leave of the subject, it is proper to mention the investigations of Jacobi, contained in his Memoirs, entitled, "Theoria nova multiplicatoris systemati æquationum differentialium vulgarium applicandi," Crelle's Journal, Vols. XXVII and XXIX. These investigations are only so far connected with the subject of this Chapter, that they are applicable to the Hamiltonian equations as a particular case. But although the results are interesting and important, they are omitted here, because the demonstration depends on the properties of functional determinants, and could not be given without a long digression.

THE END.

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CORRECTIONS.

Page 22, line 20, after "planes" insert "parallel to the cyclic planes"

24, — 32, for "$+y_2 x_3$" read "$-y_2 x_3$"

38, — 10, for "PQRS" read "PSQR"

78, — 22, for "vertical" read "virtual"

101, — 3, for "these" read "their"

179, — 3, for "line" read "conic"

180, — 9, for "vector" read "ellipsoid"

181, — 20, for "$my_0 - nz_0$" read "$ny_0 - mz_0$"

182, — 1, for "lines" read "locus"

203, — 38, for "Ω" read "Ω^2"

205, — 38, for "a" read "a^2"

223, — 7, for "m" read "M"

232, — 17, for "2" read "k^2"

251, — 24, for "b^4" and "a^4" read "ab^3" and "$a^3 b$" respectively

265, — 21, for "AB" read "AC"

332, — 29, for "O′P′" read "O′C′"

352, — 13, for "$a^2 L^2 + b^2 M^2$" read "$b^2 L^2 + a^2 M^2$"

401, — 35, for "dt" read "$d\theta$"

428, — 37, for "k^2" read "$\omega^2 k^2$"

442, — 7, omit "always"

442, — 8, after "westwards" insert "so long as θ is less than $180° - \tan^{-1}(3 \tan \lambda)$; and this deviation vanishes if $\theta = 180° - \tan^{-1}(3 \tan \lambda)$; and the deviation is eastwards if θ is greater than $180° - \tan^{-1}(3 \tan \lambda)$"

445, — 34, for "ζ" read "$\zeta - h$"

451, — 22, after "one" insert "and the greater"; omit "and .... greater"

472, — 28, for "ω" read "$\omega \sin \nu$"

557, equation (35), for "$-\dfrac{dH}{dp_i}$" read "$\dfrac{dH}{dp_i}$"

558, line 8, for "$-\dfrac{d\phi_n}{da_n}$" read "$\dfrac{d\phi_n}{da_n}$"

571, last line, for "$\dfrac{m\mu}{a}$" read "$\dfrac{m\mu}{2a}$"

572, line 5, for "$\dfrac{m\mu}{a}$" read "$\dfrac{m\mu}{2a}$"

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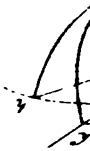

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